BER Analysis of QAM with Transmit Diversity in Rayleigh Fading Channels

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***Abstract*— In this paper, we present a log-likelihood ratio (LLR) based approach to analyze the bit error rate (BER) performance of quadrature amplitude modulation (QAM) on Rayleigh fading channels without and with transmit diversity. We derive LLRs for the individual bits forming a QAM symbol both on flat fading channels without diversity as well as on channels with transmit diversity using two transmit antennas (Alamouti's scheme) and multiple receive antennas. Using the LLRs of the individual bits forming the QAM symbol, we derive expressions for the probability of error for various bits in the QAM symbol, and hence the average BER. In addition to being used in the BER analysis, the LLRs derived can be used as soft inputs to decoders for various coded QAM schemes including turbo coded QAM with transmit diversity, as in high speed downlink packet access (HSDPA) in 3G.**

Keywords *– QAM, BER analysis, transmit diversity, log-likelihood ratio.*

I. INTRODUCTION

Multilevel quadrature amplitude modulation (M-QAM) is an attractive modulation scheme for wireless communications due to the high spectral efficiency it provides. Several works have been reported on the performance analysis of M-QAM in fading channels, where mainly the symbol error rate (SER) performance has been derived. In addition to the SER analysis, bit error rate (BER) analysis is also of interest in multilevel modulation schemes. Recent works reported in [1]-[3] provide expressions to compute the BER for M-QAM on AWGN channels. In [1], Vitthaladevuni and Alouini provide BER analysis for hierarchical 4/M-QAM on fading channels. In the 4/M-QAM scheme in [1], higher order $M-$QAM constellations are embedded by a lower order QAM constellation (4-QAM), and the M-QAM BER is obtained by using the results of the underlying 4-QAM constellation. Our focus in this paper is on the analytical evaluation of the BER performance of QAM on Rayleigh fading channels without and with transmit diversity.

The key contributions in this paper are two fold – first, we present an alternate method of deriving the BER for QAM on fading channels using log-likelihood ratios (LLRs) of the individual bits that form the QAM symbol, and second, using the LLRs, we derive the BER expressions for QAM on Rayleigh fading channels without and with transmit diversity using two transmit antennas (Alamouti's scheme [5]) and multiple receive antennas. We derive the LLRs and BER expressions for 16-QAM scheme in this paper. The analytical technique, however, is applicable to any high order ($M > 16$) QAM constellation and for any arbitrary mapping of bits to QAM symbols. Another major usefulness of the results in this paper is that the derived LLRs provide a soft metric for each bit in the mapping, which can be used as soft inputs to decoders for various coded QAM schemes. Examples of such systems include turbo coded QAM with transmit diversity in high speed downlink packet access (HSDPA) in 3G, and convolutionally coded QAM with OFDM in digital video broadcasting (DVB) and IEEE 802.11a.

The rest of the paper is organized as follows. We present the derivation of LLRs and BER expression for 16-QAM on flat Rayleigh fading channels in Section II. The derivation of the LLRs and BER expression for the case of transmit diversity is presented in Section III. Conclusions are given in Section IV.

II. LLR AND BER IN FLAT FADING

Consider the M-QAM ($M = 16$) scheme as shown in Fig. 1, where $\log_2 M = 4$ bits (r_1, r_2, r_3, r_4) are mapped on to a complex symbol $a = a_I + ja_Q$. The horizontal/vertical line pieces in Fig. 1 denote that all bits under these lines take the value 1, and the rest take the value 0. For example, the symbol with coordinates $(-3d, 3d)$ maps the 4-bit combination $r_1 = 1, r_2 = 0, r_3 = r_4 = 1$. Assuming that the transmitted symbol a undergoes multiplicative fading (the fading is assumed to be slow, frequency non-selective and remain constant over one symbol interval), the received signal y corresponding to the transmitted symbol a can be written as

$$y \;=\; ha + n, \tag{1}$$

where h is the complex fading channel coefficient with $E\{\|h\|^2\} = 1$ and the r.v's $\|h\|$'s for different symbols are assumed to be i.i.d Rayleigh distributed, and $n = n_I + jn_Q$ is a complex Gaussian r.v of zero mean and variance $\sigma^2/2$ per dimension.

A. Log-Likelihood Ratios

We define the log-likelihood ratio (LLR) of bit $r_i,\ i = 1, 2, 3, 4$ of the received symbol as [4]

$$LLR(r_i) \;=\; \log\left(\frac{Pr\{r_i = 1|y, h\}}{Pr\{r_i = 0|y, h\}}\right). \tag{2}$$

Clearly, the optimum decision rule is to decide, $\hat{r_i} = 1$ if $LLR(r_i) \geq 0$, and 0 otherwise. Define two set partitions,

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$S_i^{(1)}$ and $S_i^{(0)}$, such that $S_i^{(1)}$ comprises symbols with r_i = 1 and $S_i^{(0)}$ comprises symbols with r_i = 0 in the constellation. Then, from (2), we have

$$LLR(r_i) = \log\left(\frac{\sum_{\alpha\in S_i^{(1)}} Pr\{a=\alpha|y,h\}}{\sum_{\beta\in S_i^{(0)}} Pr\{a=\beta|y,h\}}\right). \quad (3)$$

Assume that all the symbols are equally likely and that fading is independent of the transmitted symbols. Using Bayes' rule, we then have

$$LLR(r_i) = \log\left(\frac{\sum_{\alpha\in S_i^{(1)}} f_{y|h,a}\{y|h,a=\alpha\}}{\sum_{\beta\in S_i^{(0)}} f_{y|h,a}\{y|h,a=\beta\}}\right). \quad (4)$$

Since $f_{y|h,a}\{y|h,a=\alpha\} = \frac{1}{\sigma\sqrt{\pi}}\exp\left(\frac{-1}{\sigma^2}\|y-h\alpha\|^2\right)$, (4) can be written as

$$LLR(r_i) = \log\left(\frac{\sum_{\alpha\in S_i^{(1)}} \exp\left(\frac{-1}{\sigma^2}\|y-h\alpha\|^2\right)}{\sum_{\beta\in S_i^{(0)}} \exp\left(\frac{-1}{\sigma^2}\|y-h\beta\|^2\right)}\right). \quad (5)$$

Using the approximation $\log(\sum_j \exp(-X_j)) \approx -\min_j(X_j)$, we can approximate (5) as [1]

$$LLR(r_i) = \frac{1}{\sigma^2}\left\{\min_{\beta\in S_i^{(0)}} \|y-h\beta\|^2 - \min_{\alpha\in S_i^{(1)}} \|y-h\alpha\|^2\right\}. \quad (6)$$

Define z as $z \stackrel{\Delta}{=} \frac{y}{h} = a + \frac{n}{h} = a + \widehat{n}$, where $\widehat{n}$ is a complex Gaussian r.v. with variance $\sigma^2/\|h\|^2$. Using the above definition of z into (6) and normalizing $LLR(r_i)$ by $4/\sigma^2$,

$$\begin{aligned} LLR(r_i) &= \frac{\|h\|^2}{4}\left\{\min_{\beta\in S_i^{(0)}} \|z-\beta\|^2 - \min_{\alpha\in S_i^{(1)}} \|z-\alpha\|^2\right\}. \\ &= \frac{\|h\|^2}{4}\left[\min_{\beta\in S_i^{(0)}} \left\{\|\beta\|^2 - 2z_I\beta_I - 2z_Q\beta_Q\right\} - \right. \\ &\left. \min_{\alpha\in S_i^{(1)}} \left\{\|\alpha\|^2 - 2z_I\alpha_I - 2z_Q\alpha_Q\right\}\right], \end{aligned} \quad (7)$$

where $z = z_I + jz_Q$, $\alpha = \alpha_I + j\alpha_Q$ and $\beta(k) = \beta_I + j\beta_Q$. Note that the set partitions $S_i^{(1)}$ and $S_i^{(0)}$ are delimited by horizontal or vertical boundaries. As a consequence, two symbols in different sets closest to the received symbol always lie either on the same row (if the delimiting boundaries are vertical) or on the same column (if the delimiting boundaries are horizontal). Then, for bit r_1, the two constellation symbols in $S_1^{(1)}$ and $S_1^{(0)}$ having closest distances to the received symbol satisfy the condition $\alpha_Q = \beta_Q$. Hence, for bit r_1

$$LLR(r_1) = \begin{cases} -\|h\|^2 z_I d & |z_I| \le 2d \\ 2\|h\|^2 d(d-z_I) & z_I > 2d \\ -2\|h\|^2 d(d+z_I) & z_I < -2d, \end{cases} \quad (8)$$

where $2d$ is the minimum distance between pairs of signal points. Following similar steps for bits r_2, r_3, and r_4, we get

$$LLR(r_2) = \begin{cases} -\|h\|^2 z_Q d & |z_Q| \le 2d \\ 2\|h\|^2 d(d-z_Q) & z_Q > 2d \\ -2\|h\|^2 d(d+z_Q) & z_Q < -2d, \end{cases} \quad (9)$$

[1]This is quite a standard approximation [7], and, as we will see in Sec. II-B, the analytical BER evaluated using this approximate LLR is almost the same as the BER evaluated through simulations without this approximation.

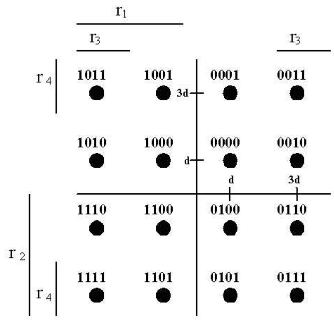

Fig. 1. 16-QAM Constellation

$$LLR(r_3) = \|h\|^2 d\{|z_I| - 2d\} \quad (10)$$

$$LLR(r_4) = \|h\|^2 d\{|z_Q| - 2d\}. \quad (11)$$

B. Derivation of Probability of Bit Error

Using the LLR(r_i)'s obtained above, we derive the analytical expression for the probability of error for the bits r_i, $i = 1, 2, 3, 4$. The probability of error for bit r_1, P_{b1}, is given by

$$P_{b1} = \frac{1}{2}\left(P_{b1|r_1=1} + P_{b1|r_1=0}\right). \quad (12)$$

Since $r_1 = 1$ implies that the real part of the transmitted symbol, a_I, can take either values $-d$ or $-3d$, and $r_1 = 0$ implies that a_I can take either values $+d$ or $+3d$, we can write the above equation as

$$\begin{aligned} P_{b1} &= P_{b1|a_I=-d}.\Pr\{a_I=-d\} + P_{b1|a_I=-3d}.\Pr\{a_I=-3d\} \\ &+ P_{b1|a_I=d}.\Pr\{a_I=d\} + P_{b1|a_I=-d}.\Pr\{a_I=3d\}, \end{aligned} \quad (13)$$

where $P_{b1|a_I=m}$ is the probability of error for bit r_1 given that the real part of the transmitted symbol takes the value m. Now $P_{b1|a_I=-d,h}$ is given by

$$\begin{aligned} P_{b1|a_I=-d,h} &= Pr\{LLR(r_1) < 0 \,|\, a_I = -d, h\} \\ &= Pr\{\hat{n}_I \ge d\} = Q\left(\frac{d\left(\sqrt{\|h\|^2}\right)}{\sigma_I}\right), \end{aligned} \quad (14)$$

where $\sigma_I^2 = \sigma^2/2$. Using the fact that $\frac{d}{\sigma_I} = \sqrt{\frac{4E_b}{5N_o}}$, where E_b is the energy per transmitted bit, we have

$$P_{b1|a_I=-d,h} = Q\left(\sqrt{\frac{4E_b\|h\|^2}{5N_o}}\right). \quad (15)$$

Unconditioning on the r.v. h, it can easily be shown that

$$P_{b1|a_I=-d} = \overline{Q\left(\sqrt{\frac{4E_b\|h\|^2}{5N_o}}\right)} = \frac{1}{2}\left(1 - \sqrt{\frac{2E_b/N_o}{5+2E_b/N_o}}\right). \quad (16)$$

On similar lines, $P_{b1|a_I=-3d}$ can be shown to be equal to

$$P_{b1|a_I=-3d} = \overline{Q\left(\sqrt{\frac{36E_b\|h\|^2}{5N_o}}\right)} = \frac{1}{2}\left(1 - \sqrt{\frac{18E_b/N_o}{5+18E_b/N_o}}\right). \quad (17)$$

It can be shown that $P_{b1|a_I=-d} = P_{b1|a_I=d}$ and $P_{b1|a_I=-3d} = P_{b1|a_I=3d}$. Hence, P_{b1} is given by

$$P_{b1} = \frac{1}{2}\left[1 - \frac{1}{2}\sqrt{\frac{2E_b/N_o}{5+2E_b/N_o}} - \frac{1}{2}\sqrt{\frac{18E_b/N_o}{5+18E_b/N_o}}\right]. \quad (18)$$

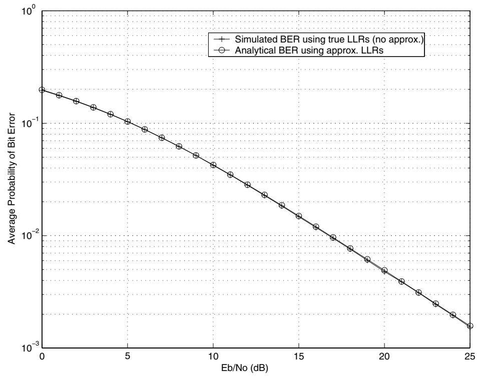

Fig. 2. Comparison of the analytical BER evaluated using approximate LLRs vs the simulated BER using the LLRs without approximation. 16-QAM on flat Rayleigh fading.

For the 16-QAM constellation considered, $P_{b1} = P_{b2}$ and $P_{b3} = P_{b4}$. The error probabilities, P_{b3} and P_{b4} can be obtained as

$$P_{b3} = P_{b4} = \frac{1}{2}\left[1 - \sqrt{\frac{2E_b/N_o}{5+2E_b/N_o}} - \frac{1}{2}\sqrt{\frac{18E_b/N_o}{5+18E_b/N_o}} + \frac{1}{2}\sqrt{\frac{50E_b/N_o}{5+50E_b/N_o}}\right]. \quad (19)$$

Using (18) and (19), we obtain the average BER, P_b, as $P_b = \frac{1}{2}\Big(P_{b1} + P_{b3}\Big)$. In Fig. 2, we compare the analytical BER evaluated using the approximate LLRs derived in the above versus the simulated BER using the LLRs without approximation, for 16-QAM on flat Rayleigh fading. It is observed that the analytically computed BER is almost the same as the simulated BER, indicating that the approximation to the LLRs results in insignificant difference between the analytically computed BER and the true BER.

III. LLR AND BER IN TRANSMIT DIVERSITY

In this section, we derive the LLRs and BER for 16-QAM on Rayleigh fading channels with transmit diversity. We consider a system with two transmit antennas (Alamouti's scheme [5]). We first analyze the case of two transmit antennas and one receive antenna. We then extend the analysis to two transmit antennas and L, $L > 1$ receive antennas.

A. Two Transmit Antennas and One Receive antenna

Let a_1, $-a_2^*$ be the symbols transmitted on the first and the second transmit antennas, respectively, during a symbol interval. During the next symbol interval, a_2, a_1^* are transmitted on the first and the second transmit antennas, respectively [5]. Assuming that the channel remains constant over two consecutive symbol intervals, the received signals during the two consecutive symbol intervals are given as

$$\begin{aligned} y_1 &= a_1h_1 - a_2^*h_2 + n_1 \\ y_2 &= a_2h_1 + a_1^*h_2 + n_2, \end{aligned} \quad (20)$$

where h_1 and h_2 are the complex fading coefficients on the path from the 1st and the 2nd transmit antennas, respectively, to the receive antenna with $\|h_1\|$, $\|h_2\|$ being Rayleigh distributed with $E\{\|h_1\|^2\} = E\{\|h_2\|^2\} = 1$, and n_1 and n_2 are complex Gaussian r.v's of zero mean and variance σ^2. Assuming perfect knowledge of the fading coefficients at the receiver, we form $\hat{a}_1$ and $\hat{a}_2$ as

$$\begin{aligned} \hat{a}_1 &= h_1^*y_1 + h_2y_2^* \\ &= \{\|h_1\|^2 + \|h_2\|^2\}a_1 + n_1h_1^* + n_2^*h_2, \end{aligned} \quad (21)$$

$$\begin{aligned} \hat{a}_2 &= h_1^*y_2 - h_2y_1^* \\ &= \{\|h_1\|^2 + \|h_2\|^2\}a_2 + n_2h_1^* - n_1^*h_2. \end{aligned} \quad (22)$$

In (21), (22), we replace $(n_1h_1^* + n_2^*h_2)$ and $(n_2h_1^* - n_1^*h_2)$ by ζ_1 and ζ_2, respectively, where ζ_1 and ζ_2 are complex Gaussian r.v's of zero mean and variance $\{\|h_1\|^2 + \|h_2\|^2\}\sigma^2$. Then

$$\begin{aligned} \hat{a}_1 &= \{\|h_1\|^2 + \|h_2\|^2\}a_1 + \zeta_1 \\ \hat{a}_2 &= \{\|h_1\|^2 + \|h_2\|^2\}a_2 + \zeta_2. \end{aligned} \quad (23)$$

1) Log-Likelihood Ratios: The derivation of the LLRs for the bits in symbol a_1 and a_2 is quite similar to that in Section II-A. We define the LLR for the bit r_i, $i = 1,2,3,4$ of symbol a_j, $j = 1,2$, as

$$\begin{aligned} LLR_{a_j}(r_i) &= \log\left(\frac{Pr\{r_i = 1|y_1,y_2,h_1,h_2\}}{Pr\{r_i = 0|y_1,y_2,h_1,h_2\}}\right) \\ &= \log\left(\frac{Pr\{r_i = 1|\hat{a}_j,h_1,h_2\}}{Pr\{r_i = 0|\hat{a}_j,h_1,h_2\}}\right). \end{aligned} \quad (24)$$

Assuming all symbols as equally likely and that the fading is independent of the transmitted symbols, using Bayes' rule,

$$LLR_{a_j}(r_i) = \log\left(\frac{\sum_{\alpha \in S_i^{(1)}} f_{\hat{a}_j|h_1,h_2,a_j}\{\hat{a}_j|h_1,h_2,a_j = \alpha\}}{\sum_{\beta \in S_i^{(0)}} f_{\hat{a}_j|h_1,h_2,a_j}\{\hat{a}_j|h_1,h_2,a_j = \beta\}}\right). \quad (25)$$

Using the conditional pdf $f_{\hat{a}_j|h_1,h_2,a_j}\{\hat{a}_j|h_1,h_2,a_j = \alpha\}$, which is given by $\frac{1}{\hat{\sigma}\sqrt{\pi}}\exp\left(\frac{-1}{\hat{\sigma}^2}\left\|\hat{a}_j - \{\|h_1\|^2 + \|h_2\|^2\}\alpha\right\|^2\right)$ where $\hat{\sigma}^2 = \sigma^2\{\|h_1\|^2 + \|h_2\|^2\}$, we obtain $LLR_{a_j}(r_i)$ as

$$LLR_{a_j}(r_i) = \frac{1}{\hat{\sigma}^2}\left[\min_{\beta \in S_i^{(0)}}\left\|\hat{a}_j - \{\|h_1\|^2 + \|h_2\|^2\}\beta\right\|^2 - \min_{\alpha \in S_i^{(1)}}\left\|\hat{a}_j - \{\|h_1\|^2 + \|h_2\|^2\}\alpha\right\|^2\right]. \quad (26)$$

Define two complex variables, $\hat{z}_j$, $j = 1,2$, as

$$\hat{z}_j = \frac{\hat{a}_j}{\|h_1\|^2 + \|h_2\|^2}. \quad (27)$$

Using (27) in (26) and normalizing by $4/\sigma^2$, we can write

$$LLR_{a_j}(r_i) = \frac{\|h_1\|^2 + \|h_2\|^2}{4}\left[\min_{\beta \in S_i^{(0)}}\|\hat{z}_j - \beta\|^2 - \min_{\alpha \in S_i^{(1)}}\|\hat{z}_j - \alpha\|^2\right]. \quad (28)$$

Following similar steps as in Sec. II-A, we obtain the following LLRs for bits r_1, r_2, r_3, r_4 of the symbol a_j.

$$LLR_{a_j}(r_1) = \begin{cases} -\{\|h_1\|^2 + \|h_2\|^2\}\hat{z}_{jI}d & |\hat{z}_{jI}| \le 2d \\ 2\{\|h_1\|^2 + \|h_2\|^2\}d(d - \hat{z}_{jI}) & \hat{z}_{jI} > 2d \\ -2\{\|h_1\|^2 + \|h_2\|^2\}d(d + \hat{z}_{jI}) & z_{jI} < -2d, \end{cases} \quad (29)$$

$$LLR_{a_j}(r_2) = \begin{cases} -\{\|h_1\|^2 + \|h_2\|^2\}\hat{z}_{jQ}d & |\hat{z}_{jQ}| \le 2d \\ 2\{\|h_1\|^2 + \|h_2\|^2\}d(d - \hat{z}_{jQ}) & \hat{z}_{jQ} > 2d \\ -2\{\|h_1\|^2 + \|h_2\|^2\}d(d + \hat{z}_{jQ}) & \hat{z}_{jQ} < -2d, \end{cases} \quad (30)$$

$$LLR_{a_j}(r_3) = \{\|h_1\|^2 + \|h_2\|^2\}d\{|\hat{z}_{jI}| - 2d\}, \quad (31)$$

$$LLR_{a_j}(r_4) = \{\|h_1\|^2 + \|h_2\|^2\}d\{|\hat{z}_{jQ}| - 2d\}. \quad (32)$$

In the above equations, $\hat{z}_{jI}$ and $\hat{z}_{jQ}$ are the real and imaginary parts of $\hat{z}_j$, respectively.

2) Probability of Bit Error: In this subsection, we derive the probability of error for the bit r_i when transmit diversity is employed. The bit error probability for bit r_1, P_{b1}, as in Sec. II-B, can be written as

$$P_{b1} = P_{b1|a_{jI}=-d} . \Pr\{a_{jI} = -d\} + P_{b1|a_{jI}=-3d} . \Pr\{a_{jI} = -3d\} + P_{b1|a_{jI}=d} . \Pr\{a_{jI} = d\} + P_{b1|a_{jI}=-d} . \Pr\{a_{jI} = 3d\}, \quad (33)$$

where a_{jI}, $j = 1, 2$ represents the real part of a_j. Now $P_{b1|a_{jI}=-d,h_1,h_2}$ is given by

$$\begin{aligned} P_{b1|a_{jI}=-d,h_1,h_2} &= Pr\{LLR_{a_j}(r_1) < 0 \,|\, a_{Ij} = -d, h_1, h_2\} \\ &= Pr\left\{\frac{\zeta_{jI}}{\|h_1\|^2 + \|h_2\|^2} \geq d\right\} \\ &= Q\left(\frac{d\left(\sqrt{\|h_1\|^2 + \|h_2\|^2}\right)}{\sigma_I}\right), \quad (34) \end{aligned}$$

where $\sigma_I^2 = \sigma^2/2$. Scaling the signal power in proportion to the number of transmit antennas, we have $\frac{d}{\sigma_I} = \sqrt{\frac{2E_b}{5N_o}}$ where E_b is the energy per bit per transmit antenna. We then have

$$P_{b1|a_{jI}=-d,h_1,h_2} = Q\left(\sqrt{\frac{2E_b\left(\|h_1\|^2 + \|h_2\|^2\right)}{5N_o}}\right). \quad (35)$$

Unconditioning the above on h_1, h_2, it can be shown that [6]

$$P_{b1|a_{jI}=-d} = \left(\frac{1-\mu_1}{2}\right)^2 \left(2 + \mu_1\right), \quad (36)$$

where μ_1 is given by $\mu_1 = \sqrt{\frac{E_b/N_o}{5+E_b/N_o}}$. Similarly, the conditional error probability $P_{b1|a_{jI}=-3d,h_1,h_2}$ is given by

$$\begin{aligned} P_{b1|a_{jI}=-3d,h_1,h_2} &= Pr\{LLR_{a_j}(r_1) < 0 \,|\, a_I = -3d, h_1, h_2\} \\ &= Pr\left\{\frac{\zeta_{jI}}{\|h_1\|^2 + \|h_2\|^2} \geq 3d\right\} \\ &= Q\left(\sqrt{\frac{18E_b\left(\|h_1\|^2 + \|h_2\|^2\right)}{5N_o}}\right). \quad (37) \end{aligned}$$

Unconditioning the above on h_1 and h_2, we get

$$P_{b1|a_{jI}=-3d} = \left(\frac{1-\mu_2}{2}\right)^2 \left(2 + \mu_2\right), \quad (38)$$

where μ_2 is given by $\mu_2 = \sqrt{\frac{9E_b/N_o}{5+9E_b/N_o}}$. It can further be shown that $P_{b1|a_I=-d} = P_{b1|a_I=d}$ and $P_{b1|a_I=-3d} = P_{b1|a_I=3d}$. Hence, the probability of error for bit r_1 is given by

$$P_{b1} = \frac{1}{2}\left(\left(\frac{1-\mu_1}{2}\right)^2 (2+\mu_1) + \left(\frac{1-\mu_2}{2}\right)^2 (2+\mu_2)\right). \quad (39)$$

For the 16-QAM constellation used, it can be shown that $P_{b1} = P_{b2}$. Using a similar approach, we can obtain the error probabilities for bits r_3 and r_4, P_{b3} and P_{b4}, as

$$\begin{aligned} P_{b3} = P_{b4} &= \frac{1}{2}\left[\frac{1}{2}(1-\mu_1)^2(2+\mu_1) + \frac{1}{4}(1-\mu_2)^2(2+\mu_2) \right. \\ &\quad \left. - \frac{1}{4}(1-\mu_3)^2(2+\mu_3)\right], \quad (40) \end{aligned}$$

where μ_3 is given by $\mu_3 = \sqrt{\frac{25E_b/N_o}{5+25E_b/N_o}}$. Using (39) and (40), we can write the average BER, P_b, as $P_b = \frac{1}{2}\left(P_{b1} + P_{b3}\right)$.

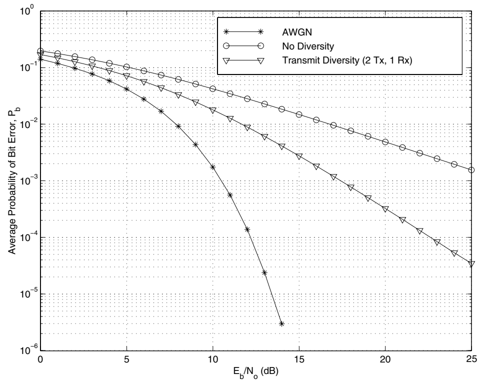

Fig. 3. BER performance of uncoded 16-QAM with transmit diversity. 2 transmit antennas and 1 receive antenna.

We computed the average BER from the above expression and plotted the numerical results in Fig. 3. Fig. 3 shows P_b as a function of E_b/N_o for 16-QAM without and with transmit diversity (2 transmit, 1 receive antenna). It can be seen that when transmit diversity is employed, the BER performance improves as expected.

B. Two Transmit Antennas and L Receive Antennas

We now consider a receiver with L, $L > 1$ receive antennas. The transmitter remains the same as discussed in Section III-A. We denote the channel fading coefficients as follows: h_{2i-1} represents the fading coefficient from transmit antenna 1 to receive antenna i, $i = 1 \cdots L$, and h_{2i} represent the fading coefficient from transmit antenna 2 to receive antenna i, $i = 1 \cdots L$. Let y_{2i-1} and y_{2i}, $i = 1 \cdots L$ be the received signal at the i^{th} antenna during two consecutive symbol intervals, respectively.

Assuming perfect knowledge of the fading coefficients at the receiver, we have (as in Sec. III-A)

$$\hat{a}_1 = \sum_{i=1}^{L}\left(h_{2i-1}^* \, y_{2i-1} + h_{2i} \, y_{2i}^*\right) \quad (41)$$

$$\hat{a}_2 = \sum_{i=1}^{L}\left(h_{2i-1}^* \, y_{2i} - h_{2i} \, y_{2i-1}^*\right). \quad (42)$$

After further simplification, $\hat{a}_1$ and $\hat{a}_2$ can be rewritten as

$$\hat{a}_1 = \left(\sum_{i=1}^{2L} \|h_i\|^2\right) a_1 + \zeta_1 \quad (43)$$

$$\hat{a}_2 = \left(\sum_{i=1}^{2L} \|h_i\|^2\right) a_2 + \zeta_2, \quad (44)$$

where ζ_1 and ζ_2 are complex Gaussian random variables with zero mean and variance $\{\sum_{i=1}^{2L} \|h_i\|^2\}\sigma^2$.

1) Log-Likelihood Ratios : Following a similar approach as in Sec. III-A.1, it can be shown that the log-likelihood ratios for bits r_1, r_2, r_3 and $r4$ are given by

$$LLR_{a_j}(r_1) = \begin{cases} -\left(\sum_{i=1}^{2L} \|h_i\|^2\right) \hat{z}_{jI} d & |\hat{z}_{jI}| \leq 2d \\ 2\left(\sum_{i=1}^{2L} \|h_i\|^2\right) d(d - \hat{z}_{jI}) & \hat{z}_{jI}(k) > 2d \\ -2\left(\sum_{i=1}^{2L} \|h_i\|^2\right) d(d + \hat{z}_{jI}) & \hat{z}_{jI}(k) < -2d \end{cases} \quad (45)$$

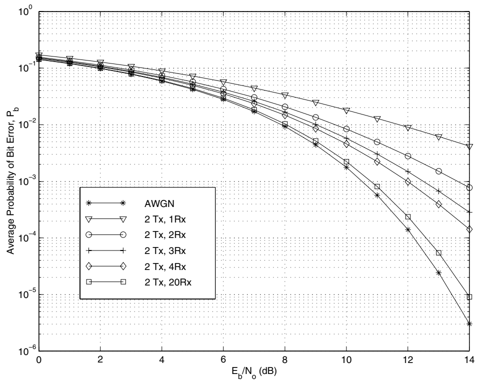

Fig. 4. BER performance of uncoded 16-QAM with transmit diversity. 2 transmit antennas and L receive antennas. $L = 1, 2, 3, 4, 20$.

$$LLR_{a_j}(r_2) = \begin{cases} -\left(\sum_{i=1}^{2L} \|h_i\|^2\right) \hat{z}_{jQ} d & |\hat{z}_{jQ}| \le 2d \\ 2\left(\sum_{i=1}^{2L} \|h_i\|^2\right) d(d - \hat{z}_{jQ}) & \hat{z}_{jQ} > 2d \\ -2\left(\sum_{i=1}^{2L} \|h_i\|^2\right) d(d + \hat{z}_{jQ}) & \hat{z}_{jQ} < -2d \end{cases} \quad (46)$$

$$LLR_{a_j}(r_3) = \Big(\sum_{i=1}^{2L} \|h_i\|^2\Big) d\{|\hat{z}_{jI}| - 2d\}, \quad (47)$$

$$LLR_{a_j}(r_4) = \Big(\sum_{i=1}^{2L} \|h_i\|^2\Big) d\{|\hat{z}_{jQ}| - 2d\}. \quad (48)$$

In the above equations, $\hat{z}_j,\ j = 1, 2$, are given by

$$\hat{z}_j = \frac{\hat{a}_j}{\sum_{i=1}^{2L} \|h_i\|^2}, \quad (49)$$

and $\hat{z}_{jI}$ and $\hat{z}_{jQ}$ are the real and imaginary parts of $\hat{z}_j$.

2) Probability of Bit Error: The probability of error can be derived following similar lines in Sec. III-A.2. The error probabilities for bits r_1, r_2 r_3 and r_4 can be derived to be:

$$P_{b1} = P_{b2} = \frac{1}{2}\Big(P_1 + P_2\Big) \quad (50)$$

$$P_{b3} = P_{b4} = \frac{1}{2}\Big(2P_1 + P_2 - P_3\Big), \quad (51)$$

where P_i, $i = 1, 2, 3$, are given by

$$P_i = \left[\frac{1}{2}\big(1 - \mu_i\big)\right]^{2L} \sum_{k=0}^{2L-1} \binom{2L-1+k}{k} \left[\frac{1}{2}\big(1 + \mu_i\big)\right]^k, \quad (52)$$

where $\mu_1 = \sqrt{\frac{E_b/N_o}{5L+E_b/N_o}}$, $\mu_2 = \sqrt{\frac{9E_b/N_o}{5L+9E_b/N_o}}$, and $\mu_3 = \sqrt{\frac{25E_b/N_o}{5L+25E_b/N_o}}$.

Fig. 4 provides the numerical results of the average BER, P_b, computed using the BER expression derived above, for the case of two transmit and multiple receive antennas. The various values of L considered are $1, 2, 3, 4$, and 20. It is seen that the performance improves as L increases due to the increased diversity order. We point out that the performance of (2-Tx,L-Rx) scheme is same as that of (1-Tx,2L-Rx) scheme. Thus our analysis provides a means to analytically evaluate the BER of QAM with 'receive-only diversity' using MRC when the number of receive antennas is even.

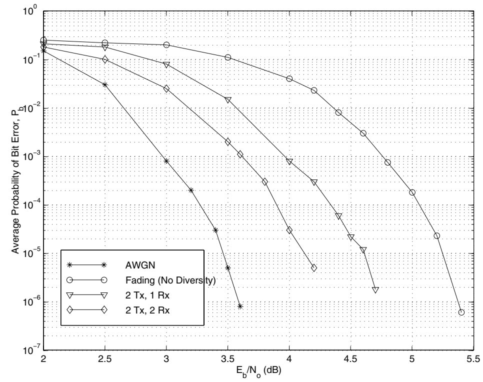

Fig. 5. BER performance of rate-1/3 turbo coded 16-QAM scheme with transmit diversity in Rayleigh fading. LLRs of bits in QAM symbols used as soft inputs to the turbo decoder.

C. LLRs as Soft Inputs to Decoders

We note that, in addition to being used in the BER analysis above, the derived LLRs for the individual bits in the QAM symbols can be used as soft inputs to the decoders in various coded QAM schemes. As an example, we employed the LLRs as soft inputs to the turbo decoder in a rate-1/3 turbo coded 16-QAM scheme in Rayleigh fading without and with transmit diversity using Alamouti scheme. Fig. 5 shows the simulated BER performance of turbo coded 16-QAM system using the derived LLRs as soft inputs to the decoder. The turbo code used in the simulations is the one specified in the 3GPP standard. Likewise, the LLRs can be used as soft inputs to decoders in DVB and IEEE 802.11a, where convolutionally coded QAM with OFDM is used.

IV. Conclusions

We analyzed the BER performance of QAM schemes in Rayleigh fading channels without and with transmit diversity. The key contributions in this paper are two fold – first, we presented an alternate method of deriving the BER for QAM on fading channels using log-likelihood ratios (LLRs) of the individual bits that form the QAM symbol, and second, using the LLRs, we derived the BER for QAM with transmit diversity in a system that uses two transmit antennas and multiple receive antennas. Although we derived the LLRs and BER for a 16-QAM scheme in this paper, the analytical technique applies to any higher order ($M > 16$) QAM constellation and for any arbitrary mapping of bits to QAM symbols. We also pointed out another major application of the LLRs derived; that is, the LLRs provide a soft metric for each bit in the mapping, which can be used as soft inputs to decoders for various coded QAM schemes, including turbo coded QAM with transmit diversity as specified in high speed downlink packet access (HSDPA) in 3G.

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