Name: ______________________________

Question:	1	2	3	4	Total
Points:	9	10	18	8	45
Score:					

E1 244 - Detection & Estimation Theory (2019) Final Exam

Instructions

- This exam has a total of 4 questions with a maximum score of 45 points. The total time is 3 hours.
- There are partial marks for subquestions, so please attempt as many parts as possible.
- Write your name at the top of this cover sheet.
- Attach your solution sheets to this cover sheet and return everything including rough work.
- No class notes, calculators or electronic aids are permitted.
- Academic dishonesty will not be tolerated.

Useful formulas, definitions, etc.

- **Generalized Likelihood Ratio Test (GLRT).** The GLRT for testing the composite hypotheses $H_0 : Y \sim p_\theta, \theta \in \Lambda_0$ vs $H_1 : Y \sim p_\theta, \theta \in \Lambda_1$ has the form

$$\frac{\max_{\theta \in \Lambda_1} p_\theta(y)}{\max_{\theta \in \Lambda_0} p_\theta(y)} \gtreqless \eta.$$

- **Constrained quadratic minimization.** For $f(x) = ax^2 + bx + c$ and $u \in \mathbb{R}$,

$$\arg\min_{x \geq u} f(x) = \max\left\{u, \arg\min_{x \in \mathbb{R}} f(x)\right\} \quad \text{and}$$
$$\arg\min_{x \leq u} f(x) = \min\left\{u, \arg\min_{x \in \mathbb{R}} f(x)\right\}.$$

- **Geometric probability distribution.** For $0 \leq \theta \leq 1$, the Geom(θ) distribution has pmf $p[i] = \theta(1-\theta)^{i-1}$, $i = 1, 2, \ldots$
- **Negative binomial probability distribution.** For $r = 1, 2, \ldots$ and $0 \leq \theta \leq 1$, the NegBin(r, θ) distribution has pmf $p[i] = \binom{i+r-1}{i} \cdot \theta^r \cdot (1-\theta)^i$, $i = 0, 1, 2, \ldots$

1. **Mixture testing**
   Consider testing the hypotheses

$$\begin{aligned} H_0 &: (Y_1,\ldots,Y_n) \overset{\text{iid}}{\sim} \text{Unif}([k]), \quad \text{vs.} \\ H_1 &: (Y_1,\ldots,Y_n) \sim P_\epsilon, \end{aligned}$$

where k is a known positive integer, $\text{Unif}([k])$ is the uniform probability distribution over the alphabet $[k] = \{1,\ldots,k\}$, $\epsilon \in (0,1)$ is known, and P_ϵ is the joint probability distribution of $Y_1,\ldots,Y_n$ defined as follows: First, $Z \sim \text{Unif}([k])$ is sampled uniformly. Then (given Z), $Y_1,\ldots,Y_n$ are sampled iid from the probability distribution P_ϵ^Z over $[k]$, under which each outcome $j \in [k]$ has probability

$$P_\epsilon^Z[j] = \begin{cases} \frac{1}{k} + \epsilon, & \text{if } j = Z \\ \frac{1}{k} - \frac{\epsilon}{k-1}, & \text{if } j \in [k] \setminus \{Z\}. \end{cases}$$

   (a) **(2 points)** For a general sequence $(y_1,\ldots,y_n)$ with elements from $[k]$, write down the probability of the sequence under each hypothesis.

   (b) **(3 points)** Write down, in the shortest possible way, a Bayes-optimal test statistic for deciding between H_0 and H_1. How does it depend on the observed sequence $y_1,\ldots,y_n$?

   (c) **(4 points)** Suppose k (the alphabet size) is quite large ($k \gg 1$), and $\epsilon \gg \frac{1}{k}$. Explain why the test statistic you devised above should reliably detect the true hypothesis as n (the sample size) becomes large.
   (Hint: Approximate the test statistic in this regime of k and ϵ, and think about what happens to it under H_0 and H_1 as $n \to \infty$).

2. **Composite hypothesis testing**
   Consider testing the hypotheses

$$H_0 : Y_1,\ldots,Y_n \overset{\text{iid}}{\sim} \mathcal{N}(\mu,1), \quad \mu \in \left(-\infty, -\frac{\epsilon}{2}\right)$$

$$\text{vs.}$$

$$H_1 : Y_1,\ldots,Y_n \overset{\text{iid}}{\sim} \mathcal{N}(\mu,1), \quad \mu \in \left(\frac{\epsilon}{2}, \infty\right).$$

   (a) **(5 points)** Write down the form of the Generalized Likelihood Ratio Test (GLRT) for this problem, in the shortest possible way. What real-valued function of the observations $y_1,\ldots,y_n$ does the test statistic depend on?

   (b) **(5 points)** Sketch graphically the GLRT test statistic vs. the real-valued function of $y_1,\ldots,y_n$, that you found above, as clearly as you can.

3. **Geometric distribution estimation**
   Consider the family of geometric probability distributions $\{\text{Geom}(\theta) : \theta \in [0,1]\}$.

   (a) **(3 points)** Is this family complete? Why/why not?

   (b) **(3 points)** Based on n iid samples $X_1,\ldots,X_n$ from $\text{Geom}(\theta)$, $\theta \in [0,1]$, can you find a sufficient statistic for θ?

(c) **(4 points)** What is the Fisher information for a single sample? Write down the Cramér-Rao lower bound for the variance of an unbiased estimator of θ based on n iid samples from the Geom(θ) distribution.

(d) **(3 points)** Suppose $X_1, \ldots, X_n$ are iid samples from Geom(θ), for some $\theta \in [0, 1]$. Find the maximum likelihood estimator (MLE) of θ as a function of $X_1, \ldots, X_n$.

(e) **(5 points)** Suppose $X_1, \ldots, X_n$ are iid samples from Geom(θ), for some $\theta \in [0, 1]$. Find the best unbiased estimator of θ as a function of $X_1, \ldots, X_n$. Provide supporting arguments as clearly as possible.
(Hint: Start with a crude unbiased estimator of θ, then improve it. You can use the fact that the sum $\sum_{i=1}^{m}(Z_i - 1)$, where Z_i are iid Geom(θ) random variables, has the NegBin(m, θ) distribution.)

4. **Linear estimation of an autoregressive process**
A zero-mean discrete-time process $\{X_t\}_{t=-\infty}^{+\infty}$ evolves as

$$X_{t+1} = \alpha X_t + \beta X_{t-1} + W_t, \tag{3}$$

where W_t is iid $\mathcal{N}(0, \sigma_1^2)$ state noise across time t. The observations from this process are given by

$$Y_t = X_t + V_t, \tag{4}$$

where V_t is iid $\mathcal{N}(0, \sigma_2^2)$ observation noise across time t. Assume that the system is steady state (i.e., the process $\{X_t\}_t$ is wide-sense stationary).

(a) **(5 points)** Can you find the steady state 0-step and 1-step autocorrelations $r_0 = \mathrm{Cov}(X_t, X_t)$ and $r_1 = \mathrm{Cov}(X_{t+1}, X_t)$, in terms of α, β and σ_1^2?
(Hint: Use equation (3) creatively.)

(b) **(3 points)** Suppose you want to find the best linear MMSE estimate of X_{t+1} in terms of the immediately preceding observations Y_t and Y_{t-1}. Describe how you would find the optimal linear coefficients in terms of r_0, r_1, α, β and σ_2^2.