E1 244: Detection and Estimation Theory (2018)
Homework 4

1. Let $X_1, X_2, \ldots, X_n$ be an iid random sample drawn from a Poisson distribution with parameter $\lambda > 0$, $X_i \sim Poi(\lambda)$. Consider the two unbiased estimators $\overline{X} = \frac{1}{n}\sum\limits_{i=1}^{n} x_i$, and $S^2 = \frac{1}{n-1}\sum\limits_{i=1}^{n} x_i^2$, of λ. Which one has lower Mean-Square-Error (MSE)?

2. Suppose $X_1, \ldots, X_n$ are iid samples from the Exponential distribution with parameter $\lambda > 0$. Find the Cramér-Rao lower bound for unbiased estimation of λ. Does there exist an unbiased estimator of λ whose variance is equal to the lower bound?

3. Suppose $X_1, X_2, \ldots, X_n$ are iid uniform observations on the interval $(\theta, \theta + 1)$, $-\infty < \theta < \infty$.
   (a) Can you find a sufficient statistic for θ?
   (b) Can you find a minimal sufficient statistic for θ?

4. *Bayes estimator*
   Suppose X is a sample from the binomial distribution with parameters $n \in \mathbb{N}$ (known) and $\theta \in [0, 1]$ (unknown). Consider the squared loss function $L(\theta, a) = (\theta - a)^2$ for $\theta, a \in [0, 1]$.
   (a) Find the Bayes estimator, as a function of X, for the squared loss function and the Beta(a, b) prior for $\theta \in [0, 1]$, i.e., $\pi(\theta) = \mathbb{1}\{0 \leq \theta \leq 1\}\, \theta^{a-1}(1-\theta)^{b-1}/B(a, b)$. (Use the Internet for properties of the Beta distribution.)
   (b) Compute the risk of the Bayes estimator you found above.

5. *Linear regression*
   Suppose the random variables $X_1, X_2, \ldots, X_n$ satisfy
   $$X_i = \alpha s_i + \eta_i, \quad i = 1, 2, \ldots, n$$
   where $s_1, s_2, \ldots, s_n$ are known and fixed constants, $\eta_1, \eta_2, \ldots, \eta_n$ are iid $\mathcal{N}(0, \sigma^2)$, σ^2 unknown.
   (a) Find a two-dimensional sufficient statistic for (α, σ^2).
   (b) Find the MLE of α and find its bias. What is the distribution of the MLE?
   (c) Show that $\frac{\sum X_i}{\sum s_i}$ is an unbiased estimator of α. Calculate its variance and compare it with the variance of the MLE of α.
   (d) Show that $\sum(X_i/s_i)$ is also an unbiased estimator of α. Calculate its variance. Finally compare the MSE of all the above estimators of α, in particular can you write an ordering of the corresponding MSEs?

6. Show that the family of Binomial(n, p) probability distributions, where $n \in \mathbb{N}$ is a fixed integer and p ranges from 0 to 1, is complete.

7. Suppose $X_1, \ldots, X_n$ are iid Ber(p), where $0 < p < 1$.

   (a) Show that the variance of MLE of p attains the Cramer-Rao lower bound.

   (b) For $n \geq 4$, show that the product $X_1X_2X_3X_4$ is an unbiased estimator of p^4. Using this, can you find the best unbiased estimator of p^4? (*Hint: Use the Rao-Blackwell theorem to improve the estimator. Observe that the Binomial family is complete.*).

8. *Gaussian conditioning or 'swiss army knife' lemma*

   (a) Prove the following useful property about Gaussians. Suppose A and B are jointly Gaussian random vectors of dimensions m and n, respectively, with

$$\begin{aligned} \mathbb{E}[A] = \mu_A, \quad \mathbb{E}[B] = \mu_b, \\ \mathrm{Cov}[A] =& \Sigma_A \text{ (invertible)}, \quad \mathrm{Cov}[B] = \Sigma_B, \text{and} \\ \mathrm{Cov}[A, B] =& \mathbb{E}[(A - \mu_A)(B - \mu_B)^T] = \Sigma_{AB} = \Sigma_{BA}^T. \end{aligned}$$

   Then, the conditional distribution of B given A is Gaussian with mean

$$\mu_{B|A} = \mu_B + \Sigma_{BA}\,\Sigma_A^{-1}(A - \mu_A)$$

   and covariance

$$\Sigma_{B|A} = \Sigma_B - \Sigma_{BA}\,\Sigma_A^{-1}\,\Sigma_{AB}.$$

   (b) If X and Y are jointly Gaussian random variables (dimension 1 vectors), then what can you say about the relationship between (a) the minimum mean-square error estimator of X based on Y and (b) the minimum mean-square error <u>linear</u> estimator of X based on Y?

9. *Recursive parameter estimation*
   Suppose X is a zero-mean Gaussian random variable with variance σ_0^2. At each time instant $k = 1, 2, \ldots$, let $Z_k = X + V_k$ denote the kth observation of X with independent Gaussian noise $V_k \sim \mathcal{N}(0, \sigma^2)$.

   (a) Find a <u>recursive</u> minimum mean-square error (MMSE) estimator $\hat{X}_k$ of X at each time k, as a function of the fresh observation Z_k and the MMSE estimator $\hat{X}_{k-1}$ at time $k - 1$.

   (b) What is the value of $\hat{X}_1$ when $\sigma^2 = 0$? Repeat for $\sigma^2 = \infty$.

10. Let X, Y_1 and Y_2 be random variables with finite mean and covariance, and denote $\sigma_{AB} = \mathrm{Cov}[A, B]$ for each pair of random variables $A, B \in \{X, Y_1, Y_2\}$. Let $L_{12}^X(Y_1, Y_2)$ denote the best linear mean-squared-error (MSE) estimator of X given Y_1, Y_2; likewise, let $L_i^X(Y_i)$ denote the best linear MSE estimator of X given Y_i, $i = 1, 2$. Suppose $\sigma_{Y_1Y_2} = \mathbb{E}[Y_1] = \mathbb{E}[Y_2] = 0$.

    (a) Assuming that the 1-step or individual best estimators $L_i^X(Y_i)$, $i = 1, 2$, along with the mean $\mathbb{E}[X]$, are known, can you express $L_{12}^X(Y_1, Y_2)$ in terms of them?

    (b) Can you express the MSE of $L_{12}^X(Y_1, Y_2)$ in terms of the given covariances?