Diversity Analysis of Time-indexed Media-based Modulation

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***Abstract*—Time-indexed media-based modulation (TI-MBM) is a block transmission scheme, where a block of N time-slots forms a frame of which only K time-slots are used for data transmission (active slots) and the remaining $N-K$ time-slots are left vacant (inactive slots). The choice of the active time-slots conveys additional information bits along with the bits conveyed by the MBM signals transmitted in the active time-slots. It has been shown that TI-MBM can achieve improved bit error performance compared to the conventional MBM scheme. However, a mathematical diversity analysis of TI-MBM has not been reported in the literature so far. In this paper, we view the time-slot indexing as coding across time (since the choice of active and inactive time-slots are dependent on each other) and carry out a mathematical analysis of the diversity performance of TI-MBM. Our analytical results show that, while conventional MBM can not extract the multipath diversity of the channel, TI-MBM can achieve higher diversity orders by extracting the channel multipath diversity for suitable choice of system parameters. We provide simulation results that support the analytically predicted diversity orders.**

Keywords *– Media-based modulation, RF mirrors, time-slot indexing, multipath diversity.*

I. Introduction

Conventionally, information bits are conveyed in a communication system by transmitting symbols from a complex modulation alphabet such as QAM. The wireless fading channel is viewed as a signal distorting medium which changes the amplitude and phase of the transmitted complex symbol in a random manner. A recently proposed scheme called the media-based modulation (MBM) takes a different approach in which the channel fades from the transmitter to the receiver are used to convey additional information bits along with the bits conveyed by the complex modulation symbols. The idea of MBM can be briefly explained as follows [1]-[6].

An MBM transmit unit (MBM-TU) consists of a transmit antenna and m_{rf} passive elements called radio frequency (RF) mirrors placed near the transmit antenna. An RF mirror either reflects (ON state) or allows (OFF state) the RF signal from the transmit antenna to pass through depending on its control input, which in turn depends on the information bits. If there are m_{rf} RF mirrors, each of them can be in either ON or OFF state, resulting in $2^{m_{rf}}$ possible ON and OFF combinations, called mirror activation patterns (MAPs). It is known that, in a rich scattering environment, even a small perturbation in the near field of the transmit antenna can result in multiple random reflections and hence result in a different end-to-end channel between the transmitter and the receiver. Therefore, in MBM, different MAPs give rise to independent channel realizations between the transmitter and the receiver. The collection of all these channel realizations corresponding to the different MAPs form the channel alphabet in MBM. In a given channel use, one of the MAPs is selected based on m_{rf} information bits and a symbol from a conventional modulation alphabet $\mathbb{A}$ (say, QAM) is transmitted using the selected MAP. The achieved rate in MBM is therefore, $\eta_{\text{MBM}} = m_{rf} + \log_2|\mathbb{A}|$ bits per channel use (bpcu).

The idea of time-slot indexing is introduced in the context of MBM in [7],[8]. Time-indexed MBM (TI-MBM) is a block transmission scheme in which time is divided into frames of N time-slots (channel uses). Out of the N time-slots in a frame, K time-slots (active slots) are selected based on $\lfloor \log_2 \binom{N}{K} \rfloor$ information bits and K MBM signal vectors are transmitted in the selected time-slots. The remaining $N-K$ time-slots are left vacant (inactive slots). It has been shown that TI-MBM can achieve significant bit error performance gains compared to conventional MBM. However, the earlier literature presents only simulation results and does not provide a mathematical analysis of the TI-MBM system performance. The aim of the present work is to fill this gap.

Our contributions in this paper can be summarized as follows. We view time-slot indexing as coding across time, since the active and inactive slots are dependent on each other thus creating dependence across time. It is known from the space-time coding literature that higher diversity orders can be achieved by extracting the multipath diversity in the channel by coding across time. Motivated by this, we carry out the diversity analysis of the TI-MBM scheme to see if TI-MBM can extract the multipath diversity of the channel. Our analysis show that, while conventional MBM in a cyclic prefixed single carrier (CPSC) system setting [5] fails to extract the multipath diversity, TI-MBM can achieve higher diversity by extracting the multipath diversity in the channel. Simulation results validate the analytically predicted diversity orders.

II. TI-MBM System model

Consider an MBM-TU with a transmit antenna and m_{rf} RF mirrors placed near it. Let n_r be the number of receive antennas. The channel is considered to be frequency selective with L multipaths and exponential power delay profile. In TI-MBM, the transmission is carried out in frames of $N+L-1$ time-slots (channel uses), where N is the length of the data part of the frame and $L-1$ time-slots are used for cyclic prefix (CP). Out of the N time-slots of the data part of the frame, only K time-slots, $1 \leq K \leq N$, are used for signal transmission (active slots) and the remaining $N-K$ time-slots are left unused (inactive slots). The K active slots are selected based on $\left\lfloor \log_2 \binom{N}{K} \right\rfloor$ information bits. A 'time-slot activation pattern' (TAP) is an N-length vector consisting of the active/inactive status of the N time-slots in a frame. In each active time-slot, one of the $N_m \triangleq 2^{m_{rf}}$ MAPs is selected using m_{rf} information bits. Then, on the selected MAP, a symbol from a modulation alphabet $\mathbb{A}$ (say, QAM) is transmitted based on $\log_2|\mathbb{A}|$ bits. The achieved rate in the TI-MBM system is, therefore, given by

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$$\eta = \frac{1}{N+L-1}\left\{K(m_{rf}+\log_2|\mathbb{A}|)+\left\lfloor \log_2\binom{N}{K}\right\rfloor\right\} \quad \text{bpcu.} \tag{1}$$

A. TI-MBM signal set

Let $\mathbb{A}_0 \triangleq \mathbb{A}\cup 0$. The MBM signal set, $\mathbb{S}_{\text{MBM}}$, is the set of $N_m \times 1$-sized MBM signal vectors given by

$$\mathbb{S}_{\text{MBM}} = \left\{s_{k,p}\in \mathbb{A}_0^{N_m};\ k=1,\cdots,N_m;\ p=1,\cdots,|\mathbb{A}|\right\}$$
$$\text{s.t}\quad s_{k,p} = \big[0,\cdots,0,\underbrace{s_p}_{k\text{th entry}}\ 0,\cdots,0\big]^T,\ s_p\in\mathbb{A}, \tag{2}$$

where k is the index of the MAP used. In TI-MBM, an MBM vector from $\mathbb{S}_{\text{MBM}}$ is transmitted in an active time-slot and the inactive time-slots are left vacant, which is equivalent to an $N_m \times 1$ zero vector being transmitted. The TI-MBM signal set is the set of $NN_m \times 1$-sized signal vectors given by

$$\mathbb{S}_{\text{TI-MBM}} = \Big\{\mathbf{x}=\big[\mathbf{x}_0^T\ \mathbf{x}_1^T\ \cdots\ \mathbf{x}_{N-1}^T\big]^T : \mathbf{x}_i\in\mathbb{S}_{\text{MBM}}\cup\mathbf{0},$$
$$\|\mathbf{x}\|_0 = K,\ \mathbf{t}_{\mathbf{x}}\in\mathbb{T}\Big\}, \tag{3}$$

where $\mathbb{T}$ denotes the set of valid TAPs and $\mathbf{t}_{\mathbf{x}}$ denotes the TAP corresponding to the transmitted vector $\mathbf{x}$.

B. TI-MBM received signal

Assuming that the channel is invariant for one frame duration, the $Nn_r \times 1$ TI-MBM received signal vector $\mathbf{y} = [\mathbf{y}_1^T\cdots\mathbf{y}_N^T]^T$ over N channel uses, after removing the CP, is given by

$$\mathbf{y} = \mathbf{H}\mathbf{x}+\mathbf{n}, \tag{4}$$

where $\mathbf{n}\sim\mathcal{CN}(0,\sigma^2\mathbf{I}_{Nn_r})$ is the $Nn_r\times 1$ noise vector, and $\mathbf{H}$ is the $Nn_r\times NN_m$ equivalent block circulant matrix given by

$$\mathbf{H} = \text{circ}[\mathbf{H}_0,\mathbf{H}_1,\cdots,\mathbf{H}_{L-1},\mathbf{0},\cdots,\mathbf{0}], \tag{5}$$

where circ$[\cdot]$ denotes the circulant operator and $\mathbf{H}_l$, $l=0,\cdots,L-1$, is the $n_r\times N_m$ channel matrix corresponding to the lth multipath whose (i,k)th entry is $h_{i,k}^{(l)}$, where $h_{i,k}^{(l)}$ denotes the channel gain from the transmit antenna to the ith receive antenna for the lth path when the kth MAP is used, $i=1,\cdots,n_r,\ k=1,\cdots,N_m,\ l=0,\cdots,L-1$. The power delay profile is assumed to follow an exponential decay model, i.e., $\mathbb{E}\big[|h_{i,k}^{(l)}|^2\big]=e^{-l}$, $l=0,1,\cdots,L-1$. The maximum likelihood (ML) detection rule for TI-MBM signal detection is given by

$$\hat{\mathbf{x}} = \underset{\mathbf{x}\in\mathbb{S}_{\text{TI-MBM}}}{\text{argmin}}\ \|\mathbf{y}-\mathbf{H}\mathbf{x}\|^2, \tag{6}$$

whose detection complexity is $\mathcal{O}(\frac{N^{K+1}(2^{m_{rf}}|\mathbb{A}|)^K n_r}{K^{K-1}})$ [8].

III. Diversity analysis

The received signal in (4) can be written in the following equivalent matrix form [9]

$$\mathbf{Y} = \mathbf{H}^{'}\mathbf{X}+\mathbf{N}, \tag{7}$$

where

$$\mathbf{X} = \begin{bmatrix}\mathbf{x}_0 & \mathbf{x}_1 & \cdots & \mathbf{x}_{N-1}\\ \mathbf{x}_{N-1} & \mathbf{x}_0 & \cdots & \mathbf{x}_{N-2}\\ \vdots & & \vdots & \\ \mathbf{x}_{N-L+1} & \mathbf{x}_{N-L+2} & \cdots & \mathbf{x}_{N-L}\end{bmatrix} \tag{8}$$

is the $LN_m\times N$ equivalent TI-MBM transmit signal matrix, $\mathbf{Y}=[\mathbf{y}_1\mathbf{y}_2\ldots\mathbf{y}_N]$ is the $n_r\times N$ received signal matrix, $\mathbf{H}^{'}=\big[\mathbf{H}_0\ \mathbf{H}_1\ \cdots\ \mathbf{H}_{L-1}\big]$ is the $n_r\times LN_m$ equivalent channel matrix, and $\mathbf{N}\in\mathbb{C}^{n_r\times N}$ is the AWGN matrix. Let $\mathbf{X}_i$ and $\mathbf{X}_j$, $i\neq j$ be two TI-MBM signal matrices. The pairwise error probability (PEP) of detecting $\mathbf{X}_j$ when $\mathbf{X}_i$ was transmitted, given the channel matrix $\mathbf{H}'$, is

$$P\Big\{\mathbf{X}_i\to\mathbf{X}_j|\mathbf{H}^{'}\Big\} = Q\left(\sqrt{\frac{\rho\|\mathbf{H}^{'}\mathbf{X}_i-\mathbf{H}^{'}\mathbf{X}_j\|_F^2}{2}}\right), \tag{9}$$

where ρ is the average signal-to-noise ratio (SNR) per receive antenna and $\|\cdot\|_F$ denotes the Frobenius norm of a matrix. Equation (9) can be upper bounded as

$$P\Big\{\mathbf{X}_i\to\mathbf{X}_j|\mathbf{H}^{'}\Big\}\leq\frac{1}{2}e^{-\mathbf{D}_{ij}\rho/4}, \tag{10}$$

where $\mathbf{D}_{ij} = \|\mathbf{H}^{'}\mathbf{X}_i-\mathbf{H}^{'}\mathbf{X}_j\|_F^2$ and we have used the inequality $Q(x)\leq\frac{1}{2}e^{-x^2/2}$. Averaging over the channel statistics, we get the unconditional PEP as [10],[11]

$$\begin{aligned}\mathbf{P}\Big\{\mathbf{X}_i\to\mathbf{X}_j\Big\} &\leq \frac{1}{2}\left(\frac{1}{\det\Big(\mathbf{I}_{LN_m}+\frac{\mathbf{C}\rho}{4}\Big)}\right)^{n_r},\\ &= \frac{1}{2}\left(\frac{1}{\prod_{r_{ij}=1}^{R_{ij}}\Big(1+\frac{\lambda_{r_{ij}}^2\rho}{4}\Big)}\right)^{n_r},\end{aligned} \tag{11}$$

where $\mathbf{C}\triangleq(\mathbf{X}_i-\mathbf{X}_j)(\mathbf{X}_i-\mathbf{X}_j)^H$, det$(\cdot)$ denotes the determinant of a matrix, and $\lambda_{r_{ij}}$ is the r_{ij}th singular value of the difference matrix $\Delta_{i,j}=(\mathbf{X}_i-\mathbf{X}_j)$ with R_{ij} being the rank of $\Delta_{i,j}$. For high SNR values, (11) can be further simplified as

$$P\Big\{\mathbf{X}_i\to\mathbf{X}_j\Big\}\leq\rho^{-n_rR_{ij}}\left(\prod_{r_{ij}=1}^{R_{ij}}\frac{1}{\lambda_{r_{ij}}^2}\right)^{n_r}. \tag{12}$$

The exponent of the SNR term ρ in (12) is $-n_rR_{ij}$. For all $i,j,i\neq j$, the PEP with the minimum value of R_{ij} governs the overall bit error rate (BER). Therefore, the achieved diversity order of TI-MBM, denoted by $D_{\text{TI-MBM}}$, is given by

$$D_{\text{ TI-MBM}} = n_r\min_{i,j\ i\neq j}\text{rank}(\Delta_{i,j}). \tag{13}$$

It should be noted that $\mathbf{X}_i$ and $\mathbf{X}_j$ are structured sparse matrices with the specific structure determined by the values of N, L, and K. Therefore, we divide the analysis into several cases to find the minimum rank of the difference matrix $\Delta_{i,j}$ and hence the diversity order of TI-MBM.

Case 1: $K=N$ (CPSC-MBM)

Lemma III.1. *The asymptotic diversity order of TI-MBM system with $K=N$ using maximum likelihood (ML) detection is n_r.*

Proof. When $K=N$, the TI-MBM system reduces to the cyclic-prefixed single-carrier MBM (CPSC-MBM) system [5]. To find the diversity order of this system, we need to find the minimum rank of the difference matrix $\Delta_{i,j}=\mathbf{X}_i-\mathbf{X}_j$. Let $\mathbf{e}_p$ denote an $N_m\times 1$ vector with its pth entry equal to one and other entries equal to zero. Then, from the MBM signal set in (2), an MBM signal vector using pth MAP and a modulation symbol $s\in\mathbb{A}$ can be written as $s\mathbf{e}_p$. With this, consider two TI-MBM transmit matrices $\mathbf{X}_i$ and $\mathbf{X}_j$ such that

$$\mathbf{X}_i = \begin{bmatrix}s_i\mathbf{e}_p & s_i\mathbf{e}_p & \cdots & s_i\mathbf{e}_p\\ s_i\mathbf{e}_p & s_i\mathbf{e}_p & \cdots & s_i\mathbf{e}_p\\ \vdots & & \vdots & \\ s_i\mathbf{e}_p & s_i\mathbf{e}_p & \cdots & s_i\mathbf{e}_p\end{bmatrix},$$

$$\mathbf{X}_j = \begin{bmatrix} s_j\mathbf{e}_p & s_j\mathbf{e}_p & \cdots & s_j\mathbf{e}_p \\ s_j\mathbf{e}_p & s_j\mathbf{e}_p & \cdots & s_j\mathbf{e}_p \\ \vdots & \vdots & & \\ s_j\mathbf{e}_p & s_j\mathbf{e}_p & \cdots & s_j\mathbf{e}_p \end{bmatrix}, \quad (14)$$

where $s_i, s_j \in \mathbb{A}$ and $s_i \neq s_j$. Let $\delta_{i,j} \triangleq s_i - s_j$. Then the difference matrix $\Delta_{i,j}$ is given by

$$\Delta_{i,j} = \begin{bmatrix} \delta_{i,j}\mathbf{e}_p & \delta_{i,j}\mathbf{e}_p & \cdots & \delta_{i,j}\mathbf{e}_p \\ \delta_{i,j}\mathbf{e}_p & \delta_{i,j}\mathbf{e}_p & \cdots & \delta_{i,j}\mathbf{e}_p \\ \vdots & \vdots & & \\ \delta_{i,j}\mathbf{e}_p & \delta_{i,j}\mathbf{e}_p & \cdots & \delta_{i,j}\mathbf{e}_p \end{bmatrix}, \quad (15)$$

whose rank is one. Therefore, from (13), the diversity order of TI-MBM with $K = N$ (i.e., CPSC-MBM) is n_r. □

The above result shows that conventional MBM, when used in a CPSC setting, does not extract the multipath diversity of the channel.

Case 2: $K = 1$, and $2 \leq L < N$

In this case, we consider TI-MBM with one active time-slot and derive the diversity orders for different choices of N and L. The results are summarized in the following Lemma.

Lemma III.2. *The asymptotic diversity order of TI-MBM with ML detection for*

1) *N even and $2 \leq L < N/2$ is Ln_r*
2) *N even and $N/2 \leq L < N$ is $\frac{Nn_r}{2}$*
3) *N odd composite and $2 \leq L < N - N/p_1$ is Ln_r, where p_1 is the smallest prime factor of N*
4) *N odd composite and $N - N/p_1 \leq L < N$ is $n_r(N - N/p_1)$, where p_1 is the smallest prime factor of N.*

Proof. The TI-MBM difference matrix $\Delta_{i,j}$ in (8) has the minimum number of non-zero rows (which is L, where $L < N$) when the TI-MBM signal matrices $\mathbf{X}_i$ and $\mathbf{X}_j$ use the same MAP in all the time-slots of a frame (say p , $1 \leq p \leq N_m$). Hence, the minimum rank of $\Delta_{i,j}$ is determined by $\mathbf{X}_i$ and $\mathbf{X}_j$ of the form

$$\mathbf{X}_i = \begin{bmatrix} s_0\mathbf{e}_p & s_1\mathbf{e}_p & \cdots & s_{N-1}\mathbf{e}_p \\ s_{N-1}\mathbf{e}_p & s_0\mathbf{e}_p & \cdots & s_{N-2}\mathbf{e}_p \\ \vdots & & \vdots & \\ s_{N-L+1}\mathbf{e}_p & s_{N-L+2}\mathbf{e}_p & \cdots & s_{N-L}\mathbf{e}_p \end{bmatrix}, \quad (16)$$

and

$$\mathbf{X}_j = \begin{bmatrix} s'_0\mathbf{e}_p & s'_1\mathbf{e}_p & \cdots & s'_{N-1}\mathbf{e}_p \\ s'_{N-1}\mathbf{e}_p & s'_0\mathbf{e}_p & \cdots & s'_{N-2}\mathbf{e}_p \\ \vdots & & \vdots & \\ s'_{N-L+1}\mathbf{e}_p & s'_{N-L+2}\mathbf{e}_p & \cdots & s'_{N-L}\mathbf{e}_p \end{bmatrix}, \quad (17)$$

where $s_t, s'_t \in \mathbb{A}_0, 0 \leq t \leq N-1$. Therefore,

$$\min_{i,j\ i\neq j} \text{rank}(\Delta_{i,j}) = \min \text{rank} \begin{bmatrix} \delta s_0\mathbf{e}_p & \delta s_1\mathbf{e}_p & \cdots & \delta s_{N-1}\mathbf{e}_p \\ \delta s_{N-1}\mathbf{e}_p & \delta s_0\mathbf{e}_p & \cdots & \delta s_{N-2}\mathbf{e}_p \\ \vdots & & \vdots & \\ \delta s_{N-L+1}\mathbf{e}_p & \delta s_{N-L+2}\mathbf{e}_p & \cdots & \delta s_{N-L}\mathbf{e}_p \end{bmatrix}, \quad (18)$$

where $\delta s_t = s_t - s'_t, 0 \leq t \leq N-1$. Since excluding the zero rows does not reduce the rank of a matrix, the above equation can be written as

$$\min_{i,j\ i\neq j} \text{rank}(\Delta_{i,j}) = \min \text{rank}(\mathbf{B}), \quad (19)$$

where

$$\mathbf{B} = \begin{bmatrix} \delta s_0 & \delta s_1 & \cdots & \delta s_{N-1} \\ \delta s_{N-1} & \delta s_0 & \cdots & \delta s_{N-2} \\ \vdots & & \vdots & \\ \delta s_{N-L+1} & \delta s_{N-L+2} & \cdots & \delta s_{N-L} \end{bmatrix}. \quad (20)$$

Since $K = 1$, the TI-MBM matrices $\mathbf{X}_i$ and $\mathbf{X}_j$ in (16) and (17) have only one non-zero entry in each row, and hence the matrix $\mathbf{B}$ can have two non-zero entries in each row. Let $\mathbf{X}_i$ and $\mathbf{X}_j$ use uth and vth time-slots, respectively, as the active slots such that $1 < u < v$. With this,

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & \cdots & s_u & \cdots & -s'_v & \cdots & \cdots 0 \\ 0 & 0 & 0 & \cdots & s_u & \cdots & -s'_v & \cdots 0 \\ \vdots & \vdots & & & \vdots & \vdots & & \\ 0 & \cdots & -s'_v & & & & s_u \cdots & 0 \end{bmatrix}. \quad (21)$$

The matrix $\mathbf{B}$ in (21) contains the first L rows of the circulant matrix $\mathbf{C}(0, 0, \cdots, s_u, \cdots, -s'_v, \cdots, 0)$. The rank of $\mathbf{C}$ is equal to $N - d$, where d is the degree of gcd$(f(y), y^N - 1)$ with $f(y) = s_u y^{u-1} - s'_v y^{v-1}$ being the associated polynomial of $\mathbf{C}$ [12]. The rank of $\mathbf{C}$ is minimum when the degree of the gcd$(f(y), y^N - 1)$, i.e., d, is maximum. This happens when the polynomials $y^N - 1$ and $f(y)$ have maximum number roots in common. The polynomials $y^N - 1$ and $f(y)$ have maximum number of roots in common when $s_u = s'_v$. That is, when $s_u = s'_v$,

$$\begin{aligned} \text{gcd}(f(y), y^N - 1) &= \text{gcd}(y^{v-u} - 1, y^N - 1) \\ &= y^{d_{max}} - 1, \end{aligned} \quad (22)$$

where d_{max} = gcd$(v - u, N)$. We now use the result in (22) to derive the diversity order of the TI-MBM system for the considered *Case 2*.

First, consider the case when N is even and $2 \leq L < N$. With this, $d_{max} = \text{gcd}(v - u, N) = N/2$, when $v - u = N/2$. Therefore,

$$\min\ \text{rank}(\mathbf{C}) = N - d_{max} = N - N/2 = N/2. \quad (23)$$

For a circulant matrix, the first r rows are linearly independent, where r is the rank of the matrix [14]. Combining this result with (23) and the fact that $\mathbf{B}$ contains the first L rows of $\mathbf{C}$, the minimum rank of $\mathbf{B}$ is given by

$$\min\ \text{rank}(\mathbf{B}) = \begin{cases} L, & \text{if } N \text{ is even and } L < N/2, \\ N/2, & \text{if } N \text{ is even and } N/2 \leq L \leq N. \end{cases} \quad (24)$$

From (19), the minimum rank of $\Delta_{i,j}$ is the same as that of $\mathbf{B}$, which is given in (24). Therefore, using (24) in (13) proves 1) and 2) of the Lemma stated above.

Next, consider the case when N is an odd composite and $2 \leq L < N$. Since N is an odd composite, it can be expressed as a product of powers of prime numbers as

$$N = p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n}, \quad (25)$$

where $2 < p_1 < p_2 < \cdots p_{n-1} < p_n < N$ are the prime factors of N and $\beta_1, \beta_2, \cdots, \beta_n \in \mathbb{Z}_+$ are their respective powers. With N as in (25), the d_{max} in (22) is N/p_1, when $v-u=N/p_1$. Therefore, when N is odd composite

$$\min \ \text{rank}(\mathbf{C}) = N - N/p_1. \tag{26}$$

As noted before, for a circulant matrix, the first r rows are linearly independent, where r is the rank of circulant matrix [14]. Combining this result with (26) and the fact that $\mathbf{B}$ contains the first L rows of $\mathbf{C}$, the minimum rank of $\mathbf{B}$ is given by

$$\min \ \text{rank}(\mathbf{B}) = \begin{cases} L \text{ if } L < N - N/p_1, \\ N - N/p_1 \text{ if } N - N/p_1 \leq L \leq N. \end{cases} \tag{27}$$

From (19), the minimum rank of $\Delta_{i,j}$ is same as that of $\mathbf{B}$, which is given in (27). Therefore, using (27) in (13) proves 3) and 4). □

Lemma III.2 shows that TI-MBM with $K=1$ can either partially or fully extract the channel multipath diversity based on the choice of N and the relation between L and N.

Case 3: N even and $K = N/2$

Lemma III.3. *The asymptotic diversity order of TI-MBM with $K = N/2$, N even, using ML detection is n_r.*

Proof. Consider two TI-MBM transmit matrices $\mathbf{X}_i$ and $\mathbf{X}_j$ with N even and $K = N/2$, given by

$$\mathbf{X}_i = \begin{bmatrix} s\mathbf{e}_q & \mathbf{0} & s\mathbf{e}_q & \cdots & s\mathbf{e}_q & \mathbf{0} \\ s\mathbf{e}_q & \mathbf{0} & s\mathbf{e}_q & \cdots & s\mathbf{e}_q & \mathbf{0} \\ \vdots & & \vdots & & & \\ s\mathbf{e}_q & \mathbf{0} & s\mathbf{e}_q & \cdots & s\mathbf{e}_q & \mathbf{0} \end{bmatrix},$$

$$\mathbf{X}_j = \begin{bmatrix} \mathbf{0} & -s\mathbf{e}_q & \cdots & \mathbf{0} & -s\mathbf{e}_q \\ \mathbf{0} & -s\mathbf{e}_q & \cdots & \mathbf{0} & -s\mathbf{e}_q \\ \vdots & & \vdots & & \\ \mathbf{0} & -s\mathbf{e}_q & \cdots & \mathbf{0} & -s\mathbf{e}_q \end{bmatrix}, \tag{28}$$

where $s, -s \in \mathbb{A}$. With this, the difference matrix $\Delta_{i,j} = \mathbf{X}_i - \mathbf{X}_j$ becomes

$$\Delta_{i,j} = \begin{bmatrix} s\mathbf{e}_q & s\mathbf{e}_q & \cdots & s\mathbf{e}_q \\ s\mathbf{e}_q & s\mathbf{e}_q & \cdots & s\mathbf{e}_q \\ \vdots & & \vdots & \\ s\mathbf{e}_q & s\mathbf{e}_q & \cdots & s\mathbf{e}_q \end{bmatrix}, \tag{29}$$

which clearly has rank one. Therefore, from (13), the diversity order of TI-MBM with $K = N/2$, N even, is n_r. □

The above result says that, when N is even, if half the number of time-slots of a TI-MBM frame are active, then such a TI-MBM system does not extract the multipath diversity.

Case 4: N is prime, $1 \leq K < N$, and $2 \leq L < N$

Lemma III.4. *The asymptotic diversity order of TI-MBM with N prime, $1 \leq K < N$, and $2 \leq L < N$ using ML detection is Ln_r.*

Proof. As mentioned in the proof of Lemma III.2, $\Delta_{i,j}$ in (8) has the minimum number of non-zero rows (which is L, where $L < N$) when the TI-MBM signal matrices $\mathbf{X}_i$ and $\mathbf{X}_j$ use the same MAP in all the time-slots of a frame (say p , $1 \leq p \leq N_m$). Hence, the minimum rank of $\Delta_{i,j}$ is determined by $\mathbf{X}_i$ and $\mathbf{X}_j$ as given in (16) and (17), respectively. Also, the minimum rank of $\Delta_{i,j}$ is the same as that of $\mathbf{B}$, as seen from (19) and (20). The matrix $\mathbf{B}$ in (20) forms the first L rows of the circulant matrix $\mathbf{C}(\delta s_0, \delta s_1, \cdots \delta s_{N-1})$. The polynomial $f(y) = \delta s_0 + \delta s_1 y + \cdots + \delta s_{N-1} y^{N-1}$ is the associated polynomial of $\mathbf{C}$ and the rank of $\mathbf{C}$ is equal to $N-d$, where d is the degree of gcd$(f(y), y^N - 1)$ [12]. The polynomial $y^N - 1$, can be expressed as

$$y^N - 1 = (y-1)\Phi_N(y), \tag{30}$$

where $\Phi_N(y) = y^{N-1} + y^{N-2} + \cdots + y + 1$ is called the Nth cyclotomic polynomial [13]. From Eisenstein's criterion [13], for N prime, $\Phi_N(y)$ can not be factored into polynomials with rational coefficients. That is, if $\Phi_N(y) = h(y)g(y)$, then the polynomials $h(y)$ and $g(y)$ have irrational coefficients. Further, Eisenstein's criterion says that, if $h(y)$ is a factor of $\Phi_N(y)$, then the only polynomial which when multiplied to $h(y)$ can result in a polynomial with rational coefficients is $g(y)$.

Let $\Phi_N(y) = h(y)g(y)$, where, as discussed above, $h(y)$ and $g(y)$ are polynomials with irrational coefficients. Also, let

$$\text{gcd}(f(y), y^N - 1) = \text{gcd}(f(y), (y-1)\Phi_N(y)) = h(y). \tag{31}$$

This means that $f(y)$ can be factored as $f(y) = h(y)p(y)$. However, since the coefficients of $f(y)$ are $\delta s_t, 0 \leq t \leq N-1$, and hence are not irrational, from Eisenstein's criterion, $p(y)$ must be equal to $kg(y)$. The condition $p(y) = kg(y)$ implies that $f(y) = k\Phi_N(y)$. However, for N odd and $K \neq N$, all the coefficients of $f(y)$, i.e., $\delta s_0, \delta s_1, \cdots, \delta s_{N-1}$, can not be equal to k for $k \neq 0$. This says that $h(y)$ cannot be a common factor for $y^N - 1$ and $f(y)$, and hence gcd$(f(y), y^N - 1) \neq h(y)$, which is a contradiction. Therefore, the only possible common factor between $y^N - 1$ and $f(y)$ is $y - 1$, resulting in gcd$(f(y), y^N - 1) = y - 1$, whose degree $d_{max} = 1$. From this, it follows that

$$\min \ \text{rank}(\mathbf{C}) = N - d_{max} = N - 1. \tag{32}$$

As noted before, for a circulant matrix, the first r rows are linearly independent, where r is the rank of circulant matrix [14]. Combining this result with (32) and the fact that $\mathbf{B}$ contains the first L rows of $\mathbf{C}$, the minimum rank of $\mathbf{B}$ is given by

$$\min \ \text{rank}(\mathbf{B}) = L, \ \text{for } N \text{ prime}, \ K \neq N, \ \text{and } L < N. \tag{33}$$

From (19), the minimum rank of $\Delta_{i,j}$ is the same as that of $\mathbf{B}$, which is given in (33). Therefore, using (33) in (13) proves that the diversity order for the considered case is Ln_r. □

The above Lemma says that, for N prime and $1 \leq K < N$, TI-MBM extracts the full multipath channel diversity.

IV. RESULTS AND DISCUSSIONS

In this section, we verify the diversity order expressions derived in the previous section through the monte-carlo simulations. Figure 1 shows the BER plots for two TI-MBM systems with the following configurations: i) System 1: $m_{rf} = 2,\ n_r = 1,\ N = 4,\ K = 2,\ L = 2$, and BPSK, and ii) System 2: $m_{rf} = 2,\ n_r = 2,\ N = 4,\ K = 4,\ L = 2$, and BPSK. While the TI-MBM System 1 uses N even and $K = N/2$ (*Case 3*), the System 2 is the CPSC-MBM system with $K = N$ (*Case 1*). From Fig. 1, it can be seen that the considered TI-MBM systems show higher slope at low-to-medium SNR values where practical values of BER are

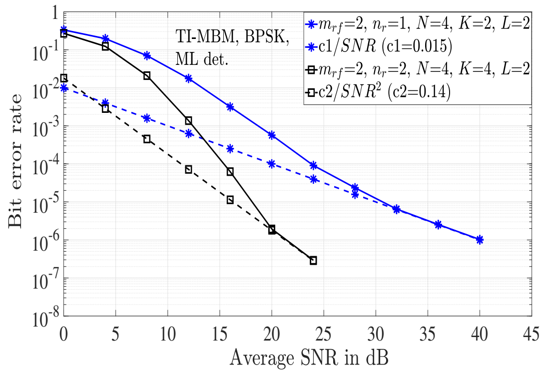

Fig. 1. BER performance of TI-MBM with $K = N$ and $K = N/2$.

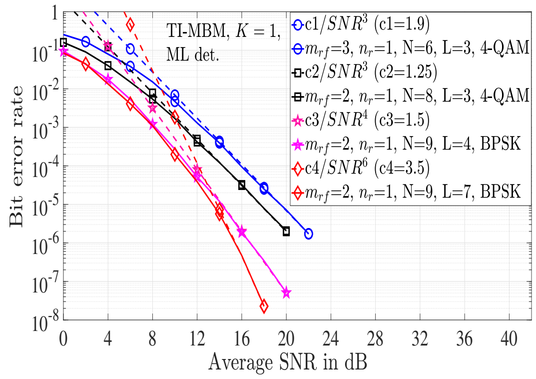

Fig. 2. BER performance of TI-MBM with $K = 1$, and $2 \leq L < N$.

achieved (10^{-4} to 10^{-6}). However, asymptotically (at high SNR values), the slope changes to n_r. This validates the analyses of *Cases 1* and *3*. This shows that, the CPSC-MBM system and the TI-MBM system with N even and $K = N/2$ can not extract the channel multipath diversity.

Figure 2 shows the BER plots for TI-MBM systems with $K = 1$ and the following configurations: i) System 1: $m_{rf} = 3$, $n_r = 1$, $N = 6$, $L = 3$, and 4-QAM, ii) System 2: $m_{rf} = 2$, $n_r = 1$, $N = 8$, $L = 3$, and 4-QAM, iii) System 3: $m_{rf} = 2$, $n_r = 1$, $N = 9$, $L = 4$, and BPSK, and iv) System 4: $m_{rf} = 2$, $n_r = 1$, $N = 9$, $L = 7$, and BPSK. System 1 has N even, $K = 1$, and $N/2 \leq L < N$. This is the case considered in 2) of Lemma III.2 (*Case 2*). The BER plot for System 1 shows a diversity order of 3, which agrees with the derived analytical diversity order value of $n_rN/2$. System 2 has N even, $K = 1$, and $2 \leq L < N/2$. This is the case 1) of Lemma III.2 (*Case 2*). The BER plot for System 2 shows a diversity order of 3, which agrees with the derived analytical diversity order value of n_rL. System 3 has N odd composite (with $p_1 = 3$), $K = 1$, and $2 \leq L < N - N/p_1$. This is the case 3) of Lemma III.2 (*Case 2*). The BER plot for System 3 shows a diversity order of 4, which confirms the derived analytical diversity order value of n_rL. System 4 has N odd composite (with $p_1 = 3$), $K = 1$, and $N - N/p_1 \leq L < N$, which is the case considered in 4) of Lemma III.2 (*Case 2*). The BER plot for System 4 shows a diversity order of 6, which validates the derived analytical diversity order value of $n_r(N - N/p_1)$. The above BER plots confirm that, for $K = 1$, if N is even and $N > 2L$ or if N is odd composite and $N > L/(1 - 1/p_1)$, then TI-MBM extracts the full channel diversity of n_rL.

Figure 3 shows the BER plots for four TI-MBM systems,

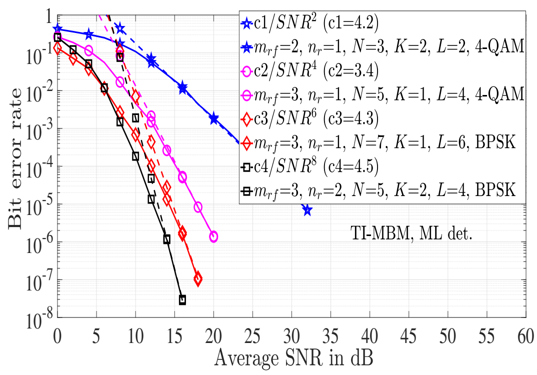

Fig. 3. BER performance of TI-MBM with N prime, $2 \leq K < N$, and $2 \leq L < N$.

all of which have N prime (3, 5, and 7) and $1 \leq K < N$. This is the case considered in Lemma III.4 (*Case 4*). From Fig. 3, it can be seen that all the systems show a diversity order equal to n_rL, which matches with the analytically derived diversity order in Lemma III.4, thus verifying the analysis. Therefore, TI-MBM with N prime and $1 \leq K < N$ also extracts the full channel multipath diversity of n_rL.

V. Conclusions

We considered time-indexed media-based modulation in a CPSC setting in ISI channels and carried out the diversity analysis. We derived closed-form expressions for the diversity order of TI-MBM for several cases of system parameter settings. Our results showed that, while the conventional CPSC-MBM cannot extract the multipath diversity of the channel, TI-MBM can achieve higher diversity order values. Some of the considered cases were shown to achieve full channel multipath diversity. We also provided simulation results that validated the analytically derived diversity order values.

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