BER Analysis of Uplink OFDMA in the Presence of Carrier Frequency and Timing Offsets

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Abstract— **In uplink orthogonal frequency division multiple access (OFDMA), carrier frequency offsets (CFO) and/or timing offsets (TO) of other users with respect to a desired user can cause significant multiuser interference (MUI). In this paper, we derive analytical bit error rate (BER) expressions that quantify the degradation of BER due to the combined effect of both CFOs as well as TOs; such an analytical BER derivation has not been reported so far for uplink OFDMA in the presence of CFOs and TOs. For the case when the desired user is perfectly frequency/time aligned (and other users have non-zero CFOs/TOs), we obtain an exact closed-form expression for the BER. For the case when the desired user also has non-zero CFO/TO, we obtain an approximate BER expression involving a single integral. Analytical and simulation results are shown to match quite well.**

Keywords – *Uplink OFDMA, carrier frequency offset, timing offset, interference, BER analysis.*

I. Introduction

Orthogonal frequency division multiple access (OFDMA) is attractive in wireless communications due its high spectral efficiency, robustness to multipath and less complexity [1]. The performance of OFDMA on the uplink depends to a large extent on how well the orthogonality among different subcarriers from different users is maintained at the receiver. In uplink OFDMA, factors including $i)$ timing offsets (TO) of different users caused due to path delay differences between different users and imperfect timing synchronization, and $ii)$ carrier frequency offsets (CFO) of different users induced by Doppler effects and/or poor oscillator alignments, can destroy the orthogonality among subcarriers at the receiver and cause multiuser interference (MUI). Several techniques have been proposed to alleviate the loss in performance due to CFOs and TOs; they include $i)$ tight closed-loop frequency/timing correction between mobile transmitters and the base station receiver [2], $ii)$ providing adequate guard interval and use of GPS timing, and $iii)$ interference cancellation techniques at the receiver [3]-[6].

Analytical characterization of the bit/symbol error performance of uplink OFDMA in the presence of large CFOs and TOs has not been addressed adequately in the literature. Most bit error rate (BER) evaluations in uplink OFDMA are based on simulations, e.g., [3]-[6]. In terms of analytical evaluation, an approximate analysis of the SNR degradation and BER of 'single user OFDM' with CFO on AWGN channels was introduced in [7]. Later, in [8], Santhanam and Tellambura presented an exact BER analysis of single user OFDM with CFO on AWGN channels. Further, making a Gaussian approximation of the inter-carrier interference, Rugini and Banelli extended the BER analysis of OFDM to frequency-selective Rayleigh and Ricean fading with CFO in [9]. However, the analyses in [7]-[9] do not consider TOs. In [10], an approximate average signal-to-interference (SIR) analysis for OFDM with TO alone (assuming zero CFO) was presented. In [11], an approximate symbol error rate (SER) analysis of OFDM with both CFO and TO is presented. However, papers [7]-[11] do not consider uplink OFDMA (i.e., 'multiuser OFDM' on the uplink).

In terms of performance analysis of uplink OFDMA, [12] and [6] derived analytical expressions for average signal-to-interference ratio (SIR) at the receiver; [12] derived SIR expressions considering only TO (assuming zero CFO), whereas [6] derived SIR expressions considering both CFOs as well as TOs. To our knowledge, however, analytical derivation of BER expressions for uplink OFDMA in the presence of both CFO as well as TO have not been reported. Our current contribution in this paper aims to fill this gap. In particular, we present the BER derivation of two cases of uplink OFDMA with CFO and TO. In the first case, we assume that the desired user's frequency and timing are perfectly aligned (i.e., zero CFO and TO for desired user), whereas the other users' CFOs and TOs are non-zero. For this case, we obtain an exact closed-form expression for BER. In the second case, we consider all users' (including desired user) CFOs and TOs are non-zeros. For this case, we obtain an approximate BER expression involving only a single integral. We show that the analytical and simulation results match quite well.

II. System Model

We consider an uplink OFDMA system with K users, where each user communicates with a BS through an independent multipath channel. We assume that there are N subcarriers in each OFDM symbol and one subcarrier can be allocated to only one user. The information symbol for the uth user on the kth subcarrier is denoted by X_k^u, $k \in S_u$, where S_u is the set of subcarriers assigned to the uth user, and $\mathbb{E}\Big[\,|X_k^u|^2\,\Big] = 1$, where $\mathbb{E}[.]$ denotes the expectation operator. Then, $\bigcup_{u=1}^K S_u = \{0, 1, \ldots, N-1\}$ and $S_u \bigcap S_v = \phi$ for $u \neq v$. The length of the cyclic prefix added is N_g sampling periods, and is assumed to be longer than the maximum channel delay spread, $L-1$, normalized by the sampling period (i.e., $N_g \geq L-1$). After IDFT processing and cyclic prefix insertion at the transmitter, the time-domain sequence of the uth user, x_n^u, is given by

$$x_n^u = \frac{1}{N}\sum_{k\in S_u} X_k^u e^{\frac{j2\pi nk}{N}}, \quad -N_g \leq n \leq N-1. \qquad (1)$$

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The uth user's signal at the receiver input, after passing through the channel, in the case of perfect synchronization, is given by

$$s_n^u = x_n^u \star h_n^u, \tag{2}$$

where $\star$ denotes linear convolution and h_n^u is the uth user's channel impulse response. It is assumed that h_n^u is non-zero only for $n = 0, \ldots, L-1$, and that all users' channels are statistically independent. We assume that h_n^u's are i.i.d. complex Gaussian with zero mean and $\mathbb{E}\Big[\left(h_{n,I}^u\right)^2\Big] = \mathbb{E}\Big[\left(h_{n,Q}^u\right)^2\Big] = 1/2L$, where $h_{n,I}^u$ and $h_{n,Q}^u$ are the real and imaginary parts of h_n^u. The channel coefficient in frequency-domain, H_k^u, is given by

$$H_k^u \;=\; \sum_{n=0}^{L-1} h_n^u e^{\frac{-j2\pi nk}{N}}, \quad \text{and} \quad \mathbb{E}\Big[|H_k^u|^2\Big] = 1. \tag{3}$$

Let ϵ_u, $u = 1, 2, \cdots, K$ denote uth user's residual CFO normalized by the subcarrier spacing, $|\epsilon_u| \leq 0.5, \forall\, u$, and let μ_u, $u = 1, 2, \cdots, K$ denote uth user's residual TO in number of sampling periods at the receiver. The DFT output on the kth carrier of the uth user at the receiver in the presence of CFOs and TOs can be written in the form

$$\begin{aligned} Y_k^u \;=\;& \underbrace{H_{k,k}^u X_k^u + \sum_{\substack{q\in\mathcal{S}_u\\ q\neq k}} H_{k,q}^u X_q^u + \sum_{q\in\mathcal{S}_u}^{u} H_{k,q}^{u,I} X_q^{u,I}}_{\text{self interference (SI)}} \\ &+ \underbrace{\sum_{\substack{v=1,\\ v\neq u}}^{K} \sum_{q\in\mathcal{S}_v} H_{k,q}^v X_q^v + H_{k,q}^{v,I} X_q^{v,I}}_{\text{MUI}} + Z_k^u, \end{aligned} \tag{4}$$

where X_q^u and $X_q^{u,I}$ are the symbols from the current and interfering frames, respectively, of uth user, Z_k^u is the output noise of variance σ_n^2, and $H_{k,q}^u$ and $H_{k,q}^{u,I}$ are defined in (6) and (11)/(12), respectively. If the TO is negative, the interfering frame will be the the previous frame; if the TO is positive, interfering frame will be the next frame. The coefficients $H_{k,q}^u$'s depend on the CFO and TO values. To write the expressions for these coefficients for $N_g = L-1$, we need to consider four different cases of TOs, referred to as *Cases* a) to d) of TOs [6], where $0 < -\mu_u \leq N_g$ for *Case a)*, $-\mu_u > N_g$ for *Case b)*, $0 < \mu_u < L$ for *Case c)*, and $\mu_u \geq L$ for *Case d)*. With l as the path index, defining

$$\Gamma_{qk}^{u,l}(n_1, n_2) \;\triangleq\; \frac{1}{N}\sum_{n=n_1}^{n_2} e^{\frac{j2\pi n(q-k+\epsilon_u)}{N}}, \tag{5}$$

the expressions for $H_{k,q}^u$ for different cases of TOs can be written as [6]

$$H_{k,q}^u \;=\; e^{\frac{i2\pi\mu_u q}{N}} \sum_{l=0}^{L-1} h_l^u e^{\frac{-i2\pi lq}{N}} \Gamma_{qk}^{u,l}(n_{\alpha_1}, n_{\alpha_2}), \tag{6}$$

where $(n_{\alpha_1}, n_{\alpha_2})$ corresponding to different cases a) to d) are given by

$$(n_{a_1}, n_{a_2}) = \begin{cases} (0, N-1), & \text{for } l \leq N_g + \mu_u, \\ (l-\mu_u-N_g, N-1), & \text{for } l > N_g + \mu_u, \end{cases} \tag{7}$$

$$(n_{b_1}, n_{b_2}) \;=\; (l-\mu_u-N_g, N-1), \;\; \forall l, \tag{8}$$

$$(n_{c_1}, n_{c_2}) = \begin{cases} (0, N-1-\mu_u+l), & \text{for } 0 \leq l \leq \mu_u - 1, \\ (0, N-1), & \text{for } l \geq \mu. \end{cases}, \tag{9}$$

and

$$(n_{d_1}, n_{d_2}) \;=\; (0, N-1+l-\mu_u), \;\; \forall l. \tag{10}$$

It is noted that in *Cases a)* and *b*), interference is only due to previous frame, and in *Cases c)* and *d*), interference is only due to next frame. Based on this observation, the expressions for $H_{k,q}^{u,I}$'s for *Cases a)* and *b*) can be written as

$$H_{k,q}^{u,I} \;=\; e^{\frac{i2\pi(\mu_u+N_g)q}{N}} \sum_{l=N_g+\mu_u+1}^{L-1} h_l^u e^{\frac{-i2\pi lq}{N}} \Gamma_{qk}^{u,l}(0, n_{\alpha_1}-1), \tag{11}$$

where n_{α_1} in (11) is n_{a_1} for *Case a)* and n_{b_1} for *Case b)*. Likewise, the expressions for $H_{k,q}^{u,I}$'s for *Cases c)* and *d*) can be written as

$$H_{k,q}^{u,I} = e^{\frac{-i2\pi(N_g-\mu_u)q}{N}} \sum_{l=0}^{\mu_u-1} h_l^u e^{\frac{-i2\pi lq}{N}} \Gamma_{qk}^{u,l}(n_{\alpha_2}+1, N-1), \tag{12}$$

where n_{α_2} in (12) is n_{c_2} for *Case c)* and n_{d_2} for *Case d)*. We note that the coefficients of any given user u, (i.e., H_{kq}^u's) are correlated, whereas the coefficients of any two different users (i.e., H_{kq}^u's and H_{kq}^v's, $u \neq v$) are uncorrelated. So, computation of the exact BER would involve M-fold integral in the case of the system with only CFOs and $2M$-fold integral for the system with both CFOs and TOs (where M is the number of subcarriers allotted to each user). To reduce this computational complexity, we adopt an approximate method to compute the BER (involving only a single integral), as outlined in the next section.

III. BER ANALYSIS

We proceed to obtain an analytical expression for the BER using the following four steps: i) since $H_{k,q}^u$'s are correlated, we obtain an estimate of each $H_{k,q}^u$ and $H_{k,q}^{u,I}$, in terms of $H_{k,k}^u$, ii) obtain expressions for the variances of SI/MUI and the SINR, conditioned on $|H_{k,k}^u|$, iii) obtain expression for the BER, conditioned on $|H_{k,k}^u|$, by assuming the estimation errors in $H_{k,q}^u$'s and $H_{k,q}^{u,I}$'s to be Gaussian, and uncondition to obtain the unconditional BER.

Step i): To obtain an estimate of $H_{k,q}^u$ in terms of $H_{k,k}^u$, We use the fact that, if two zero mean complex Gaussian random variables X and Y are correlated, an estimate of one variable (say, Y) can be obtained, in terms of the other variable, as $\widehat{Y} = \frac{C_{X,Y}}{\sigma_X^2}X$, with an estimation error, $\mathcal{E}_Y = Y - \widehat{Y}$, of variance $\sigma_Y^2 - \frac{C_{X,Y}^2}{\sigma_X^2}$, where $C_{X,Y}$ is the covariance of X and Y, and σ_X^2 and σ_Y^2 are the variances of X and Y, respectively. Using this, we can write (4) as

$$\begin{aligned} Y_k^u \;=\;& H_{k,k}^u X_k^u + \sum_{\substack{q\in\mathcal{S}_u\\ q\neq k}} \left(\frac{C_{k,q}^u}{(\sigma_k^u)^2} H_{k,k}^u + \mathcal{E}_q^u\right) X_q^u \\ &+ \sum_{q\in\mathcal{S}_u} \left(\frac{C_{k,q}^{u,I}}{(\sigma_k^u)^2} H_{k,k}^u + \mathcal{E}_q^{u,I}\right) X_q^{u,I} \\ &+ \sum_{\substack{v=1,\\ v\neq u}}^{K} \sum_{q\in\mathcal{S}_v} H_{k,q}^v X_q^v + H_{k,q}^{v,I} X_q^{v,I} + Z_k^u, \end{aligned} \tag{13}$$

Case λ	$\mathcal{A}$
$\lambda = a, b$	$\frac{1}{L^2(\sigma_k^u)^4}\Big[\sum_{\substack{q\in S_u\\ q\neq k}} \big\lvert \sum_{l=0}^{L-1} e^{\frac{i2\pi(k-q)(\mu_u-l)}{N}} \Gamma_{kk}^{u,l}(n_{\lambda_1}, n_{\lambda_2}) \Gamma_{qk}^{u,l^*}(n_{\lambda_1}, n_{\lambda_2})\big\rvert^2$ $+ \big\lvert e^{\frac{-i2\pi N_g q}{N}} \sum_{q\in S_u} \sum_{l=0}^{L-1} e^{\frac{i2\pi(k-q)(\mu_u-l)}{N}} \Gamma_{kk}^{u,l}(n_{\lambda_1}, n_{\lambda_2}) \Gamma_{qk}^{u,l^*}(0, n_{\lambda_1}-1)\big\rvert^2\Big]$
$\lambda = c, d$	$\frac{1}{L^2(\sigma_k^u)^4}\Big[\sum_{\substack{q\in S_u\\ q\neq k}} \big\lvert \sum_{l=0}^{L-1} e^{\frac{i2\pi(k-q)(\mu_u-l)}{N}} \Gamma_{kk}^{u,l}(n_{\lambda_1}, n_{\lambda_2}) \Gamma_{qk}^{u,l^*}(n_{\lambda_1}, n_{\lambda_2})\big\rvert^2$ $+ \big\lvert e^{\frac{i2\pi N_g q}{N}} \sum_{q\in S_u} \sum_{l=0}^{L-1} e^{\frac{i2\pi(k-q)(\mu_u-l)}{N}} \Gamma_{kk}^{u,l}(n_{\lambda_1}, n_{\lambda_2}) \Gamma_{qk}^{u,l^*}(n_{\lambda_2}+1, N-1)\big\rvert^2\Big]$
Case λ	$\mathcal{B}_1$
$\lambda = a, b$	$\sum_{\substack{q\in S_u\\ q\neq k}} (\sigma_q^u)^2 - \frac{1}{(\sigma_k^u)^2}\big\lvert \sum_{l=0}^{L-1} e^{\frac{i2\pi(k-q)(\mu_u-l)}{N}} \Gamma_{kk}^{u,l}(n_{\lambda_1}, n_{\lambda_2}) \Gamma_{qk}^{u,l^*}(n_{\lambda_1}, n_{\lambda_2})\big\rvert^2$ $+ \sum_{q\in S_u} (\sigma_{q,I}^u)^2 - \frac{1}{(\sigma_k^u)^2}\big\lvert e^{\frac{-i2\pi N_g q}{N}} \sum_{q\in S_u} \sum_{l=0}^{L-1} e^{\frac{i2\pi(k-q)(\mu_u-l)}{N}} \Gamma_{kk}^{u,l}(n_{\lambda_1}, n_{\lambda_2}) \Gamma_{qk}^{u,l^*}(0, n_{\lambda_1}-1)\big\rvert^2$
$\lambda = c, d$	$\sum_{\substack{q\in S_u\\ q\neq k}} (\sigma_q^u)^2 - \frac{1}{(\sigma_k^u)^2}\big\lvert \sum_{l=0}^{L-1} e^{\frac{i2\pi(k-q)(\mu_u-l)}{N}} \Gamma_{kk}^{u,l}(n_{\lambda_1}, n_{\lambda_2}) \Gamma_{qk}^{u,l^*}(n_{\lambda_1}, n_{\lambda_2})\big\rvert^2$ $+ \sum_{q\in S_u} (\sigma_{q,I}^u)^2 - \frac{1}{(\sigma_k^u)^2}\big\lvert e^{\frac{i2\pi N_g q}{N}} \sum_{q\in S_u} \sum_{l=0}^{L-1} e^{\frac{i2\pi(k-q)(\mu_u-l)}{N}} \Gamma_{kk}^{u,l}(n_{\lambda_1}, n_{\lambda_2}) \Gamma_{qk}^{u,l^*}(n_{\lambda_2}+1, N-1)\big\rvert^2$

TABLE I

EXPRESSIONS FOR $\mathcal{A}$ AND $\mathcal{B}_1$ FOR DIFFERENT TIME OFFSET CASES *a*) TO *d*).

where

$$
\begin{aligned}
C_{k,q}^u &= \mathbb{E}[H_{k,k}^u (H_{q,k}^u)^*], \quad C_{k,q}^{u,I} = \mathbb{E}[H_{k,k}^u (H_{q,k}^{u,I})^*], \\
(\sigma_k^u)^2 &= \mathbb{E}[H_{k,k}^u (H_{k,k}^u)^*], \qquad (14)
\end{aligned}
$$

where $(.)^*$ denotes the complex conjugate operation.

Step ii): Now, in (13), the total variance of all the terms which are interference to the uth user's symbol on kth subcarrier, conditioned on $H_{k,k}^u$, is obtained as

$$
\begin{aligned}
\sigma_{I|H_{k,k}}^2 = |H_{k,k}^u|^2 & \underbrace{\left(\sum_{\substack{q\in\mathcal{S}_u\\ q\neq k}} \frac{|C_{k,q}^u|^2}{(\sigma_k^u)^4} + \sum_{q\in\mathcal{S}_u} \frac{|C_{k,q}^{u,I}|^2}{(\sigma_k^u)^4} \right)}_{\triangleq\, \mathcal{A}} \\
& + \underbrace{\left(\sum_{\substack{q\in\mathcal{S}_u\\ q\neq k}} (\sigma_q^u)^2 - \frac{|C_{k,q}^u|^2}{(\sigma_k^u)^2} + \sum_{q\in\mathcal{S}_u} (\sigma_q^{u,I})^2 - \frac{|C_{k,q}^{u,I}|^2}{(\sigma_k^u)^2} \right)}_{\triangleq\, \mathcal{B}_1} \\
& + \underbrace{\sum_{\substack{v=1,\\ v\neq u}}^{K} \sum_{q\in\mathcal{S}_v} |H_{k,q}^v|^2 + |H_{k,q}^{v,I}|^2}_{\triangleq\, \mathcal{B}_2}. \qquad (15)
\end{aligned}
$$

where

$$
(\sigma_q^u)^2 = \mathbb{E}[H_{k,q}^u (H_{k,q}^u)^*], \quad (\sigma_{\mathcal{E}_q^u})^2 = (\sigma_q^u)^2 - \frac{|C_{k,q}^u|^2}{(\sigma_k^u)^2}, \qquad (16)
$$

Assuming that among K users in the system, K_λ users belong to *Case* λ), $\lambda \in a, b, c, d$, the expressions for the terms $\mathcal{A}$ and $\mathcal{B}_1$ in (15) for different TO cases are given in Table-I, where

$$
(\sigma_k^u)^2 = \frac{1}{L} \sum_{l=0}^{L} |\Gamma_{kk}^{u,l}(n_{\lambda_1}, n_{\lambda_2})|^2, \qquad (17)
$$

Case λ	$\mathcal{B}_2$
$\lambda = a$	$\sum_{\substack{v=1\\ v\neq u}}^{K_a} \Big[\sum_{\substack{q\in S_v\\ v\neq u}} \sum_{l=0}^{L-1} \Big\lvert \Gamma_{qk}^{v,l}(n_{a_1}, n_{a_2})\Big\rvert^2$ $+ \sum_{\substack{q\in S_v\\ v\neq u}} \sum_{l=N_g+\mu_v+1}^{L-1} \Big\lvert \Gamma_{qk}^{v,l}(0, n_{a_1}-1)\Big\rvert^2\Big]$
$\lambda = b$	$\sum_{\substack{v=1\\ v\neq u}}^{K_b} \Big[\sum_{\substack{q\in S_v\\ v\neq u}} \sum_{l=0}^{L-1} \Big\lvert \Gamma_{qk}^{v,l}(n_{b_1}, n_{b_2})\Big\rvert^2$ $+ \sum_{\substack{q\in S_v\\ v\neq u}} \sum_{l=0}^{L-1} \Big\lvert \Gamma_{qk}^{v,l}(0, n_{b_1}-1)\Big\rvert^2\Big]$
$\lambda = c$	$\sum_{\substack{v=1\\ v\neq u}}^{K_c} \Big[\sum_{\substack{q\in S_v\\ v\neq u}} \sum_{l=0}^{L-1} \Big\lvert \Gamma_{qk}^{v,l}(n_{c_1}, n_{c_2})\Big\rvert^2$ $+ \sum_{.}\ {}^{q\in S_v}_{v\neq u} \sum_{l=0}^{\mu_v-1} \Big\lvert \Gamma_{qk}^{v,l}(n_{c_2}+1, N-1)\Big\rvert^2\Big]$
$\lambda = d$	$\sum_{\substack{v=1\\ v\neq u}}^{K_d} \Big[\sum_{\substack{q\in S_v\\ v\neq u}} \sum_{l=0}^{L-1} \Big\lvert \Gamma_{qk}^{v,l}(n_{d_1}, n_{d_2})\Big\rvert^2$ $+ \sum_{\substack{q\in S_v\\ v\neq u}} \sum_{l=0}^{L-1} \Big\lvert \Gamma_{qk}^{v,l}(n_{d_2}+1, N-1)\Big\rvert^2\Big]$

TABLE II

EXPRESSIONS FOR $\mathcal{B}_2$ FOR DIFFERENT TIME OFFSET CASES *a*) TO *d*).

$$
(\sigma_q^u)^2 = \frac{1}{L} \sum_{l=0}^{L} |\Gamma_{qk}^{u,l}(n_{\lambda_1}, n_{\lambda_2})|^2, \qquad (18)
$$

$$
(\sigma_{q,I}^u)^2 = \begin{cases} \frac{1}{L}\sum_{l=0}^{L} |\Gamma_{qk}^{u,l}(0, n_{\lambda_1}-1)|^2, & \text{for } \lambda = a, b \\ \frac{1}{L}\sum_{l=0}^{L} |\Gamma_{qk}^{u,l}(n_{\lambda_2}+1, N-1)|^2, & \text{for } \lambda = c, d. \end{cases} \qquad (19)
$$

Similarly, the expressions for the term $\mathcal{B}_2$ in (15) for the different TO cases are given in Table-II.

Now, defining $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 + \sigma_n^2$, the SINR at the kth subcarrier of the uth user, conditioned on $H_{k,k}^u$, denoted by $\gamma_{H_{k,k}^u}$, is given by

$$\gamma_{H^u_{k,k}} = \frac{|H^u_{k,k}|^2}{\mathcal{A}|H^u_{k,k}|^2 + \mathcal{B}}. \quad (20)$$

Step iii): Now, assuming that the estimation errors $\mathcal{E}^u_q$ and $\mathcal{E}^{u,I}_q$ to be Gaussian, the conditional BER, denoted by $P_e(\gamma_{H^u_{k,k}})$, is obtained as

$$P_e(\gamma_{H^u_{k,k}}) = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{p\gamma_{H^u_{k,k}}}}^{\infty} e^{\frac{-y^2}{2}} dy, \quad (21)$$

where $p = 1$ for QPSK and $p = 2$ for BPSK. Unconditioning over the Rayleigh pdf of $|H^u_{k,k}|$, we get the unconditional BER expression as

$$P_e = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \int_{\sqrt{p\gamma_{H^u_{k,k}}}}^{\infty} e^{\frac{-y^2}{2}} dy\, f_R(r) dr. \quad (22)$$

where

$$f_R(r) = \frac{2r}{(\sigma^u_k)^2} e^{\frac{-r^2}{(\sigma^u_k)^2}} \quad \text{for } r \geq 0, \quad (23)$$

This integral in (22) can be evaluated by using the series expansion method in [9], or by using the numerical techniques. Changing the order of integration, we convert (22) into a single integral, and then use Simpson's method to evaluate it. The final single integral is of the form

$$P_e = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-y^2\left(\frac{1}{2} + \frac{\mathcal{B}}{(\sigma^u_k)^2 (p - \mathcal{A}y^2)}\right)} dy. \quad (24)$$

From the above general BER expression for uplink OFDMA with CFO and TO, consider the following two special cases.

1) Zero CFO and TO for Desired User Alone: If the desired user CFO and TO are zero, and the other users' CFO and TOs are non-zero, then there won't be any SI (so, no Gaussian approximation is needed) and only MUI occurs. For this case, $\mathcal{A} = 0$, $\mathcal{B} = \mathcal{B}_2 + \sigma^2_n$, and $(\sigma^u_k)^2 = 1$, leading to the simplification of (24) as

$$P_e = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-y^2\left(\frac{1}{2} + \frac{\mathcal{B}}{p}\right)} dy, \quad (25)$$

which, by defining $SNR = \frac{1}{\sigma^2_n}$, evaluates to

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{p \times SNR}{2 + (2\mathcal{B}_2 + p)SNR}}\right), \quad (26)$$

which is an exact closed-form BER expression.

2) Zero CFOs and TOs for all users:

If all the users are perfectly synchronized (i.e., zero CFOs and TOs for all users, $\mathcal{A} = 0$, $\mathcal{B} = \frac{1}{SNR}$ and $\sigma^2_k = 1$. In this case (24) becomes

$$P_e = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-y^2\left(\frac{1}{2} + \frac{1}{p \times SNR}\right)} dy, \quad (27)$$

which evaluates to

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{p \times SNR}{2 + p \times SNR}}\right), \quad (28)$$

which is the standard BER expression for the ideal case of zero CFOs and TOs [13].

IV. Numerical Results

In this section, we present the numerical results on the BER performance of uplink OFDMA in the presence of CFOs and TOs. We consider a system with $N = 64$, $K = 4$, $M = 16$, $L = 2$, $N_g = 1$, and interleaved allocation of subcarriers. BPSK and QPSK modulations are considered. We take the first user as the desired user and plot its BER performance obtained by both analysis as well as simulations,

Figure 1 shows the BER performance of uplink OFDMA obtained using both analysis and simulation, as a function of SNR with $\mu = [0, -5, 1, 5]$ and $\epsilon = [0, 0.1, -0.25, -0.4]$. For this parameter setting, where $\epsilon_1 = \mu_1 = 0$, the system is affected by only MUI and not by SI, and so the Gaussian approximation in the analysis is not needed. That is, the analysis becomes exact with (26) giving the exact BER. This can be verified by the very close match between the analysis and simulation plots of the BER in Fig. 1.

Figure 2 shows the BER performance of uplink OFDMA, obtained using both analysis and simulation, as a function of SNR with $\mu = [-1, -5, 1, 5]$ and $\epsilon = [0.1, 0.2, -0.15, -0.3]$, where all users (including the desired user) have non-zero CFOs and TOs. In this case, both SI as well as MUI will occur, and the Gaussian approximation of the SI terms makes the BER expression in (24) to be approximate. This can be seen by the slight mismatch between the analytical and simulated BER plots. In fact, the match between analysis and simulation is quite good.

In Fig. 3 we plot the BER performance of uplink OFDMA, obtained through analysis, as a function of SNR with $\mu = [0, -5, 1, 5]$, $\epsilon_1 = 0$ for various values of $\epsilon = \epsilon_2 = \epsilon_3 = \epsilon_4$. The figure shows that the BER degrades with increase in the CFOs of other users, as expected. Similarly, Fig. 4 shows the BER performance of uplink OFDMA, obtained using analysis, as a function of SNR with $\mu_1 = 0$, $\epsilon = [0, 0.1, -0.25, -0.4]$, for various values of $\mu = \mu_2 = \mu_3 = \mu_4$. The figure shows that the BER degrades with increase in the TOs of other users, which is also expected. The analysis presented in this paper enables such analytical quantification of the BER performance of uplink OFDMA in the presence of both CFOs as well as TOs.

V. Conclusions

We presented a BER analysis of uplink OFDMA in the presence of CFOs and TOs. For the case when the desired user is perfectly aligned in frequency/time (i.e., zero CFO/TO for the desired user) while the other users have non-zero CFOs and TOs, we obtained an exact closed-form expression for the BER. For the case when all the users (including the desired user) have non-zero CFOs and TOs, we obtained an approximate expression for the BER which involved the computation of a single integral. Analytical and simulated BER results matched well. A similar analysis can be carried out for Ricean fading as well.

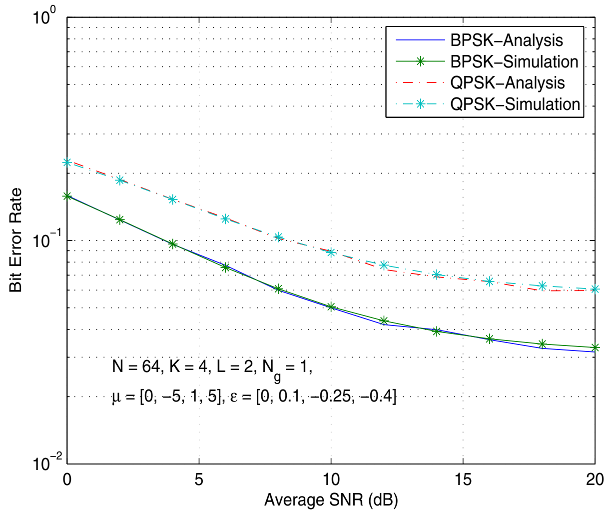

Fig. 1. BER performance of uplink OFDMA in the presence of CFOs and TOs. $N = 64$, $L = 2, N_g = 1$, 1st user is desired user, zero CFO and TO for desired user, non-zero CFOs and TOs for other users: $\epsilon = [0, 0.1, -0.25, -0.4]$, $\mu = [0, -5, 1, 5]$, BPSK and QPSK. Analysis versus simulation.

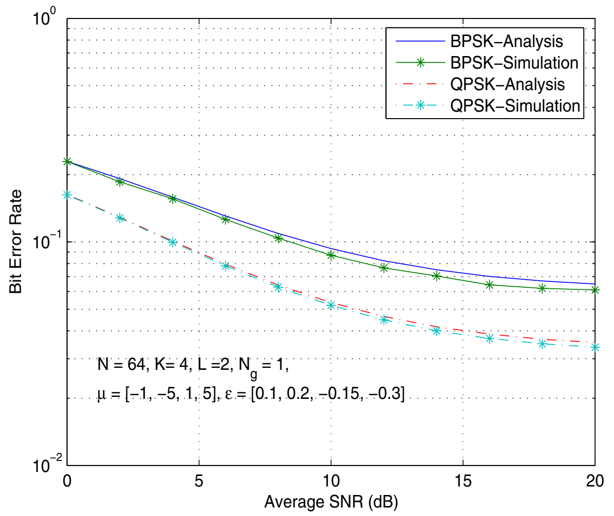

Fig. 2. BER performance of uplink OFDMA in the presence of CFOs and TOs. $N = 64$, $K = 4$, $L = 2, N_g = 1$, 1st user is desired user, all users' CFOs and TOs are non-zero: $\epsilon = [0.1, 0.2, -0.15, -0.3]$, $\mu = [-1, -5, 1, 5]$, BPSK and QPSK. Analysis versus simulation.

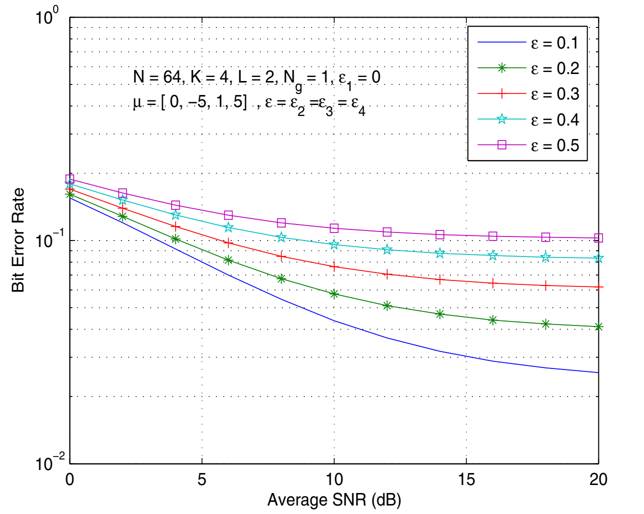

Fig. 3. BER performance of uplink OFDMA for $N = 64$, $K = 4$, $M = 16$, $L = 2, N_g = 1$, BPSK. 1st user is desired user, zero CFO and TO for desired user, all other users have have same CFO values, and $\mu = [0, -5, 1, 5]$. Analysis.

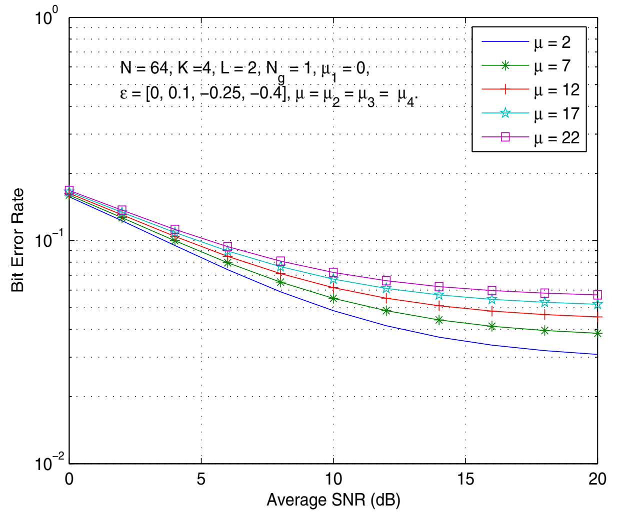

Fig. 4. BER performance of uplink OFDMA for $N = 64$, $K = 4$, $M = 16$, $L = 2, N_g = 1$, BPSK. 1st user is desired user, zero CFO and TO for desired user, all other users have same TO values, and $\epsilon = [0, 0.1, -0.25, -0.4]$. Analysis.

References

[1] S. Pietrzyk, *OFDMA for Broadband Wireless Access*, Artech House, April 2006.

[2] IEEE 802.16e-2005: IEEE standard for Local and Metropolitan Area Networks. Part 16: Air Interface for Fixed and Mobile Broadband Wireless Access Systems. Amendment 2: Physical and Medium Access Control Layers for Combined Fixed and Mobile Operation in Licensed Bands. Dec. 2005.

[3] R. Fantacci, D. Marabissi, and S. Papini, "Multiuser interference cancellation receivers for OFDMA uplink communications with carrier frequency offset," *Proc. IEEE GLOBECOM04*, pp. 2808-2812, December 2004.

[4] D. Huang and K. B. Letaief, "An interference cancellation scheme for carrier frequency offsets correction in OFDMA systems," *IEEE Trans. Commun.,* vol. 53, no. 7, pp. 1155-1165, July 2005.

[5] S. Manohar, D. Sreedhar, V. Tikia, and A. Chockalingam, "Cancellation of multiuser interference due to carrier frequency offsets in uplink OFDMA," *IEEE Trans. Wireless Commun.*, vol 6, no. 7, pp 2560-2571, July 2007.

[6] K. Raghunath and A. Chockalingam, "SIR analysis and interference cancellation in uplink OFDMA with large carrier frequency and timing offsets," *IEEE Trans. Wireless Commun.,* vol. 8, no. 5, 2202-2208, May 2009.

[7] T. Pollet, M. Van Bladel, and M. Moeneclaey, "BER sensitivity of OFDM systems to carrier frequency offset and Wiener phase noise," *IEEE Trans. Commun.,* vol. 43, pp. 191-193, Feb/Mar/Apr 1995.

[8] K. Santhanam and C. Tellambura, "Probability of error calculation of OFDM system with frequency offset," *IEEE Trans. Commun.,* vol 49, no 11, pp 1884-1888, November 2001.

[9] L. Rugini and P. Banelli, "BER of OFDM systems impaired by carrier frequency offset in multipath fading channels," *IEEE Trans. Wireless Commun.,* vol. 4, No. 5, pp 2279-2288, September 2005

[10] Y. Mostofi and D. C. Cox, "Mathematical analysis of the impact of timing synchronization errors on the performance of an OFDM system," *IEEE Trans. Commun.,* vol. 54, no. 2, pp. 226-230, February 2006.

[11] X. Wang, T. T. Tjhung, Y. Wu, and B. Caron, "SER performance evaluation and optimization of OFDM system with residual frequency and timing offsets from imperfect synchronization," *IEEE Trans. Broadcasting,* vol. 49, no. 2, pp. 170-177, June 2003.

[12] M. Park, K. Ko, H. Yoo, and D. Kong, "Performance analysis of OFDMA uplink systems with symbol timing misalignment," *IEEE Commun. Lett.,* vol. 7, no. 8, pp. 376-378, August 2003.

[13] J. G. Proakis, *Digital Communications*, 4th Ed., Mc-Graw Hill, 2001.