Full-Diversity High-Rate Non-Coherent Unitary STBCs: New Designs and Performance

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***Abstract*— In this paper, we propose new non-coherent space-time block code (STBC) designs, which achieve full transmit diversity as well as high rates. We first present a search procedure to generate full-diversity non-coherent unitary space-time codebooks for any number of transmit antennas (n_t) and time slots ($T \geq 2n_t$), using radial coordinates of points lying on the surface of a sphere in $\mathbb{R}^{T-n_t}$. The search method generates codebooks having a specified minimum chordal distance, which allows flexibility in achieving coding gain. Further, we propose another unitary design using circulant matrices for any n_t, by restricting the search only to suitable constellation sets in $\mathbb{R}^2$ for the case of $T = 2n_t$. For the latter design, we present two different optimizations; i) to maximize the minimum chordal distance, and ii) to maximize the expected chordal distance. The proposed designs use an analytical design criterion which guarantees full diversity. The proposed designs are shown to outperform Jing and Hassibi codes reported previously in the literature.**

***Keywords* – Non-coherent unitary STBCs, full-diversity, chordal distance.**

I. INTRODUCTION

The capacity of wireless multiple-input-multiple-output (MIMO) channels can be very high in scattering-rich environments compared to a single-input-single-output (SISO) channels [1]. In non-coherent communication, detection is carried out without the knowledge of the channel gains at the receiver. In fast fading situations where we cannot afford to use time slots for channel estimation, or rely on inaccurate channel knowledge (noisy observation), non-coherent space-time codes can be considered.

From a capacity point of view, calculations for a MIMO block fading channel show that unitary space-time codes are optimal for non-coherent MIMO communication with i.i.d. Gaussian noise [2]. Also, for minimizing pairwise error probability, unitary codebooks are optimal [3]. In [4], the authors used Cayley transform to obtain unitary space-time codes. However, they have no control over transmit diversity. In [5], the authors designed codes for PSK constellations, ensuring full transmit diversity, while in [6], the authors used exponential transform and coherent codes to design unitary space-time codes which require lesser number of parameters to be optimized than in [4]. Their construction guarantees full transmit diversity. In [7], the authors proposed a training based scheme which employs coherent codes that ensures full transmit-diversity and good spectral efficiency. In [8], the authors obtained non-coherent unitary codebooks by solving an optimization problem that numerically maximized the minimum principal angle.

In this paper, we propose new non-coherent unitary STBCs. Our key contributions can be summarized as follows.

- We propose a search procedure to construct full-diversity unitary codebooks having a guaranteed minimum chordal distance for any number of transmit antennas (n_t) and time slots ($T \geq 2n_t$). We also give a method to increase the codebook size if the required size is not achieved by the search procedure.
- We further propose another unitary design using circulant matrices for any n_t, by restricting the search only to suitable constellation sets in $\mathbb{R}^2$ for the case of $T = 2n_t$. For this, we present two different optimizations; i) to maximize the minimum chordal distance, and ii) to maximize the expected chordal distance.
- The proposed designs use an analytical design criterion that guarantees full transmit diversity.
- Our simulation results for 2×4, 2×5, 3×6 and 3×7 codes show that the proposed designs outperform Jing and Hassibi codes in [4] by about 1.5 dB at a codeword error rate (CER) of 10^{-2}. Also, a comparison with a training-based scheme that employs 2×2 Golden code shows that the proposed design achieves better performance by about 1 dB at a CER of 10^{-2}.

Organization of the paper: The rest of the paper is organized as follows. In Section II, we present the non-coherent MIMO system model and necessary definitions. In Section III, we present the proposed code design based on chordal distance criteria (referred to as Design-I), and its CER performance results. In Section IV, we present the code design using circulant matrices (referred to as Design-II), and its CER performance results. Conclusions are given in Section V.

II. SYSTEM MODEL

Consider a MIMO channel with n_t transmit and n_r receive antennas. We assume a block fading channel with a coherence time of T channel uses. The complex valued received signal matrix of size $n_r \times T$ is given by

$$\mathbf{Y} = \mathbf{HX} + \mathbf{N}, \tag{1}$$

where $\mathbf{X}$ is the complex valued transmitted signal matrix of size $n_t \times T$, $\mathbf{H}$ is the complex valued channel matrix of size $n_r \times n_t$, whose entries are assumed to be i.i.d. $\mathcal{CN}(0, 1)$, and $\mathbf{N}$ is the complex valued noise matrix of size $n_r \times T$, whose entries are i.i.d. $\mathcal{CN}(0, \sigma^2)$. In the absence of the knowledge of channel coefficients at the receiver, the generalized likelihood ratio test (GLRT) estimate of $\mathbf{X}$ is given by [11],[12]

$$\begin{aligned} \widehat{\mathbf{X}} &= \arg \max_{\mathbf{X} \in \xi} \sup_{\mathbf{H}} p(\mathbf{Y}|\mathbf{X}, \mathbf{H}) \\ &= \arg \max_{\mathbf{X} \in \xi} \text{tr}(\mathbf{YX}^*\mathbf{XY}^*), \end{aligned} \tag{2}$$

where ξ indicates the unitary space-time codebook used, and $(.)^*$ denotes Hermitian operation. $\mathbf{X}$ must be of rank n_t so that the codewords are decodable using the GLRT metric.

This work was supported in part by the DRDO-IISc Program on Advanced Research in Mathematical Engineering.

Our focus in this paper is to design non-coherent unitary space-time codes.

Definition 1: [11] Let $\mathbf{X}$ and $\mathbf{Y}$ be two orthonormal bases of two different subspaces $\Omega_{\mathbf{X}}$ and $\Omega_{\mathbf{Y}}$ of dimension M. Let $\mathbf{XY}^* = \mathbf{U}_X \mathbf{C} \mathbf{U}_Y^*$, where $\mathbf{U}_X$ and $\mathbf{U}_Y$ are unitary matrices of dimension $M \times M$, and $\mathbf{C}$ is a diagonal matrix with $\mathbf{C}_{ii} = \cos\theta_i$, $\theta_i \in [0, \pi/2]$, $i = 1, 2, \cdots, M$. Then θ_i's are called the principal angles between the two subspaces.

Definition 2: [11] Let $\boldsymbol{\theta} = [\theta_1 \ \theta_2 \ \cdots \ \theta_M]$ be the principal angles between the two subspaces $\Omega_{\mathbf{X}}$ and $\Omega_{\mathbf{Y}}$ of dimension M. Then, $d_c(\Omega_{\mathbf{X}}, \Omega_{\mathbf{Y}}) = \sqrt{\sum_{i=1}^{M} \sin^2\theta_i}$ is called the chordal distance between the two given subspaces.

When the channel matrix $\mathbf{H}$ is unknown at the receiver, T must be $\geq n_t$ to have at least $n_r n_t$ observation equations. Also, when $\mathbf{H}$ is unknown, it has been shown in [5] that the order of transmit diversity that can be achieved is given by

$$r = \min_{\mathbf{X}_i, \mathbf{X}_j \in \xi} [\dim(\Omega_{\mathbf{X}_i}) - \dim(\Omega_{\mathbf{X}_i} \cap \Omega_{\mathbf{X}_j})], \ \forall \ i \neq j, \quad (3)$$

where the following bound gives the maximum achievable transmit diversity:

$$r \leq \min[n_t, T - n_t]. \quad (4)$$

From (4), it is seen that T must be $\geq 2N_t$ to achieve full transmit diversity of $r = n_t$. Also, it has been shown that the r in (3) corresponds to the minimum number of non-unity diagonal values of $\mathbf{C}$ over all possible $\mathbf{X}_i, \mathbf{X}_j, i \neq j$ code word pairs (Definition 1). As in [13], we will use the chordal distance based criterion to obtain new non-coherent code designs. While the code design in [13] does not have control over the achievable transmit diversity, our codes proposed in Sec. III and IV achieve full transmit diversity.

III. Proposed Codebook Design-I

In this section, we present a search algorithm to generate codebooks that guarantee full transmit diversity. In (3), the transmit diversity r is maximized when the subspaces of $\mathbf{X}_i$ and $\mathbf{X}_j$ do not intersect $\forall \ i \neq j$. We develop a code construction which satisfies this condition of non-intersection between subspaces to achieve full transmit diversity.

A. Achieving Full Transmit Diversity

Let us consider the design of a $n_t \times T$ code, where $T \geq 2n_t$. Consider $n_t = 2$. Let $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \in \mathbb{R}^2$ be linearly independent (LI) over $\mathbb{R}$. Let $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in \mathbb{R}^{T-2}$ such that $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ are LI in $\mathbb{R}^{T-2}$, and hence in $\mathbb{C}^{T-2}$. Let $\boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4$ be another such pair. Now, let

$$\widetilde{\mathbf{X}}_1 = \begin{bmatrix} \boldsymbol{\beta}_1 & \boldsymbol{\alpha}_1 \\ \boldsymbol{\beta}_2 & \boldsymbol{\alpha}_2 \end{bmatrix}, \quad (5)$$

and

$$\widetilde{\mathbf{X}}_2 = \begin{bmatrix} \boldsymbol{\beta}_1 & \boldsymbol{\alpha}_3 \\ \boldsymbol{\beta}_2 & \boldsymbol{\alpha}_4 \end{bmatrix}. \quad (6)$$

Since $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ are assumed to be LI in $\mathbb{R}^2$, $\widetilde{\mathbf{X}}_i$'s are of rank $n_t (= 2)$, and hence GLRT decodable. The intersection of the row space of $\widetilde{\mathbf{X}}_1, \widetilde{\mathbf{X}}_2$ in $\mathbb{C}^T$ consists of elements satisfying

$$[a\boldsymbol{\beta}_1 + b\boldsymbol{\beta}_2 \quad a\boldsymbol{\alpha}_1 + b\boldsymbol{\alpha}_2] = [a'\boldsymbol{\beta}_1 + b'\boldsymbol{\beta}_2 \quad a'\boldsymbol{\alpha}_3 + b'\boldsymbol{\alpha}_4], \quad (7)$$

for some $a, b, a', b' \in \mathbb{C}$. Also, because of the LI between $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$,

$$(a - a')\boldsymbol{\beta}_1 + (b - b')\boldsymbol{\beta}_2 = \mathbf{0}, \quad \Rightarrow a = a', \ \ b = b'. \quad (8)$$

From (7) and (8), $a\boldsymbol{\alpha}_1 + b\boldsymbol{\alpha}_2 = a\boldsymbol{\alpha}_3 + b\boldsymbol{\alpha}_4$, which implies

$$a(\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_3) + b(\boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_4) = \mathbf{0}. \quad (9)$$

Now, if $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4$ be so chosen such that $\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_4$ are LI, then $a = b = 0$. This implies that the row spans of $\widetilde{\mathbf{X}}_1$ and $\widetilde{\mathbf{X}}_2$ (of dimension 2) do not intersect, and hence, from (3), codes constructed in the form (5) and (6) will achieve full transmit diversity (of 2). It can be shown that this argument can be extended for any n_t and $T \geq 2n_t$. The performance of the codes can further be improved in terms of coding gain by appropriate choice of $\boldsymbol{\alpha}$'s and $\boldsymbol{\beta}$'s. We propose to use chordal distance measure for this purpose, which is described next.

B. Choice of $\boldsymbol{\alpha}$'s and $\boldsymbol{\beta}$'s Based on Chordal Distance

For a $n_t \times T$ code design, for each code word, we need to choose the $\boldsymbol{\alpha}$ vectors; i.e., for the codeword indexed by k, choose $\boldsymbol{\alpha}_i^{(k)} = [\alpha_{i,1}^{(k)} \ \alpha_{i,2}^{(k)} \ \cdots \ \alpha_{i,T-n_t}^{(k)}]$, for $i = 1, 2, \cdots, n_t$ and $k = 1, 2, \cdots, K$, where K is the number of codewords. Also, the $\boldsymbol{\beta}$ vectors need to be chosen; i.e., choose $\boldsymbol{\beta}_i = [\beta_{i,1} \ \beta_{i,2} \ \cdots \ \beta_{i,n_t}]$, for $i = 1, 2, \cdots, n_t$ such that $\boldsymbol{\beta}_i$'s are LI. The $\boldsymbol{\alpha}$ vectors are chosen to be the Cartesian coordinates of a point lying on the surface of a sphere in $\mathbb{R}^{T-n_t}$, as follows:

$$\begin{aligned}
\alpha_{i,1}^{(k)} &= \sin(\phi_{i,1}^{(k)}) \\
\alpha_{i,2}^{(k)} &= \cos(\phi_{i,1}^{(k)}) \sin(\phi_{i,2}^{(k)}) \\
\alpha_{i,3}^{(k)} &= \cos(\phi_{i,1}^{(k)} \cos(\phi_{i,2}^{(k)}) \sin(\phi_{i,3}^{(k)}) \\
&\vdots \\
\alpha_{i,T-n_t-1}^{(k)} &= \cos(\phi_{i,1}^{(k)}) \cdots \cos(\phi_{i,T-n_t-2}^{(k)}) \sin(\phi_{i,T-n_t-1}^{(k)}) \\
\alpha_{i,T-n_t}^{(k)} &= \cos(\phi_{i,1}^{(k)}) \cdots \cos(\phi_{i,T-n_t-2)}^{(k)} \cos(\phi_{i,T-n_t-1}^{(k)}),
\end{aligned} \quad (10)$$

where $\phi_{i,n}^{(k)}$ is

$$\phi_{i,n}^{(k)} = \gamma_{i,n} + k \, \Delta_n^{(k)}, \quad (11)$$

where $\gamma_{i,n}$ is an initial angle and $\Delta_n^{(k)}$ is an angular increment. To get the first code word, we assign $\Delta_n^{(1)} = 0$ and choose $\gamma_{i,n}$ so that $\{\alpha_1^{(1)}, \alpha_2^{(1)}, \cdots, \alpha_{n_t}^{(1)}\}$ is a subset of the ordered basis of $\mathbb{R}^{T-n_t}$. Construct

$$\widetilde{\mathbf{X}}^{(1)} = \begin{bmatrix} \boldsymbol{\beta}_1 & \boldsymbol{\alpha}_1^{(1)} \\ \boldsymbol{\beta}_2 & \boldsymbol{\alpha}_2^{(1)} \\ \vdots & \vdots \\ \boldsymbol{\beta}_{n_t} & \boldsymbol{\alpha}_{n_t}^{(1)} \end{bmatrix}. \quad (12)$$

The first codeword is obtained as $\mathbf{X}_1 = (\widetilde{\mathbf{X}}^{(1)}(\widetilde{\mathbf{X}}^{(1)})^*)^{\frac{-1}{2}}\widetilde{\mathbf{X}}^{(1)}$. For generating the kth codeword rotate each element of the set $\{\boldsymbol{\alpha}_1^{(k-1)}, \boldsymbol{\alpha}_2^{(k-1)}, \cdots, \boldsymbol{\alpha}_{n_t}^{(k-1)}\}$ by an amount $\Delta_n^{(k)}$ to obtain $\{\boldsymbol{\alpha}_1^{(k)}, \boldsymbol{\alpha}_2^{(k)}, \cdots, \boldsymbol{\alpha}_{n_t}^{(k)}\}$. Form the matrix

$$\mathbf{X}' = \begin{bmatrix} \boldsymbol{\beta}_1 & \boldsymbol{\alpha}_1^{(k)} \\ \boldsymbol{\beta}_2 & \boldsymbol{\alpha}_2^{(k)} \\ \vdots & \vdots \\ \boldsymbol{\beta}_{n_t} & \boldsymbol{\alpha}_{n_t}^{(k)} \end{bmatrix}, \quad (13)$$

from which form the matrix $\mathbf{X}^{''} = (\mathbf{X}^{'}(\mathbf{X}^{'})^{*})^{-\frac{1}{2}}\mathbf{X}^{'}$. For $j = 1$ to $k-1$, compute $\mathbf{X}^{''}\mathbf{X}_j^{*}$ and obtain the SVD of the resulting matrix as $\mathbf{U}_j\mathbf{C}_j\mathbf{V}_j^{*}$. If $\forall i = 1, \cdots, n_t$ and $\forall j = 1, \cdots, k-1$,

$$(\mathbf{C}_j)_{ii} \quad < \quad 1, \tag{14}$$

and

$$\sqrt{\sum_{i=1}^{M} 1 - (\mathbf{C}_j)_{ii}^2} \quad \geq \quad d_{min}, \tag{15}$$

admit $\mathbf{X}^{''}$ as k^{th} codeword in the codebook[1]. Note that if $\widetilde{\mathbf{X}}$ is of full rank, all the eigen values of $\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}^{*}$ are strictly real and positive. Hence, $(\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}^{*})^{-\frac{1}{2}}$ exists and $\mathbf{X} = (\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}^{*})^{-\frac{1}{2}}\widetilde{\mathbf{X}}$ has the same row space as $\widetilde{\mathbf{X}}$, but now the rows $\mathbf{X}$ are orthonormal. This unitary map not only gives the capacity achieving structure but also spreads the information across all the time slots which is not the case in training based schemes. It is possible that the proposed construction method does not give the required codebook size. We address this issue in the following section.

C. Codebook Size

If the codebook size ($= K$) has still to be increased further, set $\Delta_n^{(k)} = 0$ and choose appropriate $\gamma_{i,n}$ so that $\{\alpha_1^{(k)}, \cdots, \alpha_{n_t}^{(k)}\}$ is another subset of the ordered basis of $\mathbb{R}^{T-n_t}$ for $k = K+1$. Repeat the construction described in section III-B starting with the permuted row positions of $\alpha_1^{(k)}, \cdots, \alpha_{n_t}^{(k)}$ for $k = K+1$ and vary $\phi_{i,n}^{(k)}$ for $k > K+1$. If all subsets of the set of ordered basis and all permutations are exhausted and the codebook size has still to be increased, then $\forall k = 1, \cdots, K$ and $\forall i = 1, \cdots, n_t$, we scale $\alpha_i^{(k)}$ used in forming the code words by suitable scalars $c_i^{(k)} \in \mathbb{R}$ and repeat the construction in III-B. The logic behind the scaling is as follows. If for each $k = 1$ to K and some j, $1 \leq j \leq K$, $k \neq j$,

$$\{\boldsymbol{\alpha}_1^{(k)} - \boldsymbol{\alpha}_1^{(j)}, \cdots, \boldsymbol{\alpha}_{(T-n_t)}^{(k)} - \boldsymbol{\alpha}_{(T-n_t)}^{(j)}\} \in \mathbb{R}^{T-n_t} \tag{16}$$

are LI, then it amounts to say that the determinant

$$\begin{vmatrix} \boldsymbol{\alpha}_1^{(k)} - \boldsymbol{\alpha}_1^{(j)} \\ \vdots \\ \boldsymbol{\alpha}_{(T-n_t)}^{(k)} - \boldsymbol{\alpha}_{(T-n_t)}^{(j)} \end{vmatrix} \neq 0. \tag{17}$$

Now, consider the determinant in real variables $c_1^{(j)}, \cdots, c_{T-nt}^{(j)}$

$$\begin{vmatrix} \boldsymbol{\alpha}_1^{(k)} - c_1^{(j)}\boldsymbol{\alpha}_1^{(j)} \\ \vdots \\ \boldsymbol{\alpha}_{(T-n_t)}^{(k)} - c_{T-n_t}^{(j)}\boldsymbol{\alpha}_{(T-n_t)}^{(j)} \end{vmatrix}. \tag{18}$$

Determinant (18) is not equal to zero for

$$c_1^{(j)} = \cdots = c_{T-n_t}^{(j)} = 1, \tag{19}$$

for each $k = 1, \cdots, K, k \neq j$. Now, for each k, the determinant (18) is a polynomial function in reals $c_1^{(j)}, \cdots, c_{T-n_t}^{(j)}$, say f_k. f_k is continuous in $c_i^{(j)}$'s for each $k = 1, \cdots, K$. Since, for $c_1^{(j)} = \cdots = c_{T-n_t}^{(j)} = 1$, $f_k \neq 0$ for each $k = 1, \cdots, K$, $k \neq j$, there exists however small an $\epsilon_k > 0, \sigma_k > 0, \ \epsilon_k, \sigma_k \in \mathbb{R}$ such that if

$$\sum_{i=1}^{T-n_t} |1 - c_i^{(j)}|^2 \quad < \quad \epsilon_k, \tag{20}$$

then, $\left|f_k(c_1^{(j)}, \cdots, c_{T-n_t}^{(j)}) - f_k(1, \cdots, 1)\right| < \sigma_k$. Let $\epsilon = \min(\epsilon_1, \cdots, \epsilon_{j-1}, \epsilon_{j+1}, \cdots \epsilon_K)$. Whenever $c_1^{(j)}, \cdots, c_{T-n_t}^{(j)}$ are chosen such that

$$\sum_{i=1}^{T-n_t} |1 - c_i^{(j)}|^2 \quad < \quad \epsilon, \tag{21}$$

then, $\forall k = 1, \cdots, K, k \neq j, |f_k(c_1^{(j)}, ..., c_{T-n_t}^{(j)}) - f_k(1, ..., 1)| < \sigma_k$, i.e., all $K-1$ inequalities hold simultaneously. So, there exists suitable values of $c_1^{(j)}, \cdots, c_{T-n_t}^{(j)} \neq 1$ for which the determinant given by (18) can still be guaranteed to be non-zero $\forall k = 1, \cdots, K, k \neq j$. If we impose further the restriction that for each k, $\{\boldsymbol{\alpha}_1^{(k)}, \cdots, \boldsymbol{\alpha}_{(T-n_t)}^{(k)}\}$ are LI, then we have the determinant (18) not equal to zero $\forall k = 1, \cdots, K$. Hence, the subset $\{\boldsymbol{\alpha}_1^{(k)} - c_1^{(j)}\boldsymbol{\alpha}_1^{(j)}, \cdots, \boldsymbol{\alpha}_{n_t}^{(k)} - c_{n_t}^{(j)}\boldsymbol{\alpha}_{n_t}^{(j)}\}$ of $\{\boldsymbol{\alpha}_1^{(k)} - c_1^{(j)}\boldsymbol{\alpha}_1^{(j)}, \cdots, \boldsymbol{\alpha}_{T-n_t}^{(k)} - c_{T-n_t}^{(j)}\boldsymbol{\alpha}_{T-n_t}^{(j)}\}$ can be made LI by proper choice of $c_i^{(k)}$. So, for each $k = 1, \cdots, K$, repeat the construction with the set $\{c_1^{(k)}\boldsymbol{\alpha}_1^{(k)}, \cdots, c_{nt}^{(k)}\boldsymbol{\alpha}_{nt}^{(k)}\}$ to get the required codebook size. This also explains the reason for the choice $\boldsymbol{\alpha}_i$'s to be LI in (5) and (6).

To ease the process of getting n_t LI vectors, we begin the construction with a n_t-size subset of the ordered basis of $\mathbb{R}^{T-n_t}$ and rotate $\alpha_i^{(k)}$'s by small appropriate $\Delta_n^{(k)}$ so that for each k, $\alpha_i^{(k)}$'s are LI. The choice of suitable $\Delta_n^{(k)}$ can be argued out similar to the treatment for choice of $c_i^{(k)}$, since $\alpha_i^{(k)}$'s are functions of $\phi_{i,n}^{(k)}$'s.

D. Simulation Results and Discussion

We evaluated the CER performance of the proposed codebook Design-I through simulations. Figure 1 shows the CER results for the proposed Design-I for 2×5 and 3×7 codes (i.e., $n_t = 2, 3$ and $T = 5, 7$) with $n_r = 1$ at 1 bit/channel use spectral efficiency. The codebooks are obtained by computer search using the code search parameters listed in Table I. The SNR plotted is $\frac{\mathbb{E}||\mathbf{HX}||^2}{\mathbb{E}||\mathbf{N}||^2}$. For comparison purposes, we have also plotted the performance of 2×5 and 3×7 codes of Jing and Hassibi in [4][2].

From the CER plots in Fig. 1, we observe that the proposed codes perform better compared to Jing and Hassibi codes by about 1.5 dB at a CER of 10^{-2}. This better performance is due to the flexibility in achieving coding gains in the proposed design, where the angular increment $\Delta_n^{(k)}$ along with the values of $c_i^{(k)}$ are used to control spacing d_{min} between the codewords. We further note that, looking at the plots at high SNRs in Fig. 1, it might appear that Jing and Hassibi codes have a tendency for a sharper bend at high SNRs,

[1]It is noted that, for Design-I, (14) is the condition for full-diversity and (15) controls the coding gain.

[2]While we have simulated the CER results for our codes, we did not simulate Jing and Hassibi codes. We have simply plotted the CER points available in Figs. 3 and 4 in their paper [4]. This is because Jing and Hassibi codes are not explicitly given in [4]. Even if we had obtained their codes by their search procedure, we would not be sure if the codes we obtain are same as those obtained by them. However, we have ensured that our SNR definition used in the figures is same as that in [4].

Parameters	2×5 code	3×7 code
β vectors	$\beta_1 = [1\ 0]$, $\beta_2 = [0\ 1]$	$\beta_1 = [1\ 0\ 0]$, $\beta_2 = [0\ 1\ 0]$, $\beta_3 = [0\ 0\ 1]$
$\Delta_n^{(k)}$ range	0.489 to 0.51	0.55 to 0.75
$c_i^{(k)}$ range	-4 to 4	-4 to 4
d_{min}	0.52	0.754

TABLE I

CODE SEARCH PARAMETERS FOR GENERATING THE CODEBOOKS FOR 2×5 AND 3×7 DESIGN-I CODES.

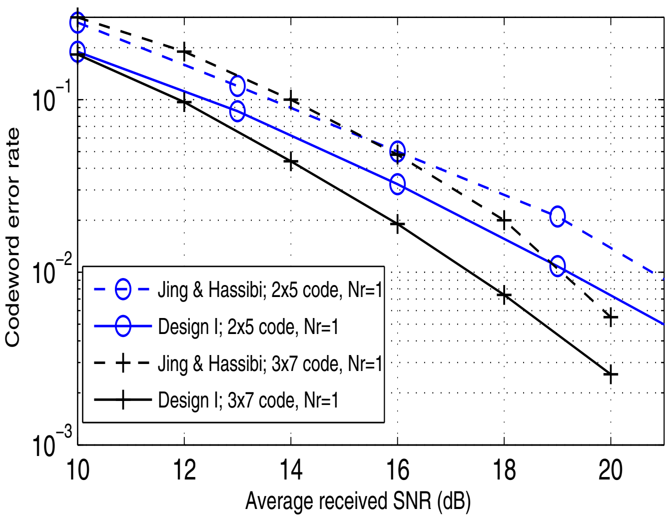

Fig. 1. Codeword error rate performance of the proposed Design-I 2×5 and 3×7 codes. $n_r = 1$.

and hence a higher diversity. We, however, clarify that i) we have analytically established that our codes achieve full-diversity (see full-diversity criteria in Sec. III-A; Eqn. (14) ensures full-diversity), ii) we have measured the slopes of our CER curves at high SNRs and verified that our codes indeed achieve the full diversity of 2 and 3 for $n_t = 2$ and 3, and iii) Jing and Hassibi codes at best can achieve full diversity, implying that their curves can at best run parallel to our curves at high SNRs. The 1.5 dB better performance achieved by the proposed codes at low SNRs is quite attractive, considering that these are the SNRs/error rates at which the vertical fall of outer turbo codes typically occur.

IV. PROPOSED CODEBOOK DESIGN-II

In this section, we present code designs restricting the search to suitable constellation sets in $\mathbb{R}^2$ (which simplifies the code search compared to Design-I). Let

$$\widetilde{\mathbf{X}} = [\ \mathbf{B}\ \ \mathbf{A}\ ], \ \mathbf{B} = \begin{bmatrix} \boldsymbol{\beta}_1 \\ \vdots \\ \boldsymbol{\beta}_{n_t} \end{bmatrix} \qquad (22)$$

where $\boldsymbol{\beta}_1, \cdots, \boldsymbol{\beta}_{n_t}$ are n_t LI row vectors in $\mathbb{C}^{n_t}$, and

$$\mathbf{A} = \begin{bmatrix} x_1 & x_2 & \cdots & x_{n_t} \\ x_{n_t} & x_1 & \cdots & x_{n_t-1} \\ \vdots & & \ddots & \\ x_2 & \cdots & x_{n_t} & x_1 \end{bmatrix}. \qquad (23)$$

is a circulant matrix. x_i's take values from a constellation set. Thus the unitary design is $\mathbf{X} = (\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}^*)^{-\frac{1}{2}}\widetilde{\mathbf{X}}$. To increase the number of symbols sent per channel use, we replace each x_i in $\mathbf{A}$ by $\sum_{k=n(i-1)+1}^{ni} x_k$. We address the issue of choosing suitable constellation sets for getting full diversity for the case of $T = 2n_t$ in the following section.

A. Optimum Choice of B and Constellation Sets

For the case $T = 2n_t$, we exploit the features of $\mathbf{A}$ of the form (23) to derive conditions for getting full diversity and optimize the choice of $\mathbf{B}$ to maximize d_{min} or the expected value of the chordal distance square ($\mathbb{E}d^2$). Let $\mathbf{B}$ be a fixed $n_t \times n_t$ invertible circulant matrix. The eigen values of $\mathbf{A}$ are

$$l_k = \sum_{j=1}^{n_t} \omega_{n_t}^{(j-1)(k-1)} x_j \ , \ k = 1, \cdots, n_t \qquad (24)$$

and $\omega_{n_t} = e^{\frac{i2\pi}{n_t}}$ [10]. Then, $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^*$, where $\mathbf{D}$ is a diagonal matrix (i.e., $\mathbf{D}_{ii} = l_i$) and $\mathbf{U}$ is the $n_t \times n_t$ DFT matrix normalized by $\sqrt{n_t}$. Similarly, $\mathbf{B} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^*$, where $\mathbf{\Lambda}$ is another diagonal matrix with $\mathbf{\Lambda}_{ii} = \lambda_i$, where λ_i are the eigen values of $\mathbf{B}$. Since $\mathbf{B}$ is full rank, it can be shown that

$$\left(\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}^*\right)^{-\frac{1}{2}} = \mathbf{U}[\ \mathbf{\Lambda}\mathbf{\Lambda}^* + \mathbf{D}\mathbf{D}^*\ ]^{-\frac{1}{2}}\mathbf{U}^*. \qquad (25)$$

Then, $\mathbf{X} = (\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}^*)^{-\frac{1}{2}}\widetilde{\mathbf{X}}$ simplifies to

$$\mathbf{X} = [\ \mathbf{C}_1 \ \ \mathbf{C}_2\ ], \qquad (26)$$

where $\mathbf{C}_1, \mathbf{C}_2$ are both circulant matrices given by

$$\mathbf{C}_1 = \mathbf{U}[\ \mathbf{\Lambda}\mathbf{\Lambda}^* + \mathbf{D}\mathbf{D}^*\ ]^{\frac{-1}{2}}\mathbf{\Lambda}\mathbf{U}^*, \qquad (27)$$

$$\mathbf{C}_2 = \mathbf{U}[\ \mathbf{\Lambda}\mathbf{\Lambda}^* + \mathbf{D}\mathbf{D}^*\ ]^{\frac{-1}{2}}\mathbf{D}\mathbf{U}^*. \qquad (28)$$

Using the arguments in Sec.III-A, for a given choice of invertible $\mathbf{B}$, it can be shown that the full diversity criteria in this case amounts to choosing constellation points such that

$$l_{k,i} \neq l_{m,i}, \qquad (29)$$

$\forall i = 1, \cdots, n_t$, $k \neq m$, and $k, m = 1, \cdots, |\xi|$, where $|\xi|$ is the codebook size. Let $\mathbf{C}$ denote the diagonal matrix obtained by SVD decomposition of $\mathbf{X}_k\mathbf{X}_m^*\mathbf{X}_m\mathbf{X}_k^*$. It can be shown that

$$\mathbf{C}_{ii} = \frac{|\lambda_i|^4 + 2|\lambda_i|^2\Re(l_{k,i}l_{m,i}^*) + |l_{k,i}|^2|l_{m,i}|^2}{|\lambda_i|^4 + |\lambda_i|^2(|l_{k,i}|^2 + |l_{m,i}|^2) + |l_{k,i}|^2|l_{m,i}|^2}. \qquad (30)$$

The chordal distance squared between codewords $\mathbf{X}_k$, $\mathbf{X}_m$ is

$$d_{k,m}^2 = \mathrm{tr}\left(\mathbf{I} - \mathbf{X}_k\mathbf{X}_m^*\mathbf{X}_m\mathbf{X}_k^*\right). \qquad (31)$$

The above equality can be shown to be reduced to

$$d_{k,m}^2 = \sum_{i=1}^{n_t} \frac{|\lambda_i|^2|l_{k,i} - l_{m,i}|^2}{(|\lambda_i|^2 + |l_{k,i}|^2)(|\lambda_i|^2 + |l_{m,i}|^2)}. \qquad (32)$$

For a given choice of constellation sets, for maximizing d_{min}^2, we numerically search for that $|\lambda_i|_{opt}^2 \in \mathbb{R}$ that maximizes the minimum of ith term of (32) over all possible codeword pairs. This process is repeated for each $i = 1, 2, \cdots, n_t$.

It may be noted that in maximizing d_{min}, we might decrease the distance between other codeword pairs severely, affecting the overall distance distribution of the codebook. So, we try to maximize the expected chordal distance squared of the codebook. We define our objective function averaged over all codeword pairs as

$$\mathbb{E}d^2 = \frac{\sum_{k=1}^{|\xi|}\sum_{m=k+1}^{|\xi|} d_{k,m}^2}{|\xi|(|\xi| - 1)} \qquad (33)$$

Setting $\frac{\partial \mathbb{E}d^2}{\partial |\lambda_i|^2} = 0$, we get

$$\sum_{k=1}^{|\xi|}\sum_{m=k+1}^{|\xi|} \frac{|l_{k,i} - l_{m,i}|^2(|l_{k,i}|^2|l_{m,i}|^2 - |\lambda_i|^4)}{(|\lambda_i|^2 + |l_{k,i}|^2)^2(|\lambda_i|^2 + |l_{m,i}|^2)^2} = 0 \qquad (34)$$

which can be solved numerically for each $i = 1, \cdots, n_t$ separately to get $|\lambda_i|^2_{opt}$. It is noted that this criteria is different from that in [4], where the expected log determinant of the difference between matrix pairs is maximized. In either of our two objective criteria, an n_t dimensional optimization problem is reduced into n_t independent 1-dimensional problems. Also, if all $|\lambda_i|_{opt}$'s are same, we reduce to the special case of scaled identity matrix. Hence, for the $T = 2n_t$ case, the choice of an optimum invertible circulant matrix can improve $\mathbb{E}d^2$ compared to using identity matrix (as used in [5]).

Choice of Constellation Sets: The kth row of a $n_t \times n_t$ DFT matrix consists of ath roots of unity, where $a = \frac{n_t}{gcd(n_t,k)}$. So the entries of the kth row of the DFT matrix belong to $\mathbb{Q}(\omega_a)$. If we let each x_j take values from $a_j\mathbb{Z}[i](a_j\mathbb{Z}[i] = \{a_j z : z \in \mathbb{Z}[i]\})$, i.e., QAM constellations scaled by a_j (for each $j = 1, \cdots, n_t$) such that $a_j \notin \mathbb{Q}(\omega_{n_t}, i)$ are irrational numbers that are $\mathbb{Q}$ LI, then the jth term in the summation of l_k in (24) belongs to $a_j\mathbb{Q}(w_a, i)$. This implies that $l_{k,i} - l_{m,i} \neq 0, \quad \forall k, m = 1, \cdots, |\xi|; k \neq m, \forall i = 1, \cdots, n_t$. Thus, there exists scaled QAM constellations for which Design-II can guarantee full diversity. A similar argument can be made for PSK constellation to ensure full diversity. Further, after scaling the constellations by appropriate irrational numbers, we propose to rotate/scale each constellation by suitable angles/(numbers $\in \mathbb{Q}$). $\mathbf{B}$ is computed using either (34) or d_{min} criteria and their respective $\mathbb{E}d^2$ or d_{min} is calculated. The optimum angle and rational number for each constellation is chosen by computer search for which $\mathbb{E}d^2$ or d_{min} is maximum.

B. Simulation results and discussion

We obtained 2×4 and 3×6 Design-II codes, where $\mathbf{B}$ is optimized to maximize $\mathbb{E}d^2$. In Fig. 2, we present the CER performance of the proposed Design-II for 3×6 code at a spectral efficiency of, $\eta = 1$ bit/channel use. The matrix $\mathbf{A}$ for this 3×6 code is

$$\mathbf{A} = \begin{bmatrix} x_1 + x_2 & x_3 + x_4 & x_5 + x_6 \\ x_5 + x_6 & x_1 + x_2 & x_3 + x_4 \\ x_3 + x_4 & x_5 + x_6 & x_1 + x_2 \end{bmatrix}, \quad (35)$$

where $x_1 \in \{\pm\frac{\pi}{3\sqrt{2}}e^{\mathbf{j}\frac{\pi}{4}}\}$, $x_2 \in \{\pm\frac{\mathbf{j}\pi}{3\sqrt{2}}e^{\mathbf{j}\frac{\pi}{4}}\}$, $x_3 \in \{\pm\frac{e^{0.25}}{\sqrt{2}}\}$, $x_4 \in \{\pm\frac{\mathbf{j}e^{0.25}}{\sqrt{2}}\}$, $x_5 \in \{\pm e^{\mathbf{j}\frac{\pi}{4}}\}$, $x_6 \in \{\pm \mathbf{j}e^{i\frac{\pi}{4}}\}$ and $\mathbf{j} = \sqrt{-1}$. We have also plotted the performance of Jing and Hassibi's 3×7 code for the same $\eta = 1$. We observe that proposed 3×6 Design-II code performs better (by about 2 dB at a CER of 10^{-2}) compared to Jing and Hassibi's 3×7 code. We present a similar comparison for 2×4 codes. For the proposed 2×4 Design-II code, $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} x_1 + x_2 & x_3 + x_4 \\ x_3 + x_4 & x_1 + x_2 \end{bmatrix}, \quad (36)$$

$x_1 \in \{\pm\frac{\pi}{3.3}e^{\mathbf{j}\frac{\pi}{18}}, \pm\frac{\pi}{3.3}e^{\mathbf{j}\frac{\pi}{2}+\mathbf{j}\frac{\pi}{18}}\}$, $x_2 \in \{\pm\frac{\sqrt{2}}{2.8}e^{\mathbf{j}\frac{\pi}{36}}, \pm\frac{\sqrt{2}}{2.8}e^{\mathbf{j}\frac{\pi}{2}+\mathbf{j}\frac{\pi}{36}}\}$, $x_3 \in \{\pm e^{-0.06}e^{\mathbf{j}\frac{\pi}{3}}, \pm e^{-0.06}e^{\mathbf{j}\frac{\pi}{2}+\mathbf{j}\frac{\pi}{3}}\}$, $x_4 \in \{\pm 0.5e^{\mathbf{j}1.17}, \pm 0.5e^{\mathbf{j}\frac{\pi}{2}+\mathbf{j}1.17}\}$.
With 2×4 codes also, the proposed Design-II codes perform better than Jing and Hassibi's codes.

We also compare the performance of the proposed 2×4 Design-II code with a training based scheme [7] using Golden code at $\eta = 1$. The CER performance of the training scheme, where the channel is MMSE estimated using an 2×2 orthogonal pilot matrix, followed by ML decoding of a 2×2 Golden code matrix (BPSK modulated) is plotted in Fig. 2. The pilot matrix as well as the data matrix have the same power. For the

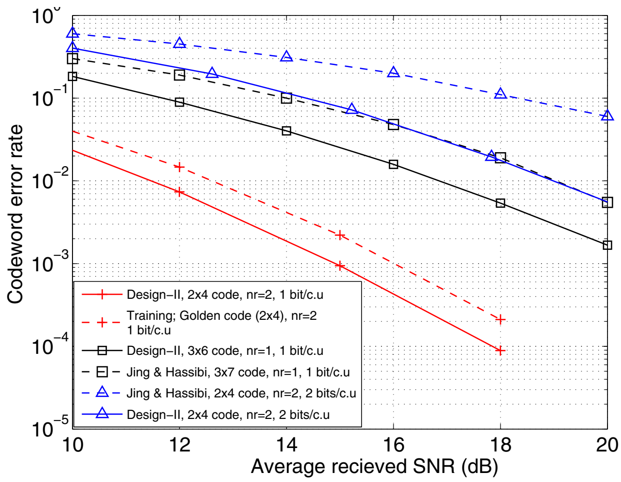

Fig. 2. Codeword error rate performance for 3×6 and and 2×4 Design-II codes. $n_r = 1, 2$.

Design-II code, $\mathbf{A}$ is given by (36), where $x_1 \in \{\pm\frac{\pi}{3\sqrt{2}}e^{\mathbf{j}\frac{\pi}{4}}\}$, $x_2 \in \{\pm\frac{\mathbf{j}\pi}{3\sqrt{2}}e^{\mathbf{j}\frac{\pi}{4}}\}$, $x_3 \in \{\pm\frac{e^{-0.05}}{\sqrt{2}}\}$, $x_4 \in \{\pm\frac{\mathbf{j}e^{-0.05}}{\sqrt{2}}\}$. The proposed 2×4 Design-II code is seen to outperform the training based scheme using Golden code by about 1 dB at 10^{-3} CER.

V. Conclusions

We presented two new non-coherent STBC designs (Designs-I and II) which achieved full diversity and high rates. For Design-I, a search technique generated full-diversity unitary STBCs whose chordal distances can be chosen to get good coding gains. The proposed algorithm can be useful in generating small-sized codebooks. Design-II simplified the code search space by restricting the constellation sets to $\mathbb{R}^2$. Our proposed designs showed improved performance than existing designs for the considered sizes of codes.

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