Name: ______________________________

Question:	1	2	3	4	Total
Points:	10	10	5	10	35
Score:					

E1 244 - Detection & Estimation Theory - Mid Term exam

Instructions

- The total time for this test is 1.5 hours.
- Write your name on this question sheet.
- Attach your solution sheets to this question sheet and return everything.
- No calculators or electronic aids are permitted.
- Academic dishonesty will not be tolerated.

Useful formulas and definitions:

- **Gaussian probability distribution.** The Gaussian probability distribution $\mathcal{N}(\mu, \sigma^2)$ is defined by the probability density function $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$, $x \in \mathbb{R}$.
- **Bayesian composite hypothesis testing.** For the composite hypothesis test $H_0 : Y \sim \mathbb{P}_\theta$, $\theta \in \Theta_0$ vs. $H_1 : Y \sim \mathbb{P}_\theta$, $\theta \in \Theta_1$, with Θ_0 and Θ_1 disjoint, let a prior distribution be π on $\Theta \equiv \Theta_0 \cup \Theta_1$, and the costs be $C[i, \theta]$ for each hypothesis $i \in \{0, 1\}$ and parameter θ. The Bayes risk of a decision rule $\delta : \Gamma \to \{0, 1\}$ is defined to be the quantity

$$\int_\Theta \left(\sum_{i=0}^{1} \mathbb{P}_\theta[\delta(Y) = i]\, C[i, \theta] \right) \pi(\theta)\, d\theta.$$

- **Series sums.** $\sum_{i=1}^n i = \frac{n(n+1)}{2}$, $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$

1. **Testing for mixtures**
Given two probability distributions P_1 and P_2, here is how random variables $Y_1, \ldots, Y_n$ are defined to be generated by the <u>mixture</u> of P_1 and P_2: First, draw $Z \sim$ Bernoulli(1/2). If $Z = 0$, then generate $Y_1, \ldots, Y_n \overset{\text{iid}}{\sim} P_1$, and if $Z = 1$, then generate $Y_1, \ldots, Y_n \overset{\text{iid}}{\sim} P_2$.

   Consider testing if $H_0 : Y_1, \ldots, Y_n$ are generated iid Bernoulli(1/2), or $H_1 : Y_1, \ldots, Y_n$ are generated by the mixture of Bernoulli(1/4) and Bernoulli(3/4).

   (a) **(5 points)** Write down a form for the Bayes-optimal test.

   (b) **(5 points)** Argue briefly (in words) what happens to the test statistic in your answer above, under each hypothesis, when n is large (i.e., why do you expect the test to work?).

2. Consider the hypothesis test

$$H_0 : Y_1, \ldots, Y_n \overset{\text{iid}}{\sim} \text{Binomial}(m, \theta_0), \quad \text{vs.}$$
$$H_1 : Y_1, \ldots, Y_n \overset{\text{iid}}{\sim} \text{Binomial}(m, \theta), \quad \theta > \theta_0,$$

   where $\theta_0 \in (0, 1)$ and m are assumed to be known constants.

   (a) **(5 points)** For a <u>fixed</u> $\theta > \theta_0$, what is the general form of the Neyman-Pearson optimal test?

   (b) **(5 points)** Does there exist a Uniformly Most Powerful (UMP) test for θ_0 vs. $\theta > \theta_0$?

3. **(5 points)** Suppose you are looking to detect the linear signal $s_k = \beta k$, $k = 1, \ldots, n$, in additive, iid Gaussian noise of mean 0 and variance σ^2, where $\beta\ \sigma^2$ are known. Determine the Neyman-Pearson optimal detector and its detection-vs-false alarm probability performance (in terms of the standard normal cdf Φ).

4. **Composite hypothesis testing**
Consider the hypothesis test

$$H_0 : X \sim \mathcal{N}(\theta, 1), \theta < 0$$
$$\text{vs.}$$
$$H_1 : X \sim \mathcal{N}(\theta, 1), \theta \geq 0,$$

   where θ is assumed to have the $\mathcal{N}(1, 1)$ prior distribution. Let the costs be uniform for $\theta < 0$ and $\theta \geq 0$, i.e., for each $i \in \{0, 1\}$ and $\theta \in \mathbb{R}$,

$$C[i, \theta] = \begin{cases} 0, & \text{if } (i = 0, \theta < 0) \text{ or } (i = 1, \theta \geq 0) \\ 1, & \text{otherwise.} \end{cases}$$

   (a) **(5 points)** What is the posterior probability distribution of θ given the observation $X = x$?

   (b) **(5 points)** What is the Bayes optimal decision rule for the observation $X = x$?