SNR Estimation in Nakagami Fading with Diversity for Turbo Decoding

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Abstract—**In this paper, we propose an online SNR estimation scheme for Nakagami-m fading channels with equal gain diversity combining. We derive the SNR estimates based on the statistical ratio of certain observables over a block of received data. An online SNR estimator for an AWGN channel has been derived by Summers and Wilson. Recently, we derived an SNR estimation scheme for Nakagami-m fading channels without diversity combining and used this estimate in the decoding of turbo codes. Now, we extend the work and solve the SNR estimation problem on Nakagami fading channels with** $L-$*branch equal gain diversity combining*. **We use our SNR estimates in the iterative decoding of turbo codes on Rayleigh fading channels ($m = 1$) with 2-branch equal gain combining. We show that the turbo decoder performance using our SNR estimates is quite close (within 0.5 dB) to the performance using perfect knowledge of the SNR and the fade amplitudes.**

Keywords – SNR estimation, Nakagami fading, Diversity, Turbo codes.

I. Introduction

Turbo codes have been shown to offer near-capacity performance on AWGN channels and significantly good performance on fully-interleaved flat Rayleigh fading channels [1], [2]. Turbo codes are typically generated using two (or more) constituent recursive systematic convolutional encoders separated by large interleavers [3]. Decoding of turbo codes involves processing each constituent code by a separate decoder and estimating the *a posteriori probability* of the various message bits. Each decoder incorporates a modified BCJR algorithm for performing symbol-by-symbol maximum *a posteriori* probability (MAP) decoding [4]. The decoders share bit-likelihood information (called extrinsic information) in an iterative fashion. That is, the bit-likelihood information computed by the first decoder is used as *a priori* information by the second decoder. Using this *a priori* information, the second decoder computes the bit-likelihood function again, which is then passed to the first decoder for the next iteration of decoding. It is noted that since the constituent encoders are separated by the turbo interleaver, the first decoder essentially provides the extrinsic information to the second decoder using only information not available to the second decoder (i.e., first encoder parity). The second decoder does likewise to the first decoder.

In addition to the bit-likelihood information, optimum decoding of turbo codes (and concatenated coding schemes of similar nature) requires knowledge of the channel signal-to-noise ratio (SNR) and channel state information (CSI) for the case of fading channels. For the AWGN case, Summers and Wilson have recently addressed the issue of the sensitivity of the turbo decoder performance to imperfect knowledge of the channel SNR, and proposed an online SNR estimation scheme [5]. It was shown that a simple estimator of SNR, based on both the sum of the squared receiver output values and square of the sum of their absolute values, can provide accurate estimates.

Performance of turbo codes on flat Rayleigh fading channels has been addressed in [2],[6],[7]. In the performance evaluation of turbo codes on fading channels, perfect knowledge of both the E_s/N_0 (channel SNR) as well as the fade amplitudes of each symbol (CSI) are typically assumed to be available at the decoder. In practice, the channel SNR needs to be estimated at the receiver for use in the turbo decoding.

A channel estimation technique suitable for decoding turbo codes on flat Rayleigh fading channels is presented in [6]. But the technique is based on sending known pilot symbols at regular intervals in the transmit symbol sequence. In [7], a channel estimator based on a low pass FIR filter is presented for flat Rayleigh and Rician fading channels. Recently, we, in [8], derived an SNR estimation scheme for Nakagami fading channels without diversity combining and used this estimate in the decoding of turbo codes. Now, we extend the work and solve the SNR estimation problem on Nakagami fading channels with $L-$*branch equal gain diversity combining*.

In this paper, we propose an online SNR estimation scheme for Nakagami-m fading with diversity combining. The proposed estimation scheme does not require the transmission of known training symbols. We derive the SNR estimates based on the statistical ratio of certain observables over a block of received data. As an example, we use our SNR estimates in the iterative decoding of turbo codes on Rayleigh fading channels ($m = 1$) with 2-branch equal gain combining. We show that the turbo decoder performance using our SNR estimates is quite close (within 0.5 dB) to the performance using perfect knowledge of the SNR and the fade amplitudes.

The rest of the paper is organized as follows. In Section II, an

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SNR estimator for L-branch equal gain combining on Nakagami-m fading is derived. The detailed derivations of the statistical parameters of interest are moved to the Appendix. In Section III, we present the modified log-MAP algorithm for decoding turbo codes with equal gain combining. Section IV presents the performance results of the turbo decoder using our proposed online estimation scheme. Conclusions are provided in Section V.

II. SNR Estimation with Diversity Combining

In this section, we derive an SNR estimate for the Nakagami-m channel with diversity combining. Let the encoded data symbols be BPSK modulated and transmitted over a Nakagami fading channel. We assume L antennas at the receiver with sufficient spacing between them so that these antennas receive signals through independent fading paths. We denote the k^{th} symbol received at the i^{th} antenna by $r_i^{(k)}$. We assume that the receiver performs equal gain combining (EGC), after coherently demodulating the received symbols on these independent diversity paths. Then, the k^{th} received symbol, v_k, at the output of the combiner is given by

$$v_k = \sum_{i=1}^{L} r_i^{(k)}, \tag{1}$$

where

$$r_i^{(k)} = \pm\alpha_i^{(k)}\sqrt{E_s} + n_i^{(k)}. \tag{2}$$

Here, $\alpha_i^{(k)}$ is the random fade experienced by k^{th} symbol on the i^{th} antenna path, E_s is the symbol energy, and $n_i^{(k)}$ is the AWGN component at the receiver front end having zero mean and variance $\sigma^2 = N_o/2$, where the two-sided power spectral density of the channel noise process is $N_0/2$ W/Hz. We assume that the α's are Nakagami-m distributed [9] and independent of the noise. Specifically,

$$p_\alpha(a) = \frac{2m^m a^{2m-1}}{\Gamma(m)} e^{-ma^2}. \tag{3}$$

In the above equation, we have normalized the second moment of the fade, $E(\alpha^2)$, to unity. The Nakagami-m distribution spans, via the m parameter, the widest range of fading among all the multipath distributions considered in this paper. For instance, it includes the one sided Gaussian distribution (m=0.5) and the Rayleigh distribution (m=1) as special cases [9]. In the limit as $m \to +\infty$, the Nakagami fading channel converges to a non-fading AWGN channel, i.e., as $m \to \infty$ the pdf approaches $\delta(a-1)$. When $m \geq 1$, a one-to-one mapping between the m parameter and the Rician factor, K, allows the Nakagami-m distribution to closely approximate the Rice distribution. The Nakagami-m distribution often gives the best fit to land-mobile and indoor-mobile multipath propagation, as well as to scintillating ionospheric radio links [9]. In order to obtain an SNR estimate for L-branch diversity combining, we propose a scheme based on the parameter z_{div} defined as

$$z_{div} = \frac{\left[E(v^2)\right]^2}{E(v^4)}. \tag{4}$$

The parameter z_{div} in Eqn. (4) can be derived in closed-form as (see the Appendix)

$$z_{div} = \frac{\left[L + \left((L^2-L)\left(\frac{\Gamma(m+\frac{1}{2})}{\sqrt{m}\Gamma(m)}\right)^2 + L\right)2\gamma\right]^2}{3L^2 + 4\Delta_m\gamma^2 + 12L\left(L + (L^2-L)\left(\frac{\Gamma(m+\frac{1}{2})}{\sqrt{m}\Gamma(m)}\right)^2\right)\gamma}, \tag{5}$$

where

$$\begin{aligned}\Delta_m &= LE(\alpha^4) + 4L(L-1)E(\alpha^3)E(\alpha) + 3L(L-1)[E(\alpha^2)]^2 + \\ &\quad 6L(L-1)(L-2)E(\alpha^2)[E(\alpha)]^2 + \\ &\quad L(L-1)(L-2)(L-3)[E(\alpha)]^4, \end{aligned}\tag{6}$$

with $E(\alpha^k) = \frac{\Gamma(m+\frac{k}{2})}{\Gamma(m)m^{\frac{k}{2}}}$. For the case when $m = 1$ (i.e., Rayleigh fading), Eqn. (5) becomes,

$$z_{div} = \frac{\left[L + \frac{L(\pi L+4-\pi)}{2}\gamma\right]^2}{3L^2 + 4\Delta_1\gamma^2 + 3L^2(\pi L+4-\pi)\gamma}. \tag{7}$$

For a given value of z (computed from a block observation of the v_n's), the corresponding estimate of γ can be found from (5). For easy implementation, an approximate relation between z_{div} and γ can be obtained through an exponential curve fitting for Eqn. (7). We use the exponential fit of the form

$$\gamma = d_3 e^{\left(d_0 e^{(d_1 z_{div})} + d_2 z_{div}\right)}, \tag{8}$$

where $d_0 = 7.25 \cdot 10^{-8}$, $d_1 = 25.94$, $d_2 = 10.44$, and $d_3 = 6.72 \cdot 10^{-3}$.

Figure 1 shows the γ versus z_{div} plots corresponding to Rayleigh fading ($m = 1$) with two-branch receiver diversity and with equal gain combining as per Eqn. (8), along with the true value plot as per Eqn. (7). It is seen that the fit is very accurate over the SNR values of interest. In order to obtain an estimate for z_{div}, we replace the expectations in Eqn. (4) with the corresponding block averages, yielding

$$\hat{z}_{div} = \frac{[\overline{v^2}]^2}{\overline{v^4}}. \tag{9}$$

Substituting (9) into (8) we get the SNR estimates, $\widehat{\gamma}$. We tested the accuracy of the fit by evaluating the mean and standard deviation of the SNR estimates $\widehat{\gamma}$, determined by over 20,000 blocks. The block sizes considered are 1000 and 5000 bits and the code rate is 1/3 (3000 and 15000 code symbols). The range of E_b/N_0 values considered is from 0 dB to 8 dB in steps of 1 dB. This corresponds to E_s/N_0 values from -4.77 dB to 3.23 dB, for rate 1/3 turbo code. The results are given in Table I. From Table I, we observe that the mean SNR estimates $\widehat{\gamma}$ through the exponential fit in Eqn. (8) are quite close to the true value of SNR, γ, and the standard deviation of the estimate reduces as the block size is increased.

III. Log-MAP Decoder with Equal Gain Combining

In this section, we modify the log-MAP decoder [10]-[13] for the case of L-branch diversity with equal gain combining. To do so, we need to calculate the transition metric defined by $\gamma_k(s,t) = \text{Prob}(\mathbf{y}_k, S_k = t | S_{k-1} = s)$, where $\mathbf{y}_k = (y_k^s, y_k^p)$, $p \in \{p_1, p_2\}$, for a rate 1/3 turbo code [3]. Here, y_k^s is the received symbol corresponding to the transmitted information symbol x_k^s, and y_k^p is the received symbol corresponding to the transmitted parity symbol, x_k^p. p_1, p_2 are the parity symbols generated by the constituent convolutional encoders [3]. Also, S_k, S_{k-1} are the encoder states at time instants k, $k-1$, respectively [15]. When the symbol x_k is transmitted, it will

True SNR, γ (dB)	Block size=1000 bits		Block Size=5000 bits	
	E[$\widehat{\gamma}$], dB	SD [$\widehat{\gamma}$], dB	E[$\widehat{\gamma}$], dB	SD[$\widehat{\gamma}$], dB
-4.77	-4.01	0.397	-4.03	0.181
-3.77	-3.32	0.395	-3.34	0.180
-2.77	-2.54	0.396	-2.57	0.178
-1.77	-1.69	0.395	-1.71	0.176
-0.77	-0.77	0.394	-0.78	0.176
0.23	0.18	0.395	0.17	0.178
1.23	1.19	0.407	1.66	0.182
2.23	2.18	0.421	2.16	0.189
3.23	3.19	0.443	3.17	0.200

TABLE I

MEAN AND STANDARD DEVIATION OF THE SNR ESTIMATE, $\widehat{\gamma}$, FOR DIFFERENT VALUES OF THE TRUE SNR, γ, FOR TWO-BRANCH DIVERSITY AND EQUAL GAIN COMBINING.

be received through L independent paths and the output of the combiner will be

$$y_k = x_k \sum_{l=1}^{L} \alpha_{k,l} + \sum_{l=1}^{L} n_{k,l}, \qquad (10)$$

where $\alpha_{k,l}$ is the random fade experienced by the k^{th} symbol at the l^{th} antenna path. Conditioning on x_k and $\alpha_{k,1}$, $\alpha_{k,2}$, $\cdots$, $\alpha_{k,L}$, we have $y_k \sim \mathcal{N}\left(x_k \sum_{l=1}^{L} \alpha_{k,l}, L\sigma^2\right)$. Applying Bayes' theorem, we can write $\gamma_k(s,t)$ as

$$\begin{aligned} \gamma_k(s,t) &= \text{Prob}(\mathbf{y}_k, S_k = t | S_{k-1} = s) \\ &= \text{Prob}(\mathbf{y}_k | S_{k-1} = s, S_k = t)\text{Prob}(S_k = t | S_{k-1} = s) \\ &= p(\mathbf{y}_k|\mathbf{x}_k)\text{Prob}(S_k = t | S_{k-1} = s) \\ &= p(\mathbf{y}_k|\mathbf{x}_k) p(x_k^s). \end{aligned} \qquad (11)$$

The last step in the above equation is due to the fact that the state transition between any given pair of states s and t uniquely determines the information bit x_k^s. Define

$$\begin{aligned} c_k(s,t) &= \log\left(\gamma_k(s,t)\right) \\ &= \log(p(\mathbf{y}_k|\mathbf{x}_k) p(x_k^s)) \\ &= \log(p(\mathbf{y}_k|\mathbf{x}_k)) + \log(p(x_k^s)). \end{aligned} \qquad (12)$$

The first term in the above equation is derived as follows. With perfect channel interleaving and knowledge of fade amplitudes, we get

$$p(\mathbf{y}_k|\mathbf{x}_k, \underline{\alpha}_k^s, \underline{\alpha}_k^p) = p(y_k^s|x_k^s, \underline{\alpha}_k^s) p(y_k^p|x_k^p, \underline{\alpha}_k^p) \qquad (13)$$

where $\underline{\alpha}_k^s = (\alpha_{k,1}^s, \alpha_{k,2}^s, \ldots, \alpha_{k,L}^s)$ and $\underline{\alpha}_k^p = (\alpha_{k,1}^p, \alpha_{k,2}^p, \ldots, \alpha_{k,L}^p)$. Upon simplifying the above expression, we arrive at

$$\begin{aligned} p(\mathbf{y}_k|\mathbf{x}_k, \underline{\alpha}_k^s, \underline{\alpha}_k^p) &= \frac{1}{(2\pi L\sigma^2)} e^{-\frac{\left(y_k^s - x_k^s \sum_{l=1}^{L} \alpha_{k,l}^s\right)^2}{2L\sigma^2}} \\ & \qquad e^{-\frac{\left(y_k^p - x_k^p \sum_{l=1}^{L} \alpha_{k,l}^p\right)^2}{2L\sigma^2}}. \end{aligned} \qquad (14)$$

Discarding all the constant terms and terms which do not depend on the code symbols $\{\mathbf{x}_k\}$, and taking logarithm of both sides of Eqn. (14), we obtain

$$\log(p(\mathbf{y}_k|\mathbf{x}_k)) = \frac{2E_s}{LN_0}\left(\sum_{l=1}^{L} y_k^s x_k^s \alpha_{k,l}^s + \sum_{l=1}^{L} y_k^p x_k^p \alpha_{k,l}^p\right). \qquad (15)$$

Defining the quantity $\hat{L}_k$,

$$\hat{L}_k = \log\left(\frac{\text{Prob}(x_k^s = +1)}{\text{Prob}(x_k^s = -1)}\right), \qquad (16)$$

and discarding all the terms independent of x_k^s, we can calculate $\log(p(x_k^s))$ as [15]

$$\log(p(x_k^s)) = \frac{\hat{L}_k x_k^s}{2}. \qquad (17)$$

Combining the results of Eqns. (15) and (17) and substituting in Eqn. (12), we obtain

$$c_k(s,t) = \frac{\hat{L}_k x_k^s}{2} + \frac{2E_s}{LN_0}\left(\sum_{l=1}^{L} y_k^s x_k^s \alpha_{k,l}^s + \sum_{l=1}^{L} y_k^p x_k^p \alpha_{k,l}^p\right). \qquad (18)$$

The above quantity $c_k(s,t)$ can then be used in the computation of the forward and backward recursion metrics in the log-MAP algorithm [15].

IV. TURBO DECODER PERFORMANCE RESULTS

Simulations were performed using the proposed online estimator to provide $\widehat{\gamma}$ for the iterative decoding of turbo codes on flat Rayleigh fading channels ($m = 1$). We consider a rate-1/3 turbo code using two 16-state (constraint length = 5) recursive systematic code (RSC) encoders with generator $(21/37)_8$. A random turbo interleaver is employed. The number of information bits per frame is 5000. The transmitted symbols are corrupted by flat Rayleigh fading and AWGN. In this paper we restrict ourselves to i.i.d. Rayleigh fading. The number of decoding iterations is set to eight.

We consider the turbo decoder performance in the case of 2-branch diversity with equal gain combining. We evaluate the turbo decoder performance using our SNR estimate derived in Section II and compare it with the performance using perfect SNR and CSI. In the ideal case, where perfect knowledge of the channel SNR as well as the symbol-by-symbol fade amplitudes (CSI) are required, the metric derived in Section III (Eqn. (18)) is used. In the non-ideal case, however, since we are estimating only the channel SNR, we propose a simple sub-optimum decoder which uses only the estimated SNR and ignore the estimation of fade amplitudes. This decoder then essentially employs the AWGN channel metric, which is equivalent to setting the fade amplitudes to unity. The metric used in this case is the following:

$$c_k^{sub-opt}(s,t) = \frac{\hat{L}_k x_k^s}{2} + 2\widehat{\gamma}\left(y_k^s x_k^s + y_k^p x_k^p\right). \qquad (19)$$

Figure 2 shows the simulated BER performance plots for 2-branch Rayleigh fading with equal gain combining. Performance in AWGN as well as 1-branch Rayleigh fading (i.e., no diversity combining) are also plotted. Note that, in Fig. 2, the AWGN performance using the estimated SNR is obtained by letting $L = 1$ and $m \to \infty$ in Eqn. (5). Likewise, the performance in 1-branch Rayleigh fading is obtained by substituting $L = 1$ and $m = 1$ in Eqn. (5). From Fig. 2 we observe that, in the case of diversity combining, the performance using our SNR estimate is very close to the perfect side information case (to within 0.5 dB).

V. Conclusion

We proposed an online SNR estimation scheme for Nakagami-m fading channels with equal gain diversity combining. We derived SNR estimates based on the statistical ratio of certain observables over a block of received data. Our generalized Nakagami fading results provided the SNR estimates on AWGN and Rayleigh fading channels as special cases of the Nakagami parameter $m = \infty$ and $m = 1$, respectively. We applied our SNR estimation scheme to the iterative decoding of turbo codes on Rayleigh fading channels with equal gain combining. It was shown that the turbo decoder performance using our SNR estimates is quite close to the performance using perfect knowledge of the fade amplitudes and E_s/N_o (to within about 0.5 dB). In addition to its application in turbo decoding on fading channels, the proposed online SNR estimation scheme could as well be applied in other problems where knowledge of the fading channel SNR is necessary.

Appendix

A. Derivation of Eqn. (5)

Here, we derive the expressions for the numerator and the denominator of Eqn. (5). Removing the superscript for convenience, the denominator $E(v^4)$ is given by

$$E(v^4) = \sum_{i=1}^{L}\sum_{j=1}^{L}\sum_{k=1}^{L}\sum_{l=1}^{L} E(r_i r_j r_k r_l). \quad (20)$$

Substituting $r_i = \alpha_i X + n_i, r_j = \alpha_j X + n_j, r_k = \alpha_k X + n_k$ and $r_l = \alpha_l X + n_l$, we obtain

$$\begin{aligned} E(v^4) &= E_s^2 \sum_{i=1}^{L}\sum_{j=1}^{L}\sum_{k=1}^{L}\sum_{l=1}^{L} E(\alpha_i\alpha_j\alpha_k\alpha_l) + \\ &E_s \sum_{i=1}^{L}\sum_{j=1}^{L} E(\alpha_i\alpha_j) \sum_{k=1}^{L}\sum_{l=1}^{L} E(n_k n_l) + \\ &E_s \sum_{i=1}^{L}\sum_{k=1}^{L} E(\alpha_i\alpha_k) \sum_{j=1}^{L}\sum_{l=1}^{L} E(n_j n_l) + \\ &E_s \sum_{i=1}^{L}\sum_{l=1}^{L} E(\alpha_i\alpha_l) \sum_{j=1}^{L}\sum_{k=1}^{L} E(n_j n_k) + \\ &E_s \sum_{j=1}^{L}\sum_{k=1}^{L} E(\alpha_j\alpha_k) \sum_{i=1}^{L}\sum_{l=1}^{L} E(n_i n_l) + \\ &E_s \sum_{j=1}^{L}\sum_{l=1}^{L} E(\alpha_j\alpha_l) \sum_{i=1}^{L}\sum_{k=1}^{L} E(n_i n_k) + \\ &E_s \sum_{k=1}^{L}\sum_{l=1}^{L} E(\alpha_k\alpha_l) \sum_{i=1}^{L}\sum_{j=1}^{L} E(n_i n_j) + \\ &\sum_{i=1}^{L}\sum_{j=1}^{L}\sum_{k=1}^{L}\sum_{l=1}^{L} E(n_i n_j n_k n_l). \end{aligned} \quad (21)$$

Since α is Nakagami-m distributed with $E(\alpha^2) = 1$, we have $E(\alpha) = \frac{\Gamma(m+\frac{1}{2})}{\sqrt{m}\Gamma(m)}$. Assuming the α's are i.i.d, and the n's are i.i.d, and assuming these groups are independent, and also independent of X, we obtain

$$\sum_{p=1}^{L}\sum_{q=1}^{L} E(\alpha_p\alpha_q) = L + (L^2 - L)\left(\frac{\Gamma(m+\frac{1}{2})}{\sqrt{m}\Gamma(m)}\right)^2, \quad (22)$$

and

$$\sum_{p=1}^{L}\sum_{q=1}^{L} E(n_p n_q) = L\sigma^2. \quad (23)$$

Defining $\Delta_m = \sum_{i=1}^{L}\sum_{j=1}^{L}\sum_{k=1}^{L}\sum_{l=1}^{L} E(\alpha_i\alpha_j\alpha_k\alpha_l)$, the expression for Δ_m can be obtained as

$$\begin{aligned} \Delta_m &= \sum_{i=1}^{L}\sum_{j=1}^{L}\sum_{k=1}^{L}\sum_{l=1}^{L} E(\alpha_i\alpha_j\alpha_k\alpha_l) \\ &= E(\alpha_1 + \alpha_2 + \cdots + \alpha_L)^4. \end{aligned} \quad (24)$$

Applying the multinomial theorem to the above equation, we get

$$\begin{aligned} \Delta_m = &\binom{L}{1} E(\alpha^4) + \binom{L}{2}\frac{4!}{1!3!} E(\alpha^3)E(\alpha) + \binom{L}{2}\frac{4!}{2!2!}[E(\alpha^2)]^2 + \\ &\binom{L}{3}\frac{4!}{1!1!2!} 3E(\alpha^2)[E(\alpha)]^2 + \binom{L}{4}\frac{4!}{1!1!1!1!}[E(\alpha)]^4. \end{aligned} \quad (25)$$

Simplifying the above equation, we arrive at

$$\begin{aligned} \Delta_m = &LE(\alpha^4) + 4L(L-1)E(\alpha^3)E(\alpha) + 3L(L-1)[E(\alpha^2)]^2 + \\ &6L(L-1)(L-2)E(\alpha^2)[E(\alpha)]^2 + \\ &L(L-1)(L-2)(L-3)[E(\alpha)]^4, \end{aligned} \quad (26)$$

where $E(\alpha^k) = \frac{\Gamma(m+\frac{k}{2})}{m^{\frac{k}{2}}\Gamma(m)}$. In the case of Rayleigh fading, $E(\alpha) = \frac{\sqrt{\pi}}{2}$, $E(\alpha^2) = 1$, $E(\alpha^3) = \frac{3\sqrt{\pi}}{4}$, and $E(\alpha^4) = 2$. Substituting these values of expectations in Eqn. (26), we get Δ_1 as

$$\Delta_1 = 2L + L(L-1)\left\{3 + \frac{3(L-1)\pi}{2} + (L-2)(L-3)\frac{\pi^2}{16}\right\}. \quad (27)$$

Next, to compute $\sum_{i=1}^{L}\sum_{j=1}^{L}\sum_{k=1}^{L}\sum_{l=1}^{L} E(n_i n_j n_k n_l)$, we use the result of Eqn. (26) with α replaced by n. Also, by recalling that the odd moments of a Gaussian random variable n with zero mean and variance σ^2 are all zero, $E(n^2) = \sigma^2$ and $E(n^4) = 3\sigma^4$, we arrive at

$$\sum_{i=1}^{L}\sum_{j=1}^{L}\sum_{k=1}^{L}\sum_{l=1}^{L} E(n_i n_j n_k n_l) = 3L^2\sigma^4. \quad (28)$$

Combining Equations (22), (23), (26) and (28), we get $E(v^4)$ as

$$\begin{aligned} E(v^4) &= E_s^2\Delta_m + 3L^2\sigma^4 + \\ &\quad 6L\left[L + (L^2 - L)\left(\frac{\Gamma(m+\frac{1}{2})}{\sqrt{m}\Gamma(m)}\right)^2\right] E_s\sigma^2 \\ &= \sigma^4\{3L^2 + \frac{E_s^2}{\sigma^4}\Delta_m + \\ &\quad 6L\left[L + (L^2 - L)\left(\frac{\Gamma(m+\frac{1}{2})}{\sqrt{m}\Gamma(m)}\right)^2\right]\frac{E_s}{\sigma^2}\}. \end{aligned} \quad (29)$$

Similarly, $E(v^2)$ can be calculated as follows

$$\begin{aligned} E(v^2) &= \sum_{i=1}^{L}\sum_{j=1}^{L} E(r_i r_j) \\ &= \sum_{i=1}^{L}\sum_{j=1}^{L} E(\alpha_i\alpha_j X^2 + \alpha_i X n_j + \alpha_j X n_i + n_i n_j) \\ &= \left[L + (L^2 - L)\left(\frac{\Gamma(m+\frac{1}{2})}{\sqrt{m}\Gamma(m)}\right)^2\right] E_s + L\sigma^2 \\ &= \sigma^2\left\{L + \left[L + (L^2 - L)\left(\frac{\Gamma(m+\frac{1}{2})}{\sqrt{m}\Gamma(m)}\right)^2\right]\frac{E_s}{\sigma^2}\right\} \quad (30) \end{aligned}$$

Squaring Eqn. (30) and dividing it by Eqn. (29), and defining $\gamma = \frac{E_s}{2\sigma^2}$, we get the Eqn. (5). By substituting $m = 1$ in Eqn. (5), we get Eqn. (7).

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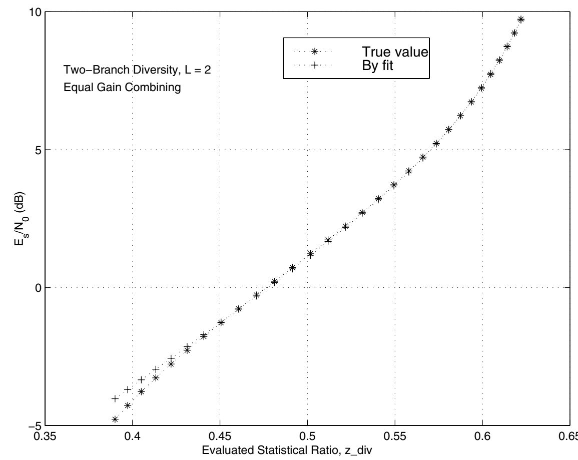

Fig. 1. γ versus z_{div} in Rayleigh fading ($m = 1$) with two-branch diversity and equal gain combining.

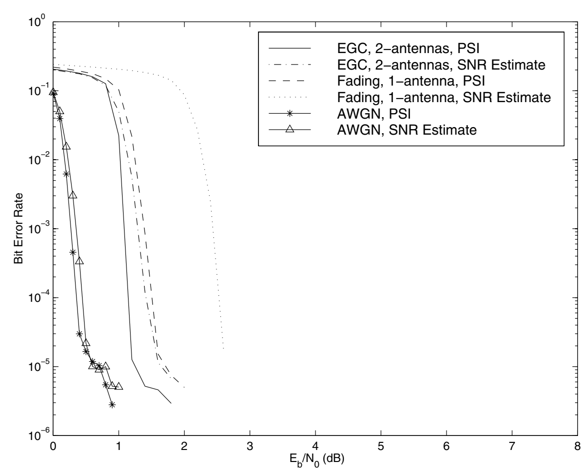

Fig. 2. Comparison of turbo decoder performance using perfect side information vs SNR estimates with and without diversity combining in Rayleigh fading. PSI refers to perfect knowledge of SNR and fade amplitudes.