E1 244: Detection and Estimation Theory (2018)
Homework 3

Signal Detection

1. Consider the model $Y_k = \theta^{1/2} s_k R_k + N_k$, $k = 1, 2, \ldots, n$, where $s_1, s_2, \ldots, s_n$ is a known signal sequence , $\theta \geq 0$ is a constant, and $R_1, R_2, \ldots, R_n, N_1, N_2, \ldots, N_n$ are i.i.d. $\mathcal{N}(0, 1)$ random variables.

   (a) Consider the hypothesis pair $H_0 : \theta = 0$ versus $H_1 : \theta = A$, where A is a known positive constant. Describe the structure of the Neyman-Pearson detector.

   (b) Consider now the hypothesis pair $H_0 : \theta = 0$ versus $H_1 : \theta > 0$. Under what conditions on $s_1, s_2, \ldots, s_n$ does a UMP test exist?

   (c) For the hypothesis pair in part (b) with general $s_1, s_2, \ldots, s_n$, what is a locally optimum detector?

Chernoff Bounding Technique

2. Derive the following inequalities (left as exercises in class): $s \in [0, 1]$

$$P_e \leq \pi_0 e^{-s\tau} \int_{\Gamma_1} (L(y))^s p_0(y) dy + \pi_1 e^{(1-s)\tau} \int_{\Gamma_0} (L(y))^s p_1(y) dy$$

   and,

$$P_e \leq \max\{\pi_0, \pi_1 e^{\tau}\} \exp\{\mu_{T,0}(s) - s\tau\}.$$

3. For a random variable T, show that the function $\mu_T(s) - s\tau$ is convex, where $\mu_T(s) = \log E[e^{sT}]$ and τ is a constant. Hints: 1) Adding a linear function does not affect convexity. 2) You may have to use the following inequality due to Hölder,

$$E[UV] \leq (E[U^p])^{1/p} (E[V^q])^{1/q}$$

   where p and q are real numbers $1 \leq p, q \leq \infty$ such that $1/p + 1/q = 1$ (Observe that this is a generalization of the Cauchy-Schwarz inequality where $p = q = 2$.)

4. *Sequential Probability Ratio testing:*
   For the sequential binary hypotheses testing problem,

$$H_0 : Y_k \overset{\text{iid}}{\sim} \mathcal{N}(1, 1), \quad k = 1, 2, \ldots$$

$$\text{vs.}$$

$$H_1 : Y_k \overset{\text{iid}}{\sim} \mathcal{N}(-1, 1), \quad k = 1, 2, \ldots,$$

find the false alarm probability(α), missed detection probability(γ) and the expected sample size for the SPRT$(0.9, 1.1)$ under both the hypotheses, using Wald's approximations as discussed in class.

5. *Estimators:*
   Let $X_1, \ldots, X_n \sim \text{Uniform}[a, b]$, where a, b are unknown parameters and $a < b$.

   (a) Derive the method of moments estimator for (a, b).

   (b) Derive the maximum likelihood estimator for (a, b).

6. *Method of moments estimation:*
   Let $X_1, X_2, ..., X_n$ be gamma distributed i.i.d random variables with parameters $\alpha > 0$ and $\theta > 0$; the probability density function for each X_i is given by

   $$f(x) = \frac{1}{\Gamma(\alpha)\,\theta^\alpha}\, x^{\alpha-1} e^{-x/\theta},\ x > 0.$$

   Find the method of moments estimators for α and θ based on $X_1, \ldots, X_n$. (You can find moment formulas for the gamma distribution on the Internet.)

7. *Minimum-variance estimation:*
   Let $X_1, ..., X_n$ be i.i.d random samples drawn from the Geometric(θ) distribution, $\theta \in \Theta = [0, 1]$, with probability mass function $p_i = \theta(1-\theta)^{i-1}$, $i = 1, 2, 3, \ldots$

   Find the Cramér-Rao lower bound for the variance of any unbiased estimator of $(1-\theta)/\theta^2$ (which happens to be the variance of the X_is).