E1 244: Detection and Estimation Theory (2018)
Homework 1

1. (a) (The minimax inequality) Show the basic min-max inequality for a function f of two variables: $\min_{a\in A}\max_{b\in B} f(a,b) \geq \max_{b\in B}\min_{a\in A} f(a,b)$.

   (b) (Minimum of linear functions is concave) Let f and g be two affine, real-valued functions defined on the interval $[u,v] \in \mathbb{R}$, i.e., $f(x) = ax+b$ and $g(x) = cx+d$ for some constants a,b,c,d. Show that the function h, defined by $h(x) = \min(f(x), g(x))$ on $[u,v]$, is concave; in other words, show that $\forall a,b \in [u,v]$ and $\lambda \in [0,1]$,

$$h(\lambda a + (1-\lambda)b) \geq \lambda h(a) + (1-\lambda)h(b).$$

2. Consider testing the hypotheses H_0 : Y has density $p_0(y) = \frac{1}{2}e^{-|y|}$, $y \in \mathbb{R}$, vs. H_1 : Y has density $p_1(y) = e^{-2|y|}$, $y \in \mathbb{R}$.

   (a) Find the Bayes rule and minimum Bayes risk for testing H_0 vs. H_1 under uniform costs and priors $(\pi_0, \pi_1) = \left(\frac{1}{4}, \frac{3}{4}\right)$.

   (b) Find the minimax rule and minimax risk under uniform costs.

3. Suppose $\mathbf{Y}$ is a random variable that under hypothesis H_0 has pdf,

$$p_0(y) = \begin{cases} \frac{2}{3}(y+1), & 0 \leq y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

   and, under hypothesis H_1 has pdf

$$p_1(y) = \begin{cases} 1, & 0 \leq y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find the Bayes rule and minimum Bayes risk for testing H_0 *versus* H_1 with uniform cost, and equal priors.

   (b) Draw the two pdfs, and identify the threshold τ in the Bayes rule assuming uniform cost, and equal priors. Discuss the effect of π_0 on the threshold τ (Hint: you can use the posterior probabilities $\pi_i(y)$ to illustrate).

   (c) Find the minimax rule and minimax risk for uniform costs.

4. For the binary channel with crossover probabilities λ_0, λ_1 discussed in class, find (a) the minimax risk, (b) a randomized decision rule $\delta(y)$ which achieves the minimax risk, and (c) the least favorable prior π_L, for each of the following cases of the channel:

   a) $\lambda_0 = 0.4, \lambda_1 = 0.2$,

b) $\lambda_0 = 0.6, \lambda_1 = 0.45$.

Assume costs to be uniform.
(You must express your decision rules explicitly in the form $\delta(y) = $ ______ for each observation y.)

5. Suppose you observe one random variable $Y \in \mathbb{R}$ and must decide between

$$H_0 : \; Y \sim \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-y^2}{2}\right)$$

versus

$$H_1 : \; Y \sim \frac{1}{2\sqrt{2\pi}} \exp\left(\frac{-y^2}{8}\right)$$

(a) Assume uniform cost assignment and find a Bayes decision rule for a general prior.

(b) Try and find the least favourable prior and minimax decision rule for this problem (assuming uniform costs). Hint: You may use the Taylor approximation of the $Q-$function around 0: $Q(x) \approx \frac{1}{2} - \frac{x}{\sqrt{2\pi}}$.