Achievable Rate Region of Gaussian Broadcast Channel with Finite Input Alphabet and Quantized Output

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***Abstract*—In this paper, we study the achievable rate region of two-user Gaussian broadcast channel (GBC) when the messages to be transmitted to both the users take values from finite signal sets and the received signal is *quantized* at both the users. We refer to this channel as *quantized broadcast channel (QBC)*. We first observe that the capacity region defined for a GBC does not carry over as such to QBC. Also, we show that the optimal decoding scheme for GBC (i.e., high SNR user doing successive decoding and low SNR user decoding its message alone) is not optimal for QBC. We then propose an achievable rate region for QBC based on two different schemes. We present achievable rate region results for the case of uniform quantization at the receivers. We find that rotation of one of the user's input alphabet with respect to the other user's alphabet marginally enlarges the achievable rate region of QBC when almost equal powers are allotted to both the users.**

Keywords – *Gaussian broadcast channel, finite input alphabet, quantized receiver, achievable rate region, successive decoding.*

I. Introduction

Communication receivers are often based on digital signal processing, where the analog received signal is quantized into finite number of bits using analog-to-digital converters (ADC) whose outputs are then processed in digital domain. These ADCs are expected to operate at high speeds in order to meet the increasing throughput and bandwidth requirements. However, at high conversion speeds, the precision of ADCs is typically low which results in loss of system performance [1]. For example, low-precision receiver quantization can cause floors in the bit error performance [2],[3]. Also, it has been shown that in a single-input single-output (SISO) point-to-point single user system with additive white Gaussian noise (AWGN), low-precision receiver quantization results in significant loss of capacity when compared to an unquantized receiver [4]. Motivated by the increasing need to investigate the effect of receiver quantization in high-throughput communication, we, in this paper, address the issue of characterizing the achievable rate region in Gaussian broadcast channel with finite input alphabet and *quantized receiver output*[1], and report some interesting results.

Gaussian broadcast channel (GBC) comes under the class of stochastically degraded broadcast channels, for which capacity is known. For a two-user GBC, it is known that the capacity is achieved when superposition coding is done at the transmitter assuming that the users' messages are from Gaussian distribution, and, at the receiver, the high SNR user does successive decoding and the low SNR user decodes its message alone considering the other user's message as noise [6]. However, the capacity region of two-user GBC when the messages to be transmitted to both the users take values from finite signal sets and the received analog signals at the users are *quantized*, is not known. Recently, achievable rate region for two-user GBC when the input messages are from finite signal sets and the received signals are *unquantized* has been studied in [7], and it is referred to as the constellation constrained (CC) capacity of GBC [8].

In the above context, our present contribution gives achievable rate region for two-user GBC with finite input alphabet *as well as quantized receiver output* (we refer to this channel as QBC - Quantized Broadcast Channel). The main results are summarized as follows.

- The capacity region defined for a GBC does not carry over as such to QBC.
- With quantization at the receiver in a GBC, the channel no more remains stochastically degraded. Therefore, the optimal decoding scheme for GBC (i.e., high SNR user alone doing successive decoding) does not necessarily result in achievable rate pairs for QBC.
- We then propose achievable rate region for QBC based on two different schemes (scheme 1 and scheme 2). In scheme 1, user 1 will do successive decoding and user 2 will not. Whereas, in scheme 2, user 2 will do successive decoding and user 1 will not. In addition, in both the schemes, the message for the user which does not do successive decoding is coded at such a rate that the message of that user can be decoded error free at both the receivers.
- Rotation of one of the user's input alphabet with respect to the other user's alphabet marginally enlarges the achievable rate region of QBC when almost equal powers are allotted to both the users.

II. System Model

We consider a two-user GBC as shown in Fig. 1. Let x_1 and x_2 denote the messages to be transmitted to the users 1 and 2, respectively. Let x_1 and x_2 take values from finite signal sets $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively. The sets $\mathcal{X}_1$ and $\mathcal{X}_2$ contain N_1 and N_2 equi-probable complex entries, respectively. Let the

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[1]We refer to Gaussian broadcast channel (GBC) with finite input alphabet and quantized receiver output as *Quantized Broadcast Channel (QBC)*.
In a related recent study in [5], we have investigated the achievable rate region of two-user Quantized MAC (QMAC), i.e., a Gaussian MAC with finite input alphabet and quantized receiver output.

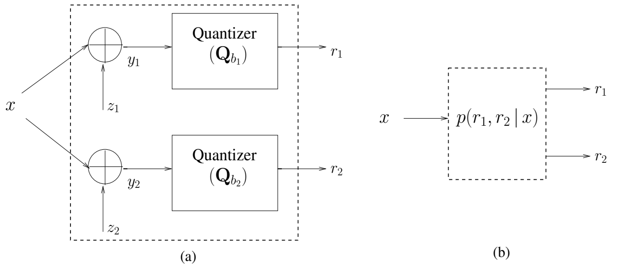

Fig. 1. (a) Two-user Gaussian broadcast channel with receiver quantization. (b) Equivalent discrete memoryless channel.

sum signal set of $\mathcal{X}_1$ and $\mathcal{X}_2$ be defined as

$$\mathcal{X} = \{x_1 + x_2 \mid x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2\}. \quad (1)$$

Let X^I and X^Q be defined as

$$X^I \triangleq \max_{a \in \mathcal{X}} |a^I|, \quad X^Q \triangleq \max_{a \in \mathcal{X}} |a^Q|, \quad (2)$$

where a^I and a^Q represent the real and imaginary components of a, respectively. Let $x \in \mathcal{X}$ be the message sent by the transmitter to the users 1 and 2 with an average power constraint P. We assume that the average power constraint on x_1 is αP and the average power constraint on x_2 is $(1-\alpha)P$, where $\alpha \in (0, 1)$. Let $z_1 \sim \mathcal{CN}(0, \sigma_1^2)$ and $z_2 \sim \mathcal{CN}(0, \sigma_2^2)$ denote the AWGN at receivers 1 and 2, respectively. The SNR at user 1 (SNR1) is P/σ_1^2 and the SNR at user 2 (SNR2) is P/σ_2^2. The received signals at user 1 and user 2 are given by

$$y_1 = x + z_1 = x_1 + x_2 + z_1, \quad (3)$$
$$y_2 = x + z_2 = x_1 + x_2 + z_2. \quad (4)$$

The received analog signals, y_1 at user 1 and y_2 at user 2, are quantized independently, resulting in outputs r_1 at user 1 and r_2 at user 2. The complex quantizer at each user is composed of two similar quantizers acting independently on the real and imaginary components of the received analog signal. The real and imaginary components of the quantized output for the users 1 and 2 are then given by

$$r_1^I = \mathbf{Q}_{b_1}(y_1^I), \quad r_1^Q = \mathbf{Q}_{b_1}(y_1^Q), \quad (5)$$
$$r_2^I = \mathbf{Q}_{b_2}(y_2^I), \quad r_2^Q = \mathbf{Q}_{b_2}(y_2^Q), \quad (6)$$

where the functions $\mathbf{Q}_{b_1}(.)$ and $\mathbf{Q}_{b_2}(.)$ model the quantizers having a resolution of b_1 and b_2 bits, respectively. $\mathbf{Q}_{b_1}(.)$ is a mapping from the set of real numbers $\mathbb{R}$ to a finite alphabet set $\mathcal{S}_{b_1}$ of cardinality 2^{b_1}, i.e.,

$$\mathbf{Q}_{b_1} : \mathbb{R} \mapsto \mathcal{S}_{b_1}, \quad \mathcal{S}_{b_1} \subset \mathbb{R}, \quad |\mathcal{S}_{b_1}| = 2^{b_1}, \quad (7)$$
$$\mathbf{Q}_{b_2} : \mathbb{R} \mapsto \mathcal{S}_{b_2}, \quad \mathcal{S}_{b_2} \subset \mathbb{R}, \quad |\mathcal{S}_{b_2}| = 2^{b_2}. \quad (8)$$

Thus, the quantized received signals r_1 at user 1 and r_2 at user 2 take values from the sets $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively:

$$\mathcal{R}_1 = \{r_1^I + j r_1^Q \mid r_1^I, r_1^Q \in \mathcal{S}_{b_1}\}, \quad |\mathcal{R}_1| = 2^{2b_1}, \quad (9)$$
$$\mathcal{R}_2 = \{r_2^I + j r_2^Q \mid r_2^I, r_2^Q \in \mathcal{S}_{b_2}\}. \quad |\mathcal{R}_2| = 2^{2b_2}. \quad (10)$$

Henceforth, we refer to the above system model as *quantized broadcast channel (QBC)*.

III. Achievable Rate Region of QBC

In this section, we derive analytical expressions for the achievable rate region of two-user QBC.

Capacity and Degradedness in GBC: The capacity region of a two-user GBC is known [9],[10], and is given by the set of all rate pairs (R_1, R_2) satisfying

$$R_1 \leq I(x_1; y_1 \,|\, x_2) \quad (11)$$
$$R_2 \leq I(x_2; y_2), \quad (12)$$

assuming $\sigma_1^2 < \sigma_2^2$, where R_1 and R_2 represent the rates achieved by users 1 and 2, respectively. The optimal input distribution that attains the capacity is Gaussian. The optimal decoding scheme is that, user 1 does successive decoding (i.e., user 1 first decodes user 2's message assuming its own message as noise and subtracts the decoded user 2's message $\hat{x}_2$ from its received signal y_1, and then decodes its own message from the subtracted signal $y_1 - \hat{x}_2$), and user 2 decodes its message by taking user 1's message as noise.

Definition 1. *A broadcast channel is said to be physically degraded [9] if $p(y_1, y_2|x) = p(y_1|x)\, p(y_2|y_1)$.*

Definition 2. *A broadcast channel is said to be stochastically degraded [9] if its conditional marginal distributions are the same as that of a physically degraded broadcast channel; that is, if there exists a distribution $p'(y_2|y_1)$ such that*

$$p(y_2|x) = \sum_{y_1} p(y_1|x)\, p'(y_2|y_1). \quad (13)$$

GBC belongs to the class of stochastically degraded broadcast channels. In the following, we show that QBC, unlike GBC, is not stochastically degraded, and hence GBC capacity expressions do not carry over to QBC.

Degradedness in QBC: If QBC is stochastically degraded, then there must exist a $p'(r_2|r_1)$ such that

$$p(r_2|x) = \sum_{r_1} p(r_1|x)\, p'(r_2|r_1) \quad (14)$$

for all r_2 and x. Towards checking the existence of such a $p'(r_2|r_1)$, we define the following.

Let $A \triangleq [A_{ij}]_{|\mathcal{X}| \times |\mathcal{R}_2|}$, $B \triangleq [B_{ik}]_{|\mathcal{X}| \times |\mathcal{R}_1|}$ and $P' \triangleq [P'_{kj}]_{|\mathcal{R}_1| \times |\mathcal{R}_2|}$ where[2] $A_{ij} \triangleq p(r_2 = \mathcal{R}_2(j) \,|\, x = \mathcal{X}(i))$, $B_{ik} \triangleq p(r_1 = \mathcal{R}_1(k) \,|\, x = \mathcal{X}(i))$ and $P'_{kj} \triangleq p(r_2 = \mathcal{R}_2(j) \,|\, r_1 = \mathcal{R}_1(k))$. Solving (14) is same as finding a matrix P' such that $A = BP'$ under the constraint $\sum_j P'_{ij} = 1, \ \forall i$ and $P'_{ij} \geq 0, \ \forall i, j$. Equivalently, this can be written as the following convex optimization problem

$$P' = \arg \min_{P \in \mathbb{R}^{|\mathcal{R}_1| \times |\mathcal{R}_2|}} \|A - BP\|_F^2 \quad s.t.$$
$$\sum_j P_{ij} = 1, \ \forall i \text{ and } P_{ij} \geq 0, \ \forall i, j, \quad (15)$$

where $\|.\|_F$ is the Frobenius norm. Observe that, the channel is stochastically degraded only if $\|A - BP'\|_F^2 = 0$. When both the users use the same quantizer resolution,

[2] Assuming that the elements of the sets $\mathcal{X}$, $\mathcal{R}_1$ and $\mathcal{R}_2$ are ordered, we denote the lth element of the sets $\mathcal{X}$, $\mathcal{R}_1$ and $\mathcal{R}_2$ by $\mathcal{X}(l)$, $\mathcal{R}_1(l)$ and $\mathcal{R}_2(l)$, respectively.

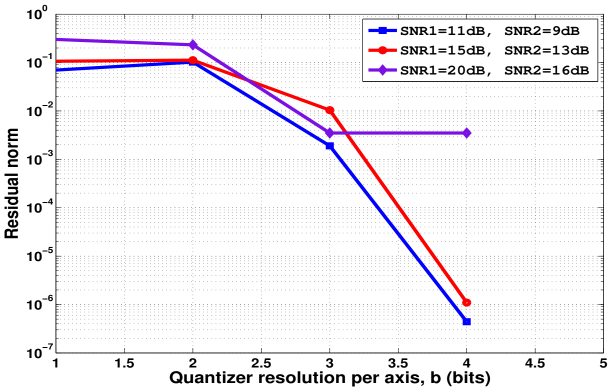

Fig. 2. Plot of the residual norm $\|A - BP'\|_F^2$ as a function of $b_1 = b_2 = b$ bits for different SNR combinations with $\alpha = 0.7$.

$b_1 = b_2 = b$, it is clear that the residual norm $\|A - BP'\|_F^2 > 0, \ \forall \ b \leq \lfloor \frac{1}{2}\log_2(|\mathcal{X}|+1) \rfloor$. For $b > \lfloor \frac{1}{2}\log_2(|\mathcal{X}|+1) \rfloor$ also, $\|A - BP'\|_F^2$ is observed to be greater than zero by numerically solving (15) using convex programming tools; this is illustrated in Fig. 2. Fig. 2 shows the plot of the residual norm, $\|A - BP'\|_F^2$, for different SNR combinations when both the users use uniform receiver quantizers of same resolution. The input alphabet of user 1 is 4-QAM. User 2 uses 45° rotated 4-QAM. Therefore, $|\mathcal{X}| = 16$, and $\lfloor \frac{1}{2}\log_2(|\mathcal{X}|+1) \rfloor = 2$. Observe that the residual norm in Fig. 2 is greater than zero even for $b > 2$ (i.e., for $b = 3, 4$), showing that the condition for stochastic degradedness is not satisfied for QBC.

A. Achievable rate region in QBC

As a consequence of QBC being not stochastically degraded, capacity expressions (11),(12) are not valid for QBC. Here, we obtain the achievable rate region for QBC based on two coding/decoding schemes. The motivation for the proposed scheme is explained below.

Motivation: We observed that, even in the presence of Gaussian noise with $\sigma_1^2 < \sigma_2^2$, $I(x_2; r_1)$ is *not always greater* than $I(x_2; r_2)$. Table I shows a listing of the mutual information for a two-user QBC when both the users use a 1-bit uniform quantizer and the input messages for both the users are from 4-QAM input alphabet at SNR1 = 10 dB and SNR2 = 7 dB. Observe that at $\alpha = 0.6$ and 0.8, $I(x_2; r_1) < I(x_2; r_2)$, which implies non-degradedness.

Mutual Information	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
$I(x_1; r_1 \mid x_2)$	0.08083	0.37272	0.93188	1.59350
$I(x_1; r_1)$	0.00893	0.15668	0.71584	1.52160
$I(x_1; r_2)$	0.03572	0.20718	0.60551	1.19670
$I(x_2; r_1)$	1.52160	0.71584	**0.15668**	**0.00893**
$I(x_2; r_2)$	1.19670	0.60551	**0.20718**	**0.03572**
$I(x_2; r_2 \mid x_1)$	1.31920	0.82872	0.43039	0.15825

TABLE I
MUTUAL INFORMATION FOR A 2-USER QBC WHEN BOTH USERS EMPLOY 1-BIT UNIFORM QUANTIZER AND INPUT MESSAGES FOR BOTH USERS ARE FROM 4-QAM ALPHABET AT SNR1= 10 DB, SNR2 = 7 DB.

Hence, user 1 can not decode user 2's message when $I(x_2; r_1) < I(x_2; r_2)$ and the rate of user 2's message is $I(x_2; r_2)$, which, in turn, implies that user 1 can not do successive decoding. However, if we set the rate of user 2 to $\min\{I(x_2; r_2),\ I(x_2; r_1)\}$, then it is guaranteed that both user 1 and user 2 can decode user 2's message and user 1 can do successive decoding.

Proposed Scheme: Based on the above observation, we obtain an achievable rate region for two-user QBC as follows. We consider two schemes characterizing two different coding/decoding procedures to get the achievable rate region.

Scheme 1: *User 1 does successive decoding and user 2 decodes its message alone.*

User 1 can achieve a rate of $I(x_1; r_1 \mid x_2)$ by successive decoding (i.e., user 1 will cancel the interference due to user 2's message and then it will decode its own message) only when it can decode user 2's message error free. From the observations made in Table I, we know that $I(x_2; r_1)$ is not always greater than $I(x_2; r_2)$ and hence, for user 1 to decode user 2's message error free, user 2's information must be restricted to a rate of $\min\{I(x_2; r_2),\ I(x_2; r_1)\}$. So, the set of achievable rate pairs $(R_1^{(1)}, R_2^{(1)})$ when user 1 does successive decoding and user 2 decodes its message alone, is given by

$$R_1^{(1)} \leq I(x_1; r_1 \mid x_2) \quad (16)$$
$$R_2^{(1)} \leq \min\{I(x_2; r_2),\ I(x_2; r_1)\}. \quad (17)$$

Scheme 2: *User 2 does successive decoding and user 1 decodes its message alone.*

User 2 can achieve a rate of $I(x_2; r_2 | x_2)$ by successive decoding only when the information to user 1 is restricted to a rate of $\min\{I(x_1; r_1),\ I(x_1; r_2)\}$. Thus, the set of achievable rate pairs $(R_1^{(2)}, R_2^{(2)})$, when user 2 does successive decoding and user 1 decodes his message alone, is given by

$$R_1^{(2)} \leq \min\{I(x_1; r_1),\ I(x_1; r_2)\} \quad (18)$$
$$R_2^{(2)} \leq I(x_2; r_2 \mid x_1). \quad (19)$$

Since any line joining a pair of achievable rate pairs in the above two schemes is also achievable by *time sharing*, we propose the achievable rate region of QBC, $\mathcal{S}$, as the set of all rate pairs (R_1, R_2) which are in the convex hull [11] of the union of the achievable rate pairs of the above two schemes. The proposed achievable rate region, $\mathcal{S}$, is then given by

$$\mathcal{S} = \{(R_1, R_2) \mid (R_1, R_2) \in conv(\ (R_1^{(1)}, R_2^{(1)}) \cup (R_1^{(2)}, R_2^{(2)}))\}, \quad (20)$$

where $conv(.)$ denotes convex hull, and $(R_1^{(1)}, R_2^{(1)})$ satisfies (16),(17) and $(R_1^{(2)}, R_2^{(2)})$ satisfies (18),(19).

The mutual information in the expressions (16), (17), (18), (19) are calculated using the probability distribution

$$\begin{aligned} & p\,(r_1 = \mathcal{R}_1(k) \,|\, x_1 = \mathcal{X}_1(l), x_2 = \mathcal{X}_2(m)) \\ & = p(r_1^I = \mathcal{R}_1^I(k),\ r_1^Q = \mathcal{R}_1^Q(k) \,|\, x_1 = \mathcal{X}_1(l), x_2 = \mathcal{X}_2(m)) \\ & = p(z_1^I \in \mathcal{F}_1(\mathcal{X}_1^I(l), \mathcal{X}_2^I(m), \mathcal{R}_1^I(k))) \\ & \qquad \times p(z_1^Q \in \mathcal{F}_1(\mathcal{X}_1^Q(l), \mathcal{X}_2^Q(m), \mathcal{R}_1^Q(k))), \end{aligned} \quad (21)$$

where $j = \sqrt{-1}$, and $\mathcal{X}_1(i)$, $\mathcal{X}_2(i)$ refer to ith elements of sets $\mathcal{X}_1$, $\mathcal{X}_2$, respectively. The region $\mathcal{F}_1(.)$ is defined as

$$\mathcal{F}_1(p, q, t) = \{n \in \mathbb{R} \,|\, \mathbf{Q}_{b_1}(p + q + n) = t\}, \tag{22}$$

and $n \sim \mathcal{N}(0, \sigma_1^2/2)$. From (21), the marginal probability distributions $p(r_1|x_1)$, $p(r_1|x_2)$ and $p(r_1)$ are calculated as

$$p(r_1 = \mathcal{R}_1(k) \,|\, x_1 = \mathcal{X}_1(l)) = \frac{1}{N_2} \sum_{m=1}^{N_2} p(r_1 = \mathcal{R}_1(k) \,|\, x_1 = \mathcal{X}_1(l), x_2 = \mathcal{X}_2(m)), \tag{23}$$

$$p(r_1 = \mathcal{R}_1(k) \,|\, x_2 = \mathcal{X}_2(m)) = \frac{1}{N_1} \sum_{l=1}^{N_1} p(r_1 = \mathcal{R}_1(k) \,|\, x_1 = \mathcal{X}_1(l), x_2 = \mathcal{X}_2(m)), \tag{24}$$

$$p(r_1 = \mathcal{R}_1(k)) = \frac{1}{N_2} \sum_{m=1}^{N_2} p(r_1 = \mathcal{R}_1(k) \,|\, x_2 = \mathcal{X}_2(m)). \tag{25}$$

Similarly, the probability distributions $p(r_2|x_1, x_2)$, $p(r_2|x_1)$, $p(r_2|x_2)$ and $p(r_2)$ can be calculated. Using the above probability distributions, the final expressions of (16)-(19) are given by Eqns. (26)-(29), which are listed in next page.

In the illustration of numerical results, we plot the boundary of the achievable rate region of two-user QBC by varying the proportion of power α allocated to each user from 0 to 1, and finding the achievable rate pairs using (20). When both x_1 and x_2 take values from the same signal set, we consider rotation of the second user's signal set by an angle θ with respect to the first user's signal set for further enlargement of the achievable rate region, i.e.,

$$\mathcal{X}_2 \triangleq \{u\, e^{j\theta} \mid u \in \mathcal{X}_1\}, \tag{30}$$

where θ is the rotation angle. We observe that, the rate expressions now become a function of θ, and hence they are explicitly denoted as $R_1^{(1)}(\theta)$, $R_2^{(1)}(\theta)$, $R_1^{(2)}(\theta)$ and $R_2^{(2)}(\theta)$. The achievable rate region of QBC with rotation, $\mathcal{S}_\theta$, is

$$\mathcal{S}_\theta = \Big\{(R_1, R_2) \mid (R_1, R_2) \in conv\Big(\bigcup_{\theta \in (0, 2\pi)} \{(R_1^{(1)}(\theta), R_2^{(1)}(\theta)) \cup (R_1^{(2)}(\theta), R_2^{(2)}(\theta))\}\Big)\Big\}. \tag{31}$$

IV. QBC WITH UNIFORM QUANTIZER

In this section, we present the achievable rate region results for two-user QBC with uniform receiver quantization.

A. *Uniform Quantizer*

A uniform b-bit quantizer, $\mathbf{Q}_b(.)$ acting on the real component of the analog received signal y is given by

$$\mathbf{Q}_b(y^I) \triangleq \begin{cases} +1, & \zeta(y^I) > (2^{b-1} - 1) \\ -1, & \zeta(y^I) < -(2^{b-1} - 1) \\ \dfrac{2\zeta(y^I) + 1}{2^b - 1}, & \text{otherwise}, \end{cases} \tag{32}$$

where $\zeta(y^I) \triangleq \left\lfloor \frac{y^I}{X^I} \frac{(2^b - 1)}{2} \right\rfloor$. The quantizer for the imaginary component, $\mathbf{Q}_b(y^Q)$, is defined likewise.

We assume that the user 1 uses a b_1-bit uniform quantizer and user 2 uses a b_2-bit uniform quantizer. Applying the above uniform quantizer to the analog received signal at the users 1 and 2, their quantized outputs on the real and imaginary components are given by

$$r_1^I = Q_{b_1}(y_1^I), \qquad r_1^Q = Q_{b_1}(y_1^Q), \tag{33}$$

$$r_2^I = Q_{b_2}(y_2^I), \qquad r_2^Q = Q_{b_2}(y_2^Q). \tag{34}$$

With the uniform quantizer defined in (33) and (34), we numerically evaluate the proposed achievable rate region of two-user QBC using (31) or (20), the results of which are discussed in the following subsection.

B. *Results and Discussion*

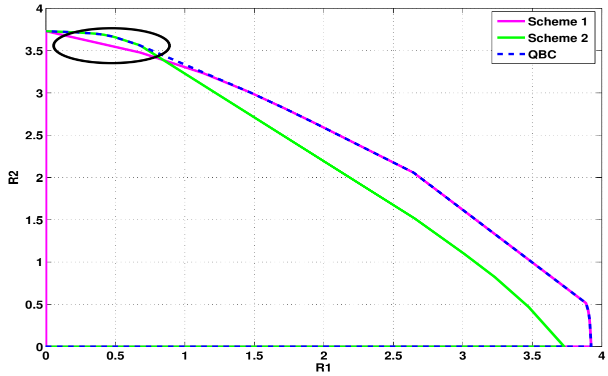

Fig. 3. Plots of the boundary of the achievable rate region of QBC when the input alphabet for user 1 is 16-QAM and user 2 is rotated 16-QAM with SNR1 = 15 dB, SNR2 = 13 and $b_1 = b_2 = 5$.

In Fig. 3, we illustrate the significance of using the two schemes instead of assuming that the user with high SNR alone does successive decoding. Fig. 3 shows the proposed achievable rate region of QBC when the input alphabet for user 1 is 16-QAM and the input alphabet for user 2 is rotated 16-QAM. Both the users use 5-bit uniform quantizer with SNR1 = 15 dB and SNR2 = 13 dB. We observe that most of the contribution to the proposed achievable rate region of QBC is due to Scheme 1, i.e., the scheme of the user with high SNR doing successive decoding and the user with low SNR decoding his message alone. However, there is an appreciable contribution to the proposed achievable rate region of QBC when the user with low SNR performs successive decoding and the user with high SNR decodes his message alone, especially when the proportion of the total transmit power allotted to that user (the one with low SNR) is more than that of the other. For instance, observe the performance in the *circled region* of Scheme 2 in Fig. 3.

Fig. 4 shows the significance of rotation on the proposed achievable rate region of QBC when both the users use uniform quantizer of same resolution, i.e., $b_1 = b_2$ at SNR1 = 10 dB and SNR2 = 7 dB. The input alphabet for user 1 is 4-QAM and the input alphabet for user 2 is rotated 4-QAM. We observe that there is no increase in the achievable rate region for a 1-bit uniform quantizer due to rotation. Also, observe that for a $b = 1$ bit uniform quantizer, *time-sharing* between the two users is sufficient to achieve the boundary points of the rate region. For $b_1 = b_2 = 2$ or 3 bit uniform quantizers, there is a small increase in the achievable rate region due to rotation only when α is around 0.5. The reason could be

$$R_1^{(1)} \leq \log_2(N_1) - \frac{1}{N_1N_2}\sum_{k=1}^{2^{2b_1}}\sum_{l_1=1}^{N_1}\sum_{m_1=1}^{N_2} p_{r_1|x_1,x_2}(\mathcal{R}_1(k) \mid \mathcal{X}_1(l_1), \mathcal{X}_2(m_1)) \log_2\left\{\frac{\sum_{l_2=1}^{N_1} p_{r_1|x_1,x_2}(\mathcal{R}_1(k) \mid \mathcal{X}_1(l_2), \mathcal{X}_2(m_1))}{p_{r_1|x_1,x_2}(\mathcal{R}_1(k) \mid \mathcal{X}_1(l_1), \mathcal{X}_2(m_1))}\right\}. \quad (26)$$

$$R_2^{(1)} \leq \min\left\{\log_2(N_2) - \frac{1}{N_1N_2}\sum_{k=1}^{2^{2b_1}}\sum_{l_1=1}^{N_1}\sum_{m_1=1}^{N_2} p_{r_1|x_1,x_2}(\mathcal{R}_1(k) \mid \mathcal{X}_1(l_1), \mathcal{X}_2(m_1)) \times \log_2\left\{\frac{\sum_{l_2=1}^{N_1}\sum_{m_2=1}^{N_2} p_{r_1|x_1,x_2}(\mathcal{R}_1(k) \mid \mathcal{X}_1(l_2), \mathcal{X}_2(m_2))}{\sum_{l_3=1}^{N_1} p_{r_1|x_1,x_2}(\mathcal{R}_1(k) \mid \mathcal{X}_1(l_3), \mathcal{X}_2(m_1))}\right\},\right.$$
$$\left.\log_2(N_2) - \frac{1}{N_1N_2}\sum_{k=1}^{2^{2b_2}}\sum_{l_1=1}^{N_1}\sum_{m_1=1}^{N_2} p_{r_2|x_1,x_2}(\mathcal{R}_2(k) \mid \mathcal{X}_1(l_1), \mathcal{X}_2(m_1)) \times \log_2\left\{\frac{\sum_{l_2=1}^{N_1}\sum_{m_2=1}^{N_2} p_{r_2|x_1,x_2}(\mathcal{R}_2(k) \mid \mathcal{X}_1(l_2), \mathcal{X}_2(m_2))}{\sum_{l_3=1}^{N_1} p_{r_2|x_1,x_2}(\mathcal{R}_2(k) \mid \mathcal{X}_1(l_3), \mathcal{X}_2(m_1))}\right\}\right\}. \quad (27)$$

$$R_1^{(2)} \leq \min\left\{\log_2(N_1) - \frac{1}{N_1N_2}\sum_{k=1}^{2^{2b_1}}\sum_{l_1=1}^{N_1}\sum_{m_1=1}^{N_2} p_{r_1|x_1,x_2}(\mathcal{R}_1(k) \mid \mathcal{X}_1(l_1), \mathcal{X}_2(m_1)) \times \log_2\left\{\frac{\sum_{l_2=1}^{N_1}\sum_{m_2=1}^{N_2} p_{r_1|x_1,x_2}(\mathcal{R}_1(k) \mid \mathcal{X}_1(l_2), \mathcal{X}_2(m_2))}{\sum_{m_3=1}^{N_2} p_{r_1|x_1,x_2}(\mathcal{R}_1(k) \mid \mathcal{X}_1(l_1), \mathcal{X}_2(m_3))}\right\},\right.$$
$$\left.\log_2(N_1) - \frac{1}{N_1N_2}\sum_{k=1}^{2^{2b_2}}\sum_{l_1=1}^{N_1}\sum_{m_1=1}^{N_2} p_{r_2|x_1,x_2}(\mathcal{R}_2(k) \mid \mathcal{X}_1(l_1), \mathcal{X}_2(m_1)) \times \log_2\left\{\frac{\sum_{l_2=1}^{N_1}\sum_{m_2=1}^{N_2} p_{r_2|x_1,x_2}(\mathcal{R}_2(k) \mid \mathcal{X}_1(l_2), \mathcal{X}_2(m_2))}{\sum_{m_3=1}^{N_2} p_{r_2|x_1,x_2}(\mathcal{R}_2(k) \mid \mathcal{X}_1(l_1), \mathcal{X}_2(m_3))}\right\}\right\}. \quad (28)$$

$$R_2^{(2)} \leq \log_2(N_2) - \frac{1}{N_1N_2}\sum_{k=1}^{2^{2b_2}}\sum_{l_1=1}^{N_1}\sum_{m_1=1}^{N_2} p_{r_2|x_1,x_2}(\mathcal{R}_2(k) \mid \mathcal{X}_1(l_1), \mathcal{X}_2(m_1)) \log_2\left\{\frac{\sum_{m_2=1}^{N_2} p_{r_2|x_1,x_2}(\mathcal{R}_2(k) \mid \mathcal{X}_1(l_1), \mathcal{X}_2(m_2))}{p_{r_2|x_1,x_2}(\mathcal{R}_2(k) \mid \mathcal{X}_1(l_1), \mathcal{X}_2(m_1))}\right\}. \quad (29)$$

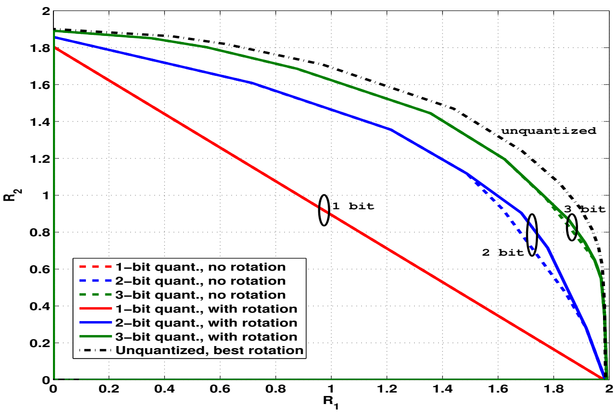

Fig. 4. Comparison of achievable rate region of QBC when both the users use uniform quantizer of same resolution i.e., $b_1 = b_2$ at SNR1 = 10 dB and SNR2 = 7 dB.

that rotation gives significant enlargement in the achievable rate region only when the sum signal set is not uniquely decodable. This happens more only when α is around 0.5. For instance, when $\alpha = 0.5$, $\mathcal{X}_1 = \mathcal{X}_2$ and thus the set $\mathcal{X}$ is not uniquely decodable. Hence, when $\alpha = 0.5$, rotation by even a small angle makes the set $\mathcal{X}$ to be uniquely decodable resulting in an increase in the achievable rate region of QBC. We have computed the proposed achievable rate region for QBC with asymmetric quantizers also, i.e., with $b_1 \neq b_2$.

V. Conclusions

We showed that the capacity expressions known for GBC are not valid for QBC as the channel is no more stochastically degraded. We proposed an achievable rate region for 2-user QBC based on two different coding/decoding procedures. We studied the proposed achievable rate region of QBC, with and without rotation of the user's input alphabet and uniform receiver quantization.

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