X- and Y-Codes for MIMO Precoding

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***Abstract*—We consider a time division duplex (TDD) $n_t \times n_r$ multiple-input multiple-output (MIMO) system with known channel state information (CSI) at both transmitter and receiver. Using singular value decomposition (SVD) precoding at the transmitter, the MIMO channels are transformed into parallel subchannels. To improve the low diversity order, we propose X- and Y-Codes, prior to SVD precoding, to pair subchannels having different diversity orders. Specifically, a pair of channels is jointly encoded using a 2×2 real matrix, which is fixed *a priori* and does not change with each channel realization. Moreover, we propose X-, Y-Precoders with the same encoding matrices as X-, Y-Codes, which adapt to each channel realization. The optimal encoding matrices for X- and Y-Codes/Precoders are derived analytically to minimize the average error probability. Finally, we see that X-, Y-Codes/Precoders indeed achieve higher diversity gains at very low encoding/decoding complexity for both well- and ill-conditioned channels, respectively, when compared to other precoding schemes in the literature. We also observe that for the Rayleigh fading channel model X- and Y-Codes/Precoders exhibit the best average error performance.**

***Index Terms*—TDD MIMO, precoding, SVD, diversity**

I. Introduction

We consider time division duplex (TDD) $n_t \times n_r$ multiple-input multiple-output (MIMO) systems with known channel state information (CSI) at both transmitter and receiver. Precoding techniques are used to provide large performance gains in such scenarios [1], [2], [3], [4], [5], [6], [7], [8], [9], [10].

In this paper we propose X- and Y-Codes, which achieve both full rate and high diversity gain at low encoding and decoding complexity. The proposed codes are based upon singular value decomposition (SVD) of the MIMO channel, which transforms it into parallel subchannels. However, the diversity gain with simple SVD precoding is limited by the smallest singular value, and is therefore low. To improve on this, we use simple linear codes prior to SVD precoding. These codes are named X- and Y-Codes due to the structure of the encoder matrix, which pairs subchannels having different diversity orders. The 2×2 encoder matrices for each pair are fixed *a priori* and do not change with each channel realization. At the receiver, maximum likelihood decoding (MLD) is used.

For X-Codes, the encoder matrices are 2×2 real orthogonal matrices and thus parameterized by a single angle. It is shown that X-Codes have better error performance than other precoders. The MLD comprises of n_r 2-dimensional real sphere decoders (SDs). However the performance of X-Codes and other precoders is observed to degrade when the subchannel pairs are ill-conditioned. This degradation along with the motivation of further complexity reduction leads us to Y-Codes, which are based on a 2×2 upper left triangular encoder matrices. Y-Codes are parameterized with 2 parameters related to the power allocated to the two subchannels. The MLD complexity of Y-Codes is shown to be the same as that of a scalar channel (i.e, same as the linear precoders in [7], [8]). For X- and Y-Codes, the parameters of the encoder matrices are optimized analytically to minimize the average error probability.

Moreover, we propose the X- and Y-Precoders, employing the same pairing structure of X- and Y-Codes, where the encoder matrices are adapted in order to minimize error probability with each channel realization. It is observed that X- and Y-Codes/Precoders perform better than other low complexity precoders for both well- and ill-conditioned channels, irrespective of the channel fading distribution.

Another precoding scheme with similar structure as X-Codes, named *E-dmin*, has been recently proposed in [12]. However, E-dmin is only optimized for 4-QAM symbols, and suffers from loss in power efficiency with higher order modulation. The error performance and decoding complexity of X- and Y-Codes are shown to be better than those of E-dmin at a lower decoding complexity.

Notations: The fields of complex numbers, real numbers and non-negative real numbers are denoted by $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{R}^+$, respectively. The real and imaginary components of a complex argument are denoted by $\Re(\cdot)$ and $\Im(\cdot)$. Superscripts ^{T} and † denote transposition and Hermitian transposition, respectively. The $n \times n$ identity matrix is denoted by $\mathbf{I}_n$, and the zero matrix is denoted by $\mathbf{0}$. $\mathbb{E}[\cdot]$ is the expectation operator, $\|\cdot\|$ denotes the Euclidean norm, and $|\cdot|$ denotes the absolute value of a complex number. Furthermore, $\lfloor c \rfloor$ denotes the largest integer less than c. The set of integers $\{a \leq k \leq b\}$ is denoted by $[a, b]$.

II. System model and SVD precoding

We consider a TDD $n_t \times n_r$ MIMO $(n_r \leq n_t)$ with perfect knowledge of CSI at both transmitter and receiver. Let $\mathbf{x} = (x_1, \ldots, x_{n_t})^T$ be the vector of symbols transmitted by the n_t transmit antennas, and let $\mathbf{H} = \{h_{ij}\}$, $i = 1, \ldots, n_r$, $j = 1, \ldots, n_t$, be the $n_r \times n_t$ channel coefficient matrix, with h_{ij} as the complex channel gain between the j-th transmit antenna and the i-th receive antenna. The standard Rayleigh flat fading model is assumed with $h_{ij} \sim \mathcal{N}_c(0, 1)$, i.e., i.i.d. complex

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Gaussian random variables with zero mean and unit variance. The received vector with n_r components is given by

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} \tag{1}$$

where $\mathbf{n}$ is a spatially uncorrelated Gaussian noise vector such that $\mathbb{E}[\mathbf{n}\mathbf{n}^\dagger] = N_0\mathbf{I}_{n_r}$. Such a system has a maximum multiplexing gain of n_r. Let the number of transmitted *information symbols* be n_s ($n_s \leq n_r$). The information bits are first mapped to the information symbol vector $\mathbf{u} = (u_1, \ldots, u_{n_s})^T \in \mathbb{C}^{n_s}$, which is then mapped to the *coded symbols* $\mathbf{z} = (z_1, \ldots, z_{n_s})^T \in \mathbb{C}^{n_s}$ using a $n_s \times n_s$ matrix $\mathbf{G}$, i.e., $\mathbf{z} = \mathbf{G}\mathbf{u} + \mathbf{u}^0$, where $\mathbf{u}^0 \in \mathbb{C}^{n_s}$ is a displacement vector used to reduce the average transmitted power. Let $\mathbf{T}$ be the $n_t \times n_s$ precoding matrix yielding $\mathbf{x} = \mathbf{T}\mathbf{z}$. The transmission power constraint is given by $\mathbb{E}[\|\mathbf{x}\|^2] = P_T$ and we define the signal-to-noise ratio (SNR) as $\gamma \triangleq P_T/N_0$.

Since the proposed codes are based upon SVD precoding, we briefly illustrate it in the following. Taking the SVD of the MIMO channel yields $\mathbf{H} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}$, where $\mathbf{U} \in \mathbb{C}^{n_r \times n_r}$, $\mathbf{\Lambda} \in \mathbb{C}^{n_r \times n_r}$, $\mathbf{V} \in \mathbb{C}^{n_r \times n_t}$, and $\mathbf{U}\mathbf{U}^\dagger = \mathbf{V}\mathbf{V}^\dagger = \mathbf{I}_{n_r}$, $\mathbf{\Lambda} = \text{diag}(\lambda_1, \ldots, \lambda_{n_r})$, with $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_{n_r} \geq 0$. Let $\tilde{\mathbf{V}} \in \mathbb{C}^{n_s \times n_t}$ be the submatrix with the first n_s rows of $\mathbf{V}$. The standard SVD precoder uses $\mathbf{T} = \tilde{\mathbf{V}}^\dagger$, $\mathbf{G} = \mathbf{I}_{n_s}$, $\mathbf{u}^0 = \mathbf{0}$ and the receiver gets $\mathbf{y} = \mathbf{H}\mathbf{T}\mathbf{u} + \mathbf{n}$. Let $\tilde{\mathbf{U}} \in \mathbb{C}^{n_r \times n_s}$ be the submatrix with the first n_s columns of $\mathbf{U}$. The receiver computes

$$\mathbf{r} = \tilde{\mathbf{U}}^\dagger\mathbf{y} = \tilde{\mathbf{\Lambda}}\mathbf{u} + \mathbf{w} \tag{2}$$

where $\mathbf{w} \in \mathbb{C}^{n_s}$ is still an uncorrelated Gaussian noise vector with $\mathbb{E}[\mathbf{w}\mathbf{w}^\dagger] = N_0\mathbf{I}_{n_s}$, $\tilde{\mathbf{\Lambda}} \triangleq \text{diag}(\lambda_1, \lambda_2, \cdots \lambda_{n_s})$, and $\mathbf{r} = (r_1, \ldots, r_{n_s})^T$. Therefore the channel is transformed by SVD precoding into n_s parallel channels $r_i = \lambda_i u_i + w_i$, $i = 1, \ldots, n_s$, with non-negative fading coefficients λ_i. The overall error performance is dominated by the minimum singular value λ_{n_s}, (see e.g. [2]). When $n_s = n_r = n_t$, the diversity order is only 1. We therefore propose pairing of subchannels as a general technique to improve significantly the diversity gain.

III. Pairing good and bad subchannels

Without loss of generality, we consider only the full-rate SVD precoding scheme with even n_r and $n_s = n_r$. The matrix $\mathbf{G} \in \mathbb{C}^{n_r \times n_r}$ is now used to pair different subchannels so as to improve the overall diversity gain. Using the precoding matrix $\mathbf{T} = \mathbf{V}^\dagger \in \mathbb{C}^{n_t \times n_r}$, the transmitted vector $\mathbf{x}$ is given by

$$\mathbf{x} = \mathbf{V}^\dagger(\mathbf{G}\mathbf{u} + \mathbf{u}^0) \tag{3}$$

where $\mathbf{G}$ is fully characterized by the list of pairings and the 2×2 encoder matrices for each pair. Let the list of pairings be $\{(i_k, j_k) \in [1, n_r] \times [1, n_r], i_k < j_k, k = 1, \ldots n_r/2\}$. The information symbols the k-th pair u_{i_k} and u_{j_k} are jointly coded using a real 2×2 matrix $\mathbf{A}_k \triangleq \{a_{k,i,j}\}$, $i, j \in [1, 2]$. Each $\mathbf{A}_k$ is a submatrix of the code matrix $\mathbf{G} \triangleq \{g_{i,j}\}$, i.e.,

$$\begin{array}{ll} g_{i_k,i_k} = a_{k,1,1} & g_{i_k,j_k} = a_{k,1,2} \\ g_{j_k,i_k} = a_{k,2,1} & g_{j_k,j_k} = a_{k,2,2} \end{array} \tag{4}$$

We shall see later, that an optimal pairing in terms of overall diversity order is one in which the k-th subchannel is paired with the $(n_r - k + 1)$-th subchannel. For example, with this pairing and $n_r = 6$, the X-Code structure is given by

$$\mathbf{G} = \begin{bmatrix} a_{1,1,1} & & & & & a_{1,1,2} \\ & a_{2,1,1} & & & a_{2,1,2} & \\ & & a_{3,1,1} & a_{3,1,2} & & \\ & & a_{3,2,1} & a_{3,2,2} & & \\ & a_{2,2,1} & & & a_{2,2,2} & \\ a_{1,2,1} & & & & & a_{1,2,2} \end{bmatrix} \tag{5}$$

and the Y-Code structure is given by

$$\mathbf{G} = \begin{bmatrix} a_{1,1,1} & & & & & a_{1,1,2} \\ & a_{2,1,1} & & & a_{2,1,2} & \\ & & a_{3,1,1} & a_{3,1,2} & & \\ & & a_{3,2,1} & & & \\ & a_{2,2,1} & & & & \\ a_{1,2,1} & & & & & \end{bmatrix} \tag{6}$$

The names X- and Y-Codes are due to the structure in (5) and (6). Let $\mathbf{u}_k \triangleq [u_{i_k}, u_{j_k}]^T$. Due to the transmit power constraint and uniform power allocation between the $n_r/2$ pairs, the encoder matrices $\mathbf{A}_k$ must satisfy

$$\mathbb{E}\left[\|\mathbf{A}_k\mathbf{u}_k + \mathbf{u}_k^0\|^2\right] = \frac{2P_T}{n_r} \tag{7}$$

where $\mathbf{u}_k^0$ is the subvector of the displacement vector $\mathbf{u}^0$ for the k-th pair. The expectation in (7) is over the distribution of the information symbol vector $\mathbf{u}_k$. The matrices $\mathbf{A}_k$ can be either fixed *a priori* (X-, Y-Codes) or can change with every channel realization (X-, Y-Precoders).

Using (1) and (3), given the received vector $\mathbf{y}$, the receiver computes

$$\mathbf{r} = \mathbf{U}^\dagger\mathbf{y} - \mathbf{\Lambda}\mathbf{u}^0 = \mathbf{\Lambda}\mathbf{G}\mathbf{u} + \mathbf{w} = \mathbf{M}\mathbf{u} + \mathbf{w} \tag{8}$$

where $\mathbf{M} \triangleq \mathbf{\Lambda}\mathbf{G}$ is the equivalent channel gain matrix and $\mathbf{w} \triangleq \mathbf{U}^\dagger\mathbf{n}$ is a noise vector with the same statistics as $\mathbf{n}$. Further, let $\mathbf{r}_k \triangleq [r_{i_k}, r_{j_k}]^T$ and $\mathbf{w}_k \triangleq [w_{i_k}, w_{j_k}]^T$. Let $\mathbf{M}_k \in \mathbb{R}^{2\times 2}$ denote the 2×2 submatrix of $\mathbf{M}$ consisting of entries in the i_k and j_k rows and columns. Then (8) can be rewritten as

$$\mathbf{r}_k = \mathbf{M}_k\mathbf{u}_k + \mathbf{w}_k, \quad k = 1, \ldots, \frac{n_r}{2} \tag{9}$$

Further, separating real and imaginary components of $\mathbf{u}_k$, we have $\Re(\mathbf{u}_k)$ and $\Im(\mathbf{u}_k) \in \mathcal{S}_k$, where $\mathcal{S}_k$ is a finite signal set in the 2-dimensional (2-D) real space. Then MLD for the k-th pair is given by

$$\begin{aligned} \Re(\hat{\mathbf{u}}_k) &= \arg\min_{\Re(\mathbf{u}_k)\in\mathcal{S}_k} \|\Re(\mathbf{r}_k) - \mathbf{M}_k\Re(\mathbf{u}_k)\|^2 \\ \Im(\hat{\mathbf{u}}_k) &= \arg\min_{\Im(\mathbf{u}_k)\in\mathcal{S}_k} \|\Im(\mathbf{r}_k) - \mathbf{M}_k\Im(\mathbf{u}_k)\|^2 \end{aligned} \tag{10}$$

where $\hat{\mathbf{u}}_k$ is the output of the ML detector for the k-th pair.

With Rayleigh fading and MLD given by (10), it can be shown using the union bounding technique, that δ_k, the diversity order of the k-th pair is lower bounded by

$$\delta_k \geq (n_r - i_k + 1)(n_t - i_k + 1) \tag{11}$$

Let δ_{ord} denote the overall diversity order achieved. It can be shown that $\delta_{ord} \geq \min_k(\delta_k)$. We can therefore conclude that the pairing of subchannels $i_k = k, j_k = (n_r - k + 1)$, $k = 1, \ldots, n_r/2$ yields the best lower bound on δ_{ord}, which is given by

$$\delta_{ord} \geq \left(\frac{n_r}{2} + 1\right)\left(n_t - \frac{n_r}{2} + 1\right) \tag{12}$$

Note that this pairing results in a cross-form matrix $\mathbf{G}$.

IV. X-Codes and X-Precoders

In [14], we presented the design and optimization of X-Codes and also showed that the ML decoding complexity is that of n_r 2-dimensional real SDs [11]. For X-Codes, the encoder matrices are 2×2 real orthogonal matrices parameterized by a single angle, and are given by

$$\mathbf{A}_k = \begin{bmatrix} \cos(\theta_k) & \sin(\theta_k) \\ -\sin(\theta_k) & \cos(\theta_k) \end{bmatrix} \quad k = 1, \ldots n_r/2 \tag{13}$$

Each symbol in $\mathbf{u}$ takes values from a regular M^2-QAM constellation which consists of the M-PAM constellation

$$\mathcal{S} \triangleq \{\tau(2i - (M-1)) \ |i = 0, 1, \cdots (M-1)\}$$

used in quadrature on the real and the imaginary components of the channel, where

$$\tau \triangleq \sqrt{\frac{3P_T}{2n_r(M^2-1)}}$$

For optimal error performance of the k-th pair, we minimize w.r.t. θ_k the union bound on the average probability of error. The optimal θ_k, denoted by θ_k^*, is then given by

$$\theta_k^* = \arg \max_{\theta_k \in [0, \frac{\pi}{4}]} \min_{(p,q) \in \mathbb{S}_M} (p^2 + q^2)\cos^2(\theta_k - \varphi_{p,q}) \tag{14}$$

where $\mathbb{S}_M \triangleq \{(p,q) \neq (0,0) | 0 \leq |p| \leq (M-1), 0 \leq |q| \leq (M-1)\}$ and $\varphi_{p,q} \triangleq \tan^{-1}\left(\frac{q}{p}\right)$. For X-Precoders, the angle θ_k varies at each channel realization. Therefore, for X-Precoders θ_k is a function of the channel gain $(\lambda_{i_k}, \lambda_{j_k})$. The best angle for the k-th pair is given by

$$\tilde{\theta}_k(\lambda_{i_k}, \lambda_{j_k}) = \arg \max_{\theta_k \in [0, \frac{\pi}{4}]} \min_{(p,q) \in \mathbb{S}_M} d_k^2(p, q, \theta_k) \tag{15}$$

where

$$\begin{aligned} d_k^2(p,q,\theta_k) &\triangleq (p^2+q^2)(\lambda_{i_k}^2 \cos^2(\theta_k - \varphi_{p,q})) \\ &+ (p^2+q^2)(\lambda_{j_k}^2 \sin^2(\theta_k - \varphi_{p,q})) \end{aligned} \tag{16}$$

is the *effective Euclidean distance* between a two different information vectors $\Re(\mathbf{u}_k)$ and $\Re(\mathbf{v}_k)$. No closed form analytic expression exists for the optimization problem in (15) except for small values of M (see [15]).

It is observed that the error performance at high SNR is dependent on the minimum value of the effective Euclidean distance $d_k^2(p, q, \theta_k)$ over all $(p, q) \in \mathbb{S}_M$. Since $\lambda_{i_k} \geq \lambda_{j_k}$, the first term in (16) provides the larger contribution to the effective Euclidean distance. Therefore, codes should be designed so that the minimum separation of any two code vectors is larger along the first component. This, along with the goal of further complexity reduction, motivates the idea of Y-Codes.

V. Y-Codes and Y-Precoder

We consider encoder matrices $\mathbf{A}_k$ of the form

$$\mathbf{A}_k = \begin{bmatrix} a_k & 2a_k \\ 2b_k & 0 \end{bmatrix} \tag{17}$$

where $a_k, b_k \in \mathbb{R}^+$. For Y-Codes, $\mathcal{S}_k$ is the set of pairs of integers defined by the Cartesian product

$$\mathcal{S}_k \triangleq \left\{[0,1] \times \left[0, \frac{M}{2} - 1\right]\right\}$$

Both the real and imaginary components of the displacement vector for the k-th pair are given by

$$\Re(\mathbf{u}_k^0) = \Im(\mathbf{u}_k^0) = \left[-\frac{(M-1)a_k}{2}, -b_k\right]^T \tag{18}$$

We consider the 2-D codebook generated by applying $\mathbf{A}_k$ to the elements of $\mathcal{S}_k$ and adding a displacement of $\mathbf{u}_k^0$. The M code vectors in this codebook, $v = 1, \ldots, M$, are given by

$$Y_k(v) = \left[a_k\left(v - 1 - \frac{M-1}{2}\right), b_k(-1)^v\right]^T \quad v = 1, \ldots, M \tag{19}$$

and are represented in Fig. 1 by the black dots for $M = 8$. Due to the transmit power constraint in (7), both a_k and b_k satisfy

$$b_k^2 + a_k^2 \frac{M^2-1}{12} = \frac{P_T}{n_r} \tag{20}$$

Gray mapping is used for mapping the information bits to code vectors. For Y-Codes, the parameters a_k and b_k are fixed *a priori*, whereas, for the Y-Precoders, these are chosen every time the channel changes. The MLD is the same for both the real and imaginary components of $\mathbf{u}_k$ and therefore without loss of generality, we only discuss it for the real component. We first partition the 2-D received signal space $\mathbb{R}^2$ into $\left(\frac{M}{2} + 1\right)$ regions:

$$R_0 : \left\{[x,y]^T \in \mathbb{R}^2 \ | -\infty \leq \left(\frac{x}{\lambda_{i_k} a_k} + \frac{M-1}{2}\right) \leq 1\right\}$$

$$R_{\frac{M}{2}} : \left\{[x,y]^T \in \mathbb{R}^2 \ | (M-1) \leq \left(\frac{x}{\lambda_{i_k} a_k} + \frac{M-1}{2}\right) \leq +\infty\right\}$$

$$R_i : \left\{[x,y]^T \in \mathbb{R}^2 \ | (2i-1) \leq \left(\frac{x}{\lambda_{i_k} a_k} + \frac{M-1}{2}\right) \leq (2i+1)\right\}$$

where $i \in \left[1, \frac{M}{2} - 1\right]$. In Fig. 1, we illustrate the 5 regions with $M = 8$ for the real component of the k-th pair. We next show a low complexity MLD algorithm for Y-Codes. We first decode the received vector to the appropriate region R_i. This is easily accomplished by a rounding operation on the first component of the received vector. For example, in Fig. 1, the received vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ belong to R_0, R_1, and R_3 respectively. It can be shown that, once we know the region of the received vector, it is possible to directly find the ML code vector by checking a few linear relations between the 2 components of the received vector. Therefore the MLD complexity of Y-Codes is the same as that of a scalar channel. For example in Fig. 1, the received vector $\mathbf{p}_3$ is to the right of the perpendicular bisector between $Y_k(6)$ and $Y_k(8)$. The vector $\mathbf{p}_3$ is also above the perpendicular bisector between $Y_k(7)$ and $Y_k(8)$. From these two checks it can be easily concluded that the ML code vector is $Y_k(8)$.The ML decision regions are outlined in Fig. 1.

Thanks to the simple structure of the ML decision regions, it is possible to get analytic expression for the exact error probability of the k-the pair, in terms of certain integrals of the $Q(\cdot)$ function. These analytic expressions are presented in [15]. However no closed form expression exists for the error probability, and the integrals have to be computed numerically. For optimal error performance, the error probability is minimized w.r.t. (a_k, b_k) subject to the power constraint in (20). This minimization is done numerically and can be computed off-line since (a_k, b_k) are fixed *a priori*.

For Y-Precoders, the optimal (a_k, b_k) have to be computed for a given channel realization. Since evaluating the exact error probability is highly prohibitive, we consider minimizing the truncated union bound on the error probability for the k-th pair. This minimization is equivalent to maximizing the effective minimum distance w.r.t. (a_k, b_k) subject to the power constraint in (20). For Y-Codes, the effective minimum distance is given by

$$d^2_{k,\min}(a_k, b_k) \triangleq \min_{v \neq w} \left(\lambda^2_{i_k} a^2_k (v-w)^2 + \lambda^2_{j_k} b^2_k ((-1)^v - (-1)^w)^2 \right) \tag{21}$$

where v and w are distinct indices of the codebook. The optimal choice of (a_k, b_k), denoted by (a^*_k, b^*_k), is given by

$$(a^*_k, b^*_k) = \begin{cases} \left(\sqrt{\frac{12 P_T}{n_r(M^2-1)}}, 0 \right) & \beta^2_k \geq \frac{M^2-1}{3} \\ \left(\sqrt{\frac{4 P_T}{3 n_r(\beta^2_k + M')}}, \beta_k \sqrt{\frac{P_T}{n_r(\beta^2_k + M')}} \right) & \beta^2_k < \frac{M^2-1}{3} \end{cases} \tag{22}$$

where $M' = \frac{M^2-1}{9}$ and $\beta_k = \lambda_{i_k}/\lambda_{j_k}$.

VI. Simulation Results and Complexity

We assume $n_r = n_t$. Comparisons are made with 1) the E-dmin precoder in [12]), 2) the Arithmetic mean BER precoder (ARITH-MBER) in [7], 3) the Equal Energy linear precoder (EE) in [8], 4) the TH precoder in [6], and 5) CI precoder [3].

In Fig. 2, we plot the error performance of all precoding schemes for a 2×2 MIMO system with $\gamma = 26$ dB, as a function of the channel condition number $\beta = \lambda_1/\lambda_2$. We fix ${\lambda_1}^2 + {\lambda_2}^2 = 1$, and the target spectral efficiency to 8 bps/Hz with 16-QAM signalling. We observe that X-Precoder has the best error performance for low values of β (well-conditioned channel), whereas Y-Precoder has the best performance for high values of β (ill-conditioned channel). Therefore, a combination of X- and Y-Precoder would have

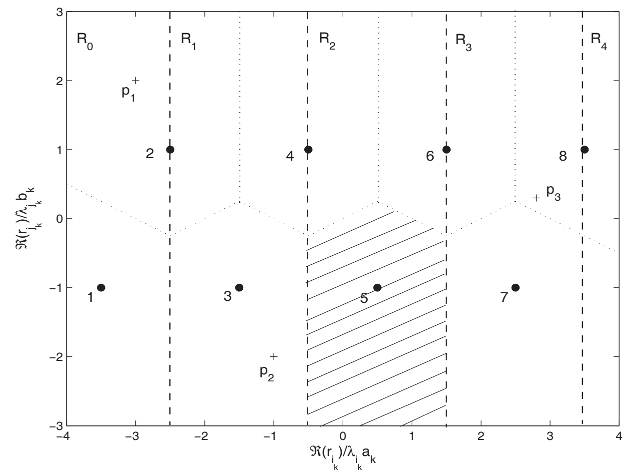

Fig. 1. Received signal space for the real component of the k-th pair. For $M = 8$ we have 5 regions demarcated by vertical dashed lines. The scaled codebook vectors are labeled with their corresponding index. Dotted lines demarcate the boundary of the ML decision regions. The hatched area illustrates the ML decision region of $Y_k(5)$.

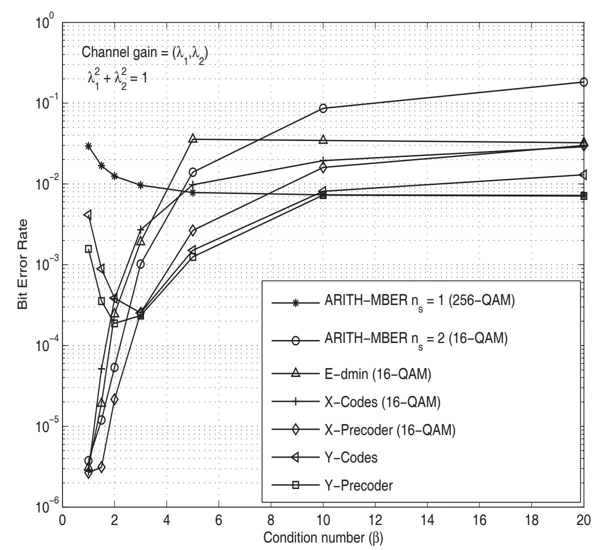

Fig. 2. Effect of the channel condition number on error performance of various precoders for a 2×2 system and a target spectral efficiency of 8bps/Hz.

the best error performance, when compared to other precoders considered, *irrespective of the channel fading distribution*.

When the channel coefficients are Rayleigh distributed, both E-dmin and X-, Y-Codes have the best diversity order among all precoders considered (except CI). The CI scheme achieves infinite diversity, but it suffers from power enhancement at the transmitter. However, we shall later see that E-dmin achieves this diversity gain at a much higher complexity than X-, Y-Codes.

In Fig. 3, we plot the average bit error rate (BER) for $n_r = n_t = 4$ at spectral efficiency of 16 bps/Hz with 16-QAM signalling. Rayleigh fading is assumed. It is observed that Y-Precoder performs the best. The ARITH-MBER and E-dmin perform 2.6 and 3.5 dB worse than Y-Precoder at a

BER of 10^{-3}. E-dmin has poor performance since the precoder proposed in [12] has been optimized for 4-QAM.

It is also observed through simulations that with Rayleigh fading, Y-Codes perform much better than X-Codes, though we do not present illustrations due to lack of space. We also observed that X-Precoder performs significantly better than X-Codes, when higher order modulation is used. However, Y-Precoder was found to be only marginally better than Y-Codes, and this is due to the fact that the optimization of the encoder matrices was done on an upper bound to the exact error probability. Please refer to [15] for a more detailed discussion. We next discuss the encoding and decoding complexity of the proposed codes in comparison with that of the other schemes.

The encoding complexity of all the schemes is $O(n_r n_t)$ due to the preprocessing filter. However in terms of actual number of operations, CI, E-dmin and X-, Y-Codes would have the lowest encoding complexity, since 1) linear precoders need to compute an extra pre-processing matrix (in addition to SVD); 2) THP does successive interference pre-cancelation (in addition to QR). The decoding complexity of all the schemes have a square dependence on n_r, due to the post-processing matrix filter at the receiver. The linear precoders, CI and THP employ post processing at the receiver, which enables independent decoding for each subchannel. E-dmin and X-Codes use sphere decoding to jointly decode pairs of subchannels. ML decoding for X-Codes is accomplished by using n_r 2-D real SDs as compared to $\frac{n_r}{2}$ 4-D real SDs required by E-dmin. The average complexity of SD is cubic in the number of dimensions [13], and therefore X-Codes have a much lower decoding complexity compared to E-dmin. The ML decoding complexity of Y-Codes is even lower, and is equal to the ML decoding complexity of a scalar channel (i.e., same as the linear precoders, CI and THP).

VII. Conclusion

We proposed X-, Y-Codes/Precoders which can achieve full-rate and high diversity at low complexity by pairing the subchannels prior to SVD precoding. One way of pairing the subchannels is to use rotation based encoding (X-Codes/Precoders) for well-conditioned channels, while the other is to use an upper left triangular code generator matrix (Y-Codes/Precoders) for ill-conditioned channels. We observe that the proposed scheme achieves better error performance than other existing precoders at a very low complexity.

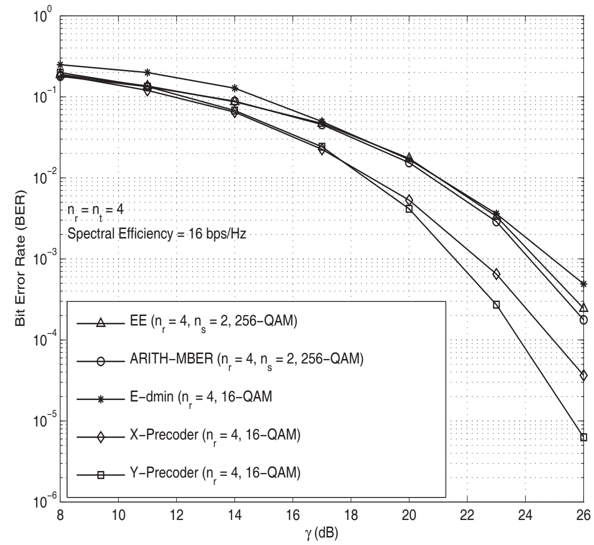

Fig. 3. BER comparison between various precoders for $n_r = n_t = 4$ and target spectral efficiency of 8, 16 bps/Hz.

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