# Bounds on the Performance of Turbo Codes on Nakagami Fading Channels with Diversity Combining

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*Abstract*—**In this paper, we derive bit error performance bounds for turbo codes on Nakagami fading channels with diversity combining. We first derive the average pairwise error probability expressions for turbo codes with maximal ratio combining (MRC), equal gain combining (EGC), and selection combining (SC) on Nakagami fading channels. Using the pairwise error probability expressions and the union bounding technique, we then obtain bounds on the bit error probability of turbo codes for MRC, EGC, and SC diversity schemes. We present the modified log-MAP turbo decoder for MRC, EGC, and SC, and compare the analytical bounds with simulation results for the special case of 2-antenna diversity with Nakagami parameter $m = 1$ (i.e., Rayleigh fading). Results indicate that the EGC scheme with turbo coding performs close to MRC scheme for i.i.d Rayleigh fading.**

*Keywords* – Turbo codes, Union bound, Nakagami fading, Diversity.

## I. INTRODUCTION

Turbo codes have been shown to offer near-capacity performance on AWGN channels and significantly enhanced performance on fully-interleaved flat Rayleigh fading channels [1],[2]. Bit error probability bounds for turbo codes on AWGN channels have been derived in [3] and [4] using transfer function bounding techniques. The same bounding technique is extended to flat Rayleigh fading channels with and without channel correlation, in [2]. It is well known that diversity reception can combat the performance degradation due to channel fading [5]. In this paper, we derive the performance bounds for turbo codes on generalized Nakagami fading channels with $L$-branch diversity reception. Our contributions in this paper are: $a$) we derive a closed form expression for the pairwise error probability of the codewords with maximum-likelihood decoding on Nakagami-$m$ fading channels with maximal ratio combining (MRC), $b$) for equal gain combining (EGC) and selection combining (SC), we provide exact analytical expressions (not in closed form) as well as simple bounds (in closed form) on the pairwise probability of error, and $c$) we derive average bit error probability bounds based on the pairwise error probability expressions and union bounding techniques for MRC, EGC, and SC. We evaluate the analytical bit error probability bounds for a $(1, 7/5, 7/5)_8$ rate-1/3 turbo code with diversity schemes and compare them with simulation results. We observe that, among other things, with turbo coding, EGC performs close to MRC for i.i.d Rayleigh fading with two branch diversity combining.

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The rest of the paper is organized as follows. In Section 2, we present the system model and derive the pairwise error probability expressions for MRC, EGC, and SC on Nakagami-$m$ fading channels. In Section 3, the union bound on the bit error probability of turbo codes with maximum likelihood decoding and uniform turbo interleaving is presented. Section 4 presents the modified log-MAP BCJR algorithm for decoding turbo codes in MRC, EGC, and SC diversity schemes. In Section 5, we provide numerical results of the analytical bounds as well as the simulation results for the special case of Nakagami parameter $m = 1$ (Rayleigh fading). Conclusions are provided in Section 6.

## II. SYSTEM MODEL

We assume that the transmitted data symbols are BPSK modulated which are coherently demodulated at the receiver. The receiver employs $L$ antennas for diversity combining to mitigate the multipath fading effect. Let $X_k$ be the transmitted symbol sequence. Then the received symbol sequence on the $l^{th}$ antenna, $r_k^{(l)}$, is given by

$$r_k^{(l)} = \alpha_k^{(l)} X_k + n_k^{(l)} \qquad k = 1, 2, \ldots, N;\ l = 1, 2, \ldots, L, \qquad (1)$$

where $\alpha_k^{(l)}$ is the fade random variable, and $n_k^{(l)}$ is the AWGN component associated with the $l^{th}$ antenna path when the transmitted data symbol is $X_k \in \{-1, +1\}$. The data symbols $X_k$ are assumed to have unit energy and the $n_k^{(l)}$ random variables are assumed to be i.i.d with zero mean and variance $\sigma^2 = N_0/2E_s$, where $N_0/2E_s$ is the two-sided power spectral density of the underlying random process $n(t)$. The data block length is $N$ code symbols. Further, the fade random variables $\alpha_k^{(l)}$s are assumed to be i.i.d and Nakagami-$m$ distributed with the pdf

$$f_\alpha(x) = \frac{2m^m}{\Gamma(m)} e^{-mx^2} x^{2m-1}, \qquad x \geq 0, \tag{2}$$

where $\Gamma(m)$ is the standard Gamma integral [5]. Here, we normalized the second moment of fade, $E(\alpha^2)$, to unity. We assume that $m$ takes positive integer values[1]. The fade amplitudes $\alpha_k^{(l)}$s are assumed to remain constant over one code symbol duration. In the following subsections, we derive the pairwise error probability for MRC, EGC, and SC diversity schemes.

### A. Maximal Ratio Combining

In MRC, the receiver weights the incoming signals on $L$ antennas by the respective conjugates of the complex fading random variables. For the BPSK case, the output of the combiner $r_k$ is given by

$$r_k = \sum_{l=1}^{L} \alpha_k^{(l)} r_k^{(l)} = X_k \sum_{l=1}^{L} (\alpha_k^{(l)})^2 + \sum_{l=1}^{L} \alpha_k^{(l)} n_k^{(l)}. \tag{3}$$

Let $\mathbf{X}_0 = (x_1, x_2, \cdots, x_N)$ be the transmitted code symbol sequence and let the receiver decode this sequence incorrectly as $\mathbf{X}_j = (x_1^j, x_2^j, \cdots, x_N^j)$. Then the pairwise error probability of decoding $\mathbf{X}_0$ incorrectly as $\mathbf{X}_j$, $P_2^{MRC}(d|\underline{\alpha})$, is given by [7]

$$\begin{aligned} P_2^{MRC}(d|\underline{\alpha}) &= \mathrm{P}_E^{(d)}(\mathbf{X}_0 \to \mathbf{X}_j|\underline{\alpha}) \\ &= Q\left(\frac{\sqrt{\sum_{n=1}^{d}\sum_{l=1}^{L}[\alpha_n^{(l)}]^2}}{\sigma}\right), \end{aligned} \tag{4}$$

where $\underline{\alpha} = (\alpha_1^{(1)}, \alpha_1^{(2)}, \ldots, \alpha_d^{(L-1)}, \alpha_d^{(L)})$, $d$ is the number of positions in which $\mathbf{X}_0$ differs from $\mathbf{X}_j$, and $Q(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{u=x}^{\infty} e^{-\frac{u^2}{2}}\, du$. To evaluate the average pairwise error probability, $\overline{P^{MRC}}_2(d)$, we have to average Eqn. (4) over $\underline{\alpha}$. To do this, we use the alternative form of $Q(\cdot)$ when the argument takes nonnegative values [6], given by

$$Q(x) = \frac{1}{\pi} \int\limits_{\theta=0}^{\frac{\pi}{2}} e^{-\left(\frac{x^2}{2\sin^2\theta}\right)} d\theta, \qquad x \geq 0. \tag{5}$$

Then, $\overline{P^{MRC}}_2(d)$ is given by [7]

$$\begin{aligned} \overline{P^{MRC}}_2(d) &= \int\limits_{\alpha_1^{(1)}} \int\limits_{\alpha_1^{(2)}} \cdots \int\limits_{\alpha_d^{(L-1)}} \int\limits_{\alpha_d^{(L)}} Q\left(\frac{\sqrt{\sum_{n=1}^{d}\sum_{l=1}^{L}[\alpha_n^{(l)}]^2}}{\sigma}\right) \\ &\quad f_{\alpha_1^{(1)}}(\alpha_1^{(1)}) \cdots f_{\alpha_d^{(L)}}(\alpha_d^{(L)}) d\alpha_1^{(1)} \ldots d\alpha_d^{(L)} \\ &= \frac{1}{\pi} \int\limits_{\theta=0}^{\frac{\pi}{2}} [\int\limits_{x=0}^{\infty} \frac{2m^m}{\Gamma(m)} e^{-mx^2} x^{2m-1} e^{-\left(\frac{x^2\gamma}{\sin^2\theta}\right)} dx]^{Ld}\, d\theta, \end{aligned} \tag{6}$$

[1] In [7], we have generalized the analysis for any real value of $m \geq 0.5$

where $\gamma = \frac{1}{2\sigma^2} = \frac{E_s}{N_o}$. By substituting $x^2(m + \frac{\gamma}{\sin^2\theta}) = t$ and noting that $\Gamma(m) = \int\limits_{t=0}^{\infty} e^{-t} t^{m-1} dt$, we can further simplify Eqn. (6) as

$$\overline{P^{MRC}}_2(d) = \frac{1}{\pi} \int\limits_{\theta=0}^{\frac{\pi}{2}} \left[\frac{\sin^2\theta}{\sin^2\theta + \frac{\gamma}{m}}\right]^{Ldm} d\theta. \tag{7}$$

To evaluate (7) in closed-form, we recall the result [8]

$$\begin{aligned} I_n(c) &\triangleq \int\limits_{\theta=0}^{\frac{\pi}{2}} \left[\frac{\sin^2\theta}{\sin^2\theta + c}\right]^n d\theta \\ &= \pi[P(c)]^n \sum_{k=0}^{n-1} \binom{n-1+k}{k} [1-P(c)]^k, \end{aligned} \tag{8}$$

where $P(x) = \frac{1}{2}\left(1 - \sqrt{\frac{x}{x+1}}\right)$, $x \geq 0$. Using Eqn. (8) in (7), $\overline{P^{MRC}}_2(d)$ can be evaluated in closed-form as

$$\overline{P^{MRC}}_2(d) = \left[P\left(\frac{\gamma}{m}\right)\right]^{Lmd} \sum_{k=0}^{Lmd-1} \binom{Lmd-1+k}{k} \left[1 - P\left(\frac{\gamma}{m}\right)\right]^k. \tag{9}$$

### B. Equal Gain Combining

In EGC, the combiner cophases and equally weights the signals on each branch and combines them. So the output of the combiner, $r_k$, is given by

$$r_k = \sum_{l=1}^{L} r_k^{(l)} = X_k \sum_{l=1}^{L} \alpha_k^{(l)} + \sum_{l=1}^{L} n_k^{(l)}. \tag{10}$$

The conditional pairwise error probability of decoding $\mathbf{X}_0$ incorrectly as $\mathbf{X}_j$, $P_2^{EGC}(d|\underline{\alpha})$, is given by

$$\begin{aligned} P_2^{EGC}(d|\underline{\alpha}) &= \mathrm{P}_E^{(d)}(\mathbf{X}_0 \to \mathbf{X}_j|\underline{\alpha}) \\ &= Q\left(\frac{\sqrt{\sum_{n=1}^{d}[\sum_{l=1}^{L}\alpha_n^{(l)}]^2}}{\sqrt{L}\sigma}\right). \end{aligned} \tag{11}$$

The average pairwise error probability, $\overline{P^{EGC}}_2(d)$, averaged over $\underline{\alpha}$ is given by

$$\begin{aligned} \overline{P^{EGC}}_2(d) &= \int\limits_{\alpha_1^{(1)}} \int\limits_{\alpha_1^{(2)}} \cdots \int\limits_{\alpha_d^{(L-1)}} \int\limits_{\alpha_d^{(L)}} Q\left(\frac{\sqrt{\sum_{n=1}^{d}[\sum_{l=1}^{L}\alpha_n^{(l)}]^2}}{\sqrt{L}\sigma}\right) \\ &\quad f_{\alpha_1^{(1)}}(\alpha_1^{(1)}) \cdots f_{\alpha_d^{(L)}}(\alpha_d^{(L)}) d\alpha_1^{(1)} \ldots d\alpha_d^{(L)}. \end{aligned} \tag{12}$$

Defining $v_n = \sum_{l=1}^{L} \alpha_n^{(l)}$, the above integral can be written as

$$\begin{aligned} \overline{P^{EGC}}_2(d) &= \int\limits_{v_1} \cdots \int\limits_{v_d} Q\left(\frac{\sqrt{\sum_{n=1}^{d} v_n^2}}{\sqrt{L}\sigma}\right) \\ &\quad f_{v_1}(v_1) \cdots f_{v_d}(v_d) dv_1 \ldots dv_d. \end{aligned} \tag{13}$$

Using Eqn. (5) and observing that $v_1, v_2, \cdots, v_d$ are all i.i.d, we can reduce the above expression to

$$\overline{P^{EGC}}_2(d) = \frac{1}{\pi}\int_{\theta=0}^{\frac{\pi}{2}} [I(\theta)]^d \, d\theta, \tag{14}$$

where

$$I(\theta) \triangleq \int_{v=0}^{\infty} e^{-\left(\frac{\gamma v^2}{L\sin^2\theta}\right)} f_v(v)dv. \tag{15}$$

Applying Parseval's theorem [5],[9] to the integral $I(\theta)$, we get

$$I(\theta) = \frac{1}{2\pi}\int_{\omega=-\infty}^{+\infty} \text{FT}\left(e^{-(\frac{\gamma v^2}{L\sin^2\theta})}\right)\Phi_v^*(\omega)d\omega, \tag{16}$$

where FT$(\cdot)$ is the Fourier transform operator and $\Phi_v(\omega)$ is the characteristic function (CHF). The Fourier transform [2] of $e^{-\left(\frac{\gamma v^2}{L\sin^2\theta}\right)}$ is given by

$$\begin{aligned}\text{FT}\left(e^{-\left(\frac{\gamma v^2}{L\sin^2\theta}\right)}\right) &= \int_{v=0}^{\infty} e^{-\left(\frac{\gamma v^2}{L\sin^2\theta}\right)} e^{j\omega v}dv \\ &= \sqrt{\frac{\pi L}{\gamma}}\sin\theta e^{-\frac{L\omega^2\sin^2\theta}{4\gamma}}. \end{aligned}\tag{17}$$

The CHF of the random variable $v$, $E[e^{j\omega v}]$, can be obtained as

$$\begin{aligned}\Phi_v(\omega) &= E[e^{j\omega v}] = E\left[e^{j\omega\sum_{l=1}^{L}\alpha_l}\right] \\ &= \prod_{l=1}^{L} E[e^{j\omega\alpha_l}] = \left(E[e^{j\omega\alpha}]\right)^L = [\Phi_\alpha(\omega)]^L. \end{aligned}\tag{18}$$

The CHF of the Nakagami distributed random variable $\alpha$ can be computed as follows [7]

$$\begin{aligned}\Phi_\alpha(\omega) &= \int_{a=0}^{\infty} e^{j\omega a}\frac{2m^m}{\Gamma(m)}e^{-ma^2}a^{2m-1}da \\ &= e^{-\frac{\omega^2}{4m}}\frac{m^m}{\Gamma(m)\sqrt{m}}\sum_{k=0}^{2m-1}\frac{\binom{2m-1}{k}}{(\sqrt{m})^k}\left(\frac{j\omega}{2m}\right)^{2m-1-k} \\ &\quad \Gamma\left(\frac{-\omega^2}{4m}, \frac{k+1}{2}\right), \end{aligned}\tag{19}$$

where

$$\Gamma(a,n) = \int_{u=a}^{\infty} e^{-u}u^{n-1}du \tag{20}$$

[2] We define the FT of a function $f(t)$ as $F(\omega) = E[e^{j\omega t}] = \int_{-\infty}^{\infty} f(t)e^{j\omega t}dt$ to be consistent with the definition of characteristic function and its inverse FT used in [5]. This differs from the usual definition of FT by a negative sign in the exponential.

with $\Gamma(0,n) = \Gamma(n)$. Since $I(\theta)$ and FT $\left(e^{-\left(\frac{\gamma v^2}{L\sin^2\theta}\right)}\right)$ are real, we can write $I(\theta)$ in Eqn. (16) as

$$I(\theta) = \frac{1}{\pi}\int_{\omega=0}^{\infty} \text{FT}\left(e^{-\left(\frac{\gamma v^2}{L\sin^2\theta}\right)}\right)\Re\{\Phi_v^*(\omega)\}d\omega, \tag{21}$$

where $\Re\{\cdot\}$ denotes the real part of the complex number. Upon substituting $I(\theta)$ of Eqn. (21) into Eqn. (14) we get the exact analytical expression for $\overline{P^{EGC}}_2(d)$. The resulting expression can be computed numerically (which involves computation of multi-dimensional integrals). For the special case of $L = 2$, however, we have derived an alternate expression for $\overline{P^{EGC}}_2(d)$, which requires only a single numerical integration [7].

### C. Selection Combining

In SC the output of the selection combiner is expressed as

$$r_k = a_k X_k + n_k, \tag{22}$$

where $a_k = \max\left(\alpha_k^{(1)}, \alpha_k^{(2)}, \ldots, \alpha_k^{(L)}\right)$, and $n_k$ is Gaussian distributed with zero mean and variance $\sigma^2$. With this statistic at the output of the demodulator, the conditional pairwise error probability, $P_2^{SC}(d|\underline{a})$, conditioned on the fading random variables $a_1, a_2, \ldots, a_N$, is given by

$$P_2^{SC}(d|\underline{a}) = \text{P}_E^{(d)}(\mathbf{X}_0 \to \mathbf{X}_j|\underline{a}) = Q\left(\frac{\sqrt{\sum_{n=1}^{d} a_n^2}}{\sigma}\right). \tag{23}$$

Let $\eta_n = a_n^2 = \max(\alpha_n^{2(1)}, \alpha_n^{2(2)}, \ldots, \alpha_n^{2(L)})$ and $\underline{\eta} = (\eta_1, \eta_2, \ldots, \eta_d)$. With this, we can simplify the above expression as

$$\begin{aligned}\overline{P^{SC}}_2(d) &= \frac{1}{\pi}\int_{\theta=0}^{\frac{\pi}{2}} e^{-\left(\frac{\sum_{n=1}^{d} a_n^2}{2\sigma^2\sin^2\theta}\right)} f_{a_1}(a_1)f_{a_2}(a_2)\ldots f_{a_d}(a_d) \\ &\quad da_1 da_2 \ldots da_d d\theta \\ &= \frac{1}{\pi}\int_{\theta=0}^{\frac{\pi}{2}}\left[\int_{x=0}^{\infty} f_\eta(x)e^{-\left(\frac{\gamma x}{\sin^2\theta}\right)}\right]^d d\theta. \end{aligned}\tag{24}$$

The pdf of $\eta$, $f_\eta(x)$, can be obtained by first deriving the c.d.f $F_\eta(x)$ and then differentiating it with respect to $x$ as follows.

$$\begin{aligned}F_\eta(x) &= \text{Prob}\left(\max(a_1^2, a_2^2, \ldots, a_L^2) \le x\right) \\ &= \left[\frac{1}{\Gamma(m)}\int_{u=0}^{mx} e^{-u}u^{m-1}du\right]^L. \end{aligned}\tag{25}$$

Differentiating the above with respect to $x$, we get the p.d.f, $f_\eta(x)$ as

$$f_\eta(x) = L\left[\frac{\Gamma(m) - \Gamma(mx,m)}{\Gamma(m)}\right]^{L-1}\frac{m^m}{\Gamma(m)}e^{-mx}x^{m-1}, \tag{26}$$

where $\Gamma(\cdot,\cdot)$ is defined by Eqn. (20). Upon substituting Eqn. (26) in Eqn. (24), we obtain the expression for $\overline{P^{SC}}_2(d)$ as

$$\overline{P^{SC}}_2(d) = \frac{L^d}{\pi\Gamma(m)^{Ld}} \int_{\theta=0}^{\frac{\pi}{2}} \left(\frac{\sin^2\theta}{\sin^2\theta + \frac{\gamma}{m}}\right)^d \cdot \left[\int_{u=0}^{\infty} e^{-u} u^{m-1} \left(\Gamma(m) - \Gamma\left(\frac{u\sin^2\theta}{\sin^2\theta+\frac{\gamma}{m}}, m\right)\right)^{L-1} du\right]^d d\theta. \quad (27)$$

The above equation can be computed numerically. However, for the special case of Nakagami parameter $m = 1$ (i.e., Rayleigh fading), an alternate expression for $\overline{P^{SC}}_2(d)$, which is much simpler to compute than Eqn. (27), can be derived as follows:

The p.d.f $f_\eta(x)$ for the case of Rayleigh fading can be obtained by substituting $m = 1$ in Eqn. (26), i.e.,

$$f_\eta(x) = L(1-e^{-x})^{L-1}e^{-x}. \quad (28)$$

Now,

$$\begin{aligned} P_2^{SC}(d|\underline{\eta}) &= Q\left(\frac{\sqrt{\sum_{n=1}^{d}\eta_n}}{\sigma}\right) \\ \overline{P^{SC}}_2(d) &= E_{\underline{\eta}}\left[P_2^{SC}(d|\underline{\eta})\right] \\ &= \frac{L^d}{\pi}\int_{\theta=0}^{\frac{\pi}{2}}\left[\sum_{l=0}^{L-1}\binom{L-1}{l}(-1)^l\frac{\sin^2\theta}{(1+l)\sin^2\theta+\gamma}\right]^d d\theta. \quad (29) \end{aligned}$$

The above expression can be evaluated by numerical integration for given $\gamma$, $L$ and $d$. Note that computation of Eqn. (27) requires evaluation of two integrals, whereas Eqn. (29) requires only a single integral.

## III. Union Bound on BER

To determine the bit error performance of turbo codes in high SNR regions where the “error floor” occurs, we require long BER simulation runs or an analytical performance bounding technique. The upper bound on the average bit error probability for turbo codes on AWGN channels was developed in [3] and [4], and was later extended to Rayleigh fading channels in [2]. Following the same notation in [2], we obtain upper bounds on the average bit error probability for turbo codes on generalized Nakagami fading diversity channels with MRC, EGC, and SC.

The traditional union upper bound for the maximum likelihood (ML) decoding of an $(N, K)$ block code can be derived as follows. Without loss of generality, we assume that the all-zero codeword was sent, so the upper bound on the probability of word error is given by

$$P_w \leq \sum_{d=1}^{N} A(d)P_2(d), \quad (30)$$

where $A(d)$ is the number of codewords with Hamming weight $d$ and $P_2(d)$ is the probability of incorrectly decoding to a codeword with weight $d$. For a turbo code with a fixed interleaver, the construction of $A(d)$ requires exhaustive search. To avoid this, [3] and [4] propose an average upper bound averaged over all possible interleavers. With this framework, the average weight distribution is given by

$$\overline{A(d)} = \sum_{i=1}^{K}\binom{K}{i}p(d|i), \quad (31)$$

where $\binom{K}{i}$ is the number of input words with Hamming weight $i$ and $p(d|i)$ is the probability that an input word with Hamming weight $i$ produces a codeword with Hamming weight $d$. Substituting Eqn. (31) in Eqn. (30), the average upper bound for word and bit error probabilities can be written as

$$\begin{aligned} \overline{P_w} &\leq \sum_{d=d_{min}}^{N}\overline{A(d)}P_2(d) = \sum_{d=d_{min}}^{N}\sum_{i=1}^{K}\binom{K}{i}p(d|i)P_2(d) \\ &= \sum_{i=1}^{K}\binom{K}{i}E_{d|i}[P_2(d)], \quad (32) \end{aligned}$$

and

$$\overline{P_b} \leq \sum_{i=1}^{K}\frac{i}{K}\binom{K}{i}E_{d|i}[P_2(d)], \quad (33)$$

respectively, where $E_{d|i}[\cdot]$ is an expectation with respect to the distribution $p(d|i)$. The distribution $p(d|i)$ can be computed from the state transition matrix of the constituent RSC encoders [3],[4]. With the above formulation, the bit error performance of turbo codes on Nakagami fading channels with diversity combining can be evaluated by substituting $\overline{P^{MRC}}_2(d)$, $\overline{P^{EGC}}_2(d)$, $\overline{P^{SC}}_2(d)$ for $P_2(d)$ in Eqn. (33), for MRC, EGC and SC diversity schemes, respectively.

## IV. Log-MAP Turbo Decoder with Diversity

In this section, we modify the log-MAP decoder for the case of $L$-branch diversity combining. To do so, we need to calculate the transition metric defined by $\gamma_k(s,t) = \text{Prob}(\mathbf{y}_k, S_k = t|S_{k-1} = s)$, where $\mathbf{y}_k = (y_k^s, y_k^p)$ for a rate-1/3 turbo code. Here, $y_k^s$ is the received symbol corresponding to the transmitted information symbol $x_k^s$, and $y_k^p$ represents the received symbol corresponding to the transmitted parity symbol, $x_k^p$. Here $p \in \{p_1, p_2\}$, where $p_1$ signifies the first parity and $p_2$ signifies the second parity. It is to be noted that, for the first decoder the received symbols due to transmitted symbol and parity symbol (i.e.,first parity) have the same time alignment, whereas for the second decoder the received symbols are due to the interleaved version of the transmitted symbols and again have the same time alignment with the second parity symbol. Also, $S_k$, $S_{k-1}$ are the encoder states at time instants $k$, $k-1$, respectively [7]. When the symbol $x_k$ is transmitted, it is received through $L$ independent paths, and the output of the combiner will be

$$MRC: \ y_k = x_k\sum_{l=1}^{L}[\alpha_k^{(l)}]^2 + \sum_{l=1}^{L}n_k^{(l)}\alpha_k^{(l)} \quad (34)$$

$$EGC: \; y_k = x_k \sum_{l=1}^{L} \alpha_k^{(l)} + \sum_{l=1}^{L} n_k^{(l)} \tag{35}$$

$$SC: \; y_k = x_k \max\left(\alpha_k^{(1)}, \alpha_k^{(2)}, \ldots, \alpha_k^{(L)}\right) + n_k. \tag{36}$$

Here, $n_k$ has the same distribution as $n_k^{(l)}$, i.e., it is distributed Gaussian with zero mean and variance $\sigma^2$. Applying Bayes' theorem, we can write $\gamma_k(s,t)$ as

$$\begin{aligned} \gamma_k(s,t) &= \text{Prob}(\mathbf{y}_k, S_k = t | S_{k-1} = s) \\ &= \text{Prob}(\mathbf{y}_k | S_{k-1} = s, S_k = t)\text{Prob}(S_k = t | S_{k-1} = s) \\ &= p(\mathbf{y}_k|\mathbf{x}_k)\text{Prob}(S_k = t|S_{k-1} = s) = p(\mathbf{y}_k|\mathbf{x}_k)p(x_k^s). \end{aligned} \tag{37}$$

In the above, $\text{Prob}(\mathbf{y}_k|S_{k-1} = s, S_k = t) = p(\mathbf{y}_k|\mathbf{x}_k)$ because the knowledge of states at times $k-1$ and $k$ is equivalent to knowing the transmitted code symbol vector $\mathbf{x}_k$. Also, $\text{Prob}(S_k = t|S_{k-1} = s) = p(x_k^s)$ because, for a rate-$1/n$ RSC code, the state transition between any given pair of states $s$ and $t$ uniquely determines the information bit $x_k^s$. Now, define

$$\begin{aligned} c_k(s,t) &= \log\left(\gamma_k(s,t)\right) = \log(p(\mathbf{y}_k|\mathbf{x}_k)p(x_k^s)) \\ &= \log(p(\mathbf{y}_k|\mathbf{x}_k)) + \log(p(x_k^s)). \end{aligned} \tag{38}$$

Defining the quantity $\hat{L}_k$ as

$$\hat{L}_k = \log\left(\frac{\text{Prob}(x_k^s = +1)}{\text{Prob}(x_k^s = -1)}\right), \tag{39}$$

and discarding all the terms independent of $x_k^s$, we can calculate $\log(p(x_k^s))$ as [7]

$$\log(p(x_k^s)) = \frac{\hat{L}_k x_k^s}{2}. \tag{40}$$

In the following subsections, we derive the first term in the Eqn. (38) for the three diversity cases of interest, i.e., MRC, EGC, and SC diversity schemes.

### A. MRC Diversity

For MRC, conditioning on $x_k, \alpha_k^{(1)}, \alpha_k^{(2)}, \cdots, \alpha_k^{(L)}$, $y_k \sim \mathcal{N}\left(x_k \sum_{l=1}^{L}[\alpha_k^{(l)}]^2, \sigma^2 \sum_{l=1}^{L}[\alpha_k^{(l)}]^2\right)$. With perfect knowledge of the fade amplitudes, we get

$$p(\mathbf{y}_k|\mathbf{x}_k, \underline{\alpha}_k^s, \underline{\alpha}_k^p) = p(y_k^s|x_k^s, \underline{\alpha}_k^s)p(y_k^p|x_k^p, \underline{\alpha}_k^p), \tag{41}$$

where $\underline{\alpha}_k^s = (\alpha_k^{(1),s}, \alpha_k^{(2),s}, \ldots, \alpha_k^{(L),s})$, and $\underline{\alpha}_k^p = (\alpha_k^{(1),p}, \alpha_k^{(2),p}, \ldots, \alpha_k^{(L),p})$. Here, $\alpha_k^{(l),s}$ and $\alpha_k^{(l),p}$ denote the fade amplitudes experienced by the $k^{th}$ data symbol, and the corresponding parity symbol, respectively, on the $l^{th}$ antenna path. Upon simplifying the above expression, discarding all the constant terms and terms which do not depend on the code symbols $\{\mathbf{x}_k\}$, and taking logarithm on both sides of Eqn. (41), we obtain

$$\log(p(\mathbf{y}_k|\mathbf{x}_k)) = \frac{2E_s}{N_0}\left(y_k^s x_k^s + y_k^p x_k^p\right). \tag{42}$$

Combining the results of Eqns. (42) and (40) and substituting in Eqn. (38), we obtain

$$c_k(s,t) = \frac{\hat{L}_k x_k^s}{2} + \frac{2E_s}{N_0}\left(y_k^s x_k^s + y_k^p x_k^p\right). \tag{43}$$

### B. EGC Diversity

For EGC, conditioning on $x_k, \alpha_k^{(1)}, \alpha_k^{(2)}, \ldots, \alpha_k^{(L)}$, $y_k \sim \mathcal{N}\left(x_k \sum_{l=1}^{L} \alpha_k^{(l)}, L\sigma^2\right)$. With perfect knowledge of the fade amplitudes, we get

$$p(\mathbf{y}_k|\mathbf{x}_k, \underline{\alpha}_k^s, \underline{\alpha}_k^p) = p(y_k^s|x_k^s, \underline{\alpha}_k^s)p(y_k^p|x_k^p, \underline{\alpha}_k^p), \tag{44}$$

where $\underline{\alpha}_k^s = (\alpha_k^{(1),s}, \alpha_k^{(2),s}, \ldots, \alpha_k^{(L),s})$, and $\underline{\alpha}_k^p = (\alpha_k^{(1),p}, \alpha_k^{(2),p}, \ldots, \alpha_k^{(L),p})$. Upon simplifying the above expression, discarding all the constant terms and terms which do not depend on the code symbols $\{\mathbf{x}_k\}$, and taking logarithm on both sides of Eqn. (44), we obtain

$$\log(p(\mathbf{y}_k|\mathbf{x}_k)) = \frac{2E_s}{LN_0}\left(\sum_{l=1}^{L} y_k^s x_k^s \alpha_k^{(l),s} + \sum_{l=1}^{L} y_k^p x_k^p \alpha_k^{(l),p}\right). \tag{45}$$

Combining the results of Eqns. (45) and (40) and substituting in Eqn. (38), we obtain

$$c_k(s,t) = \frac{\hat{L}_k x_k^s}{2} + \frac{2E_s}{LN_0}\left(\sum_{l=1}^{L} y_k^s x_k^s \alpha_k^{(l),s} + \sum_{l=1}^{L} y_k^p x_k^p \alpha_k^{(l),p}\right). \tag{46}$$

### C. SC Diversity

For SC, conditioning on $x_k, \alpha_k^{(1)}, \alpha_k^{(2)}, \cdots, \alpha_k^{(L)}$, $y_k \sim \mathcal{N}(x_k \max(\alpha_k^{(1)}, \alpha_k^{(2)}, \cdots, \alpha_k^{(L)}), \sigma^2)$. Let us define $\alpha_k^s = \max(\alpha_k^{(1),s}, \alpha_k^{(2),s}, \cdots, \alpha_k^{(L),s})$, and $\alpha_k^p = \max(\alpha_k^{(1),p}, \alpha_k^{(2),p}, \cdots, \alpha_k^{(L),p})$. With perfect knowledge of the fade amplitudes, we get

$$p(\mathbf{y}_k|\mathbf{x}_k, \alpha_k^s, \alpha_k^p) = p(y_k^s|x_k^s, \alpha_k^s)p(y_k^p|x_k^p, \alpha_k^p). \tag{47}$$

Upon simplifying the above expression, discarding all the constant terms and terms which do not depend on the code symbols $\{\mathbf{x}_k\}$, and taking logarithm on both sides of Eqn. (47), we obtain

$$\log(p(\mathbf{y}_k|\mathbf{x}_k)) = \frac{2E_s}{N_0}\left(y_k^s x_k^s \alpha_k^s + y_k^p x_k^p \alpha_k^p\right). \tag{48}$$

Combining the results of Eqns. (48) and (40) and substituting in Eqn. (38), we obtain

$$c_k(s,t) = \frac{\hat{L}_k x_k^s}{2} + \frac{2E_s}{N_0}\left(y_k^s x_k^s \alpha_k^s + y_k^p x_k^p \alpha_k^p\right). \tag{49}$$

The quantity $c_k(s,t)$, derived in Eqns. (43), (46), (49), can be used in the computation of the forward and backward recursion metrics in the simulation of log-MAP algorithm for decoding turbo codes [7].

## V. Results and Discussion

We evaluated the bit error performance of a $(1, 7/5, 7/5)_8$ rate-1/3 turbo code on fading channels with MRC, EGC, and SC, using the pairwise error probability and the bit error probability bounds derived in Sections II and III. We also evaluated the bit error performance using simulations. All the analytical and simulation results are obtained for the special case of 2-antenna diversity ($L = 2$) with Nakagami parameter $m = 1$ (i.e., Rayleigh fading).

In Fig. 1, the analytical bit error performance of the $(1, 7/5, 7/5)_8$ rate-1/3 turbo code based on the union bounds is presented for various scenarios. In particular, performance is shown for $a$) AWGN, $b$) fading without diversity ($L = 1$), and $c$) fading with 2-antenna diversity using MRC, EGC, and SC ($L = 2$). The input data block length is 100 bits (i.e., 300 code symbols). It is noted that the EGC scheme with turbo coding performs very close to the MRC scheme.

In Fig. 2, we plot the analytical bit error performance results versus the simulation results for the $(1, 7/5, 7/5)_8$ rate-1/3 turbo code with block length of 100 bits. The modified log-MAP turbo decoder for diversity, presented in Section IV, is used in the simulations. In the simulations, perfect channel state information is assumed at the decoder. The number of turbo iterations is set to eight. From Figure 2, we see that the analytical performance of the turbo code agrees very well with the simulation results for the SNR values above a threshold value of the $E_b/N_o$ determined by the computational cutoff rate $R_0$ [3],[4],[2]. We can also observe that the performance of the turbo code on fading channels with EGC diversity is very close to the MRC diversity, both in analysis as well as in simulation. Moreover, the implementation of EGC diversity is relatively simple compared to MRC diversity. In [10], we gave a practical method of estimating the channel SNR for EGC diversity on Nakagami fading channels and showed that the performance with estimated channel SNR is inferior to that with perfect knowledge of the channel SNR by only 0.8 dB. Like the bit error probability bounds for AWGN and fading with no diversity presented in [3] and [2], the bit error probability bounds derived for diversity in this paper are found to be loose in the low SNR regions, where tighter bounds need to be developed.

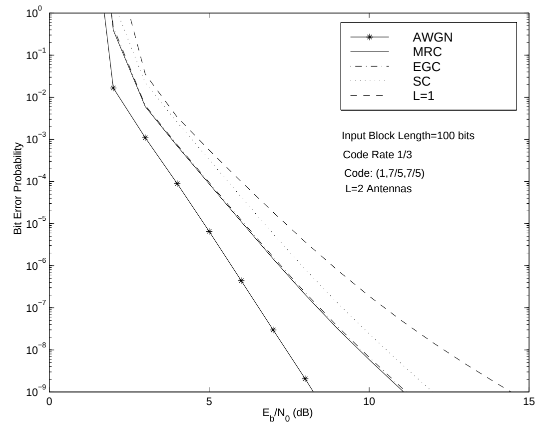


Fig. 1. Comparison of union bounds on the bit error probability for MRC, EGC, SC diversity ($L = 2$) for i.i.d fading ($m = 1$). Block length=100 bits. AWGN and fading with no diversity ($L = 1$) results also shown.

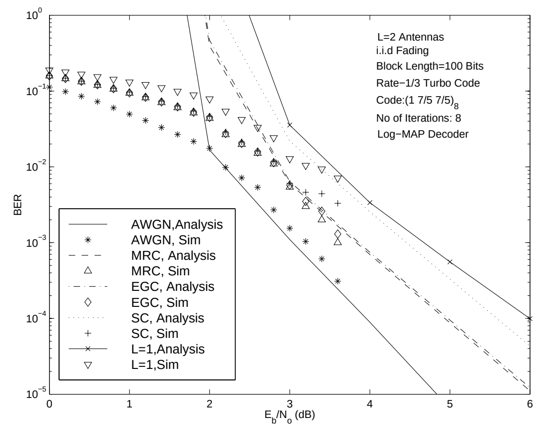


Fig. 2. Comparison of union bounds on the bit error probability *versus* simulation results for various diversity schemes. Block length=100 bits, $m = 1$. AWGN and fading with no diversity results also shown.

## VI. CONCLUSIONS

We derived performance bounds for turbo codes on Nakagami fading channels with diversity combining. We derived average pairwise error probability expressions for turbo codes with MRC, EGC, and SC diversity schemes on Nakagami fading channels. Using the pairwise error probabilities, we derived the bit error performance using the union bounding technique. We compared the analytical bounds with the simulation results for the 2-antenna Rayleigh fading case. It was found that the simulation and analytical results are close for high SNR values. The EGC diversity scheme with turbo coding was found to perform as well as the MRC diversity scheme.

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