E1 244: Detection and Estimation Theory (2019)
Homework 2

1. For question 1 in homework 1, obtain the Neyman-Pearson optimal test for given false-alarm level $\alpha \in (0,1)$. Note: You must explicitly show how the false alarm probability of your test is adjusted.

2. For question 2 in homework 1, obtain the Neyman-Pearson optimal test for given false-alarm level $\alpha \in (0,1)$. Note: You must explicitly show how the false alarm probability of your test is adjusted.

3. Consider the following pair of hypotheses on a sequence $Y_1, Y_2, \ldots, Y_n$ of independent random variables,

$$H_0 : Y_k \sim \mathcal{N}(0, \sigma^2), \quad k = 1, 2, \ldots, n,$$
$$H_1 : Y_k \sim \mathcal{N}(1, \sigma^2), \quad k = 1, 2, \ldots, n,$$

   where $\sigma > 0$ is unknown.

   Does there exist a uniformly most powerful test? If so, find it and show that it is a UMP. If not, show why not.

4. Consider the M-ary decision problem: $(\Gamma = \mathcal{R}^n)$

$$\begin{aligned} H_0 :&\ \underline{Y} = \underline{N} + \underline{s}_0 \\ H_1 :&\ \underline{Y} = \underline{N} + \underline{s}_1 \\ &\ \vdots \\ H_{M-1} :&\ \underline{Y} = \underline{N} + \underline{s}_M \end{aligned}$$

   where $\underline{s}_0, \underline{s}_1, \ldots, \underline{s}_M$ are known signals with equal energies, $\|\underline{s}_0\|^2 = \|\underline{s}_1\|^2 = \cdots = \|\underline{s}_{M-1}\|^2$.

   (a) Assuming $\underline{N} = \mathcal{N}(\underline{0}, \sigma^2 \mathbf{I})$, find the decision rule achieving minimum error probability when all hypotheses are equally likely.

   (b) Assuming further that the signal are orthogonal, show that minimum error probability is given by

$$P_e = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\Phi(x)]^{M-1} e^{-(x-d)^2/2} \, dx$$

   where $d^2 = \|\underline{s}_0\|^2/\sigma^2$, $x = \underline{s}_0^T \underline{Y}/(\sigma\|\underline{s}_0\|)$.

5. Suppose we have a real observation Y and binary hypotheses described by the following pair of pdfs:

$$H_0 : p_0(y) = \begin{cases} (1 - |y|) & \text{if } |y| \leq 1 \\ 0 & \text{if } |y| > 1. \end{cases}$$

$$H_1 : p_1(y) = \begin{cases} (2 - |y|)/4 & \text{if } |y| \leq 2 \\ 0 & \text{if } |y| > 2. \end{cases}$$

Find the Neyman-Pearson test for H_0 vs H_1 with false alarm probability α. Find the corresponding power of the test.

6. Let $X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} \text{Cauchy}(\theta, 1)$. i.e., their joint pdf is given by:

$$p_\theta(\mathbf{x}) = \frac{1}{\pi^n} \prod_{i=1}^{n} \frac{1}{1 + (x_i - \theta)^2}, \quad \mathbf{x} \equiv (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Consider the problem of testing,

$$H_0 : \theta = 0 \text{ versus } H_1 : \theta > 0.$$

Does there exist a UMP test? If so, find it and show that it is a UMP. If not, show why not and find a Locally Most Powerful test (LMP) for the above problem.

7. (**MATLAB programming assignment**) This exercise asks you to implement a hypothesis test and evaluate its performance by Monte Carlo simulation.

Imagine that you are a communication receiver designer where the receiver needs to resolve between the following hypotheses:

$$H_0 : Y \sim \mathcal{N}(\mu_0, 1),$$
$$\text{versus}$$
$$H_1 : Y \sim \mathcal{N}(\mu_1, 1)$$

where $\mu_0 = 0$. Hypothesis H_0 may represent no signal sent and hypothesis H_1 may represent a nontrivial signal sent.

For each possible value of SNR $d \in \{0, 0.1, 0.2, \ldots, 1.9, 2\}$ (recall: $d := (\mu_1 - \mu_0)/\sigma$), and each possible value of prior $\pi_0 \in \{0.1, 0.2, 0.3, 0.4\}$, you need to carry out the following.

(a) Generate 10^5 independent samples of the observation Y, according to the Bayes prior $\pi_0, 1 - \pi_0$ on the two hypotheses.

(b) Run a Bayes-optimal decision rule on each sample, record the cost of the decision under a uniform cost structure, and find the empirical average cost of the decision rule on all the samples. This is the empirical Bayes risk of your receiver.

Plot the empirical Bayes risk that you obtained in part (b) against SNR (i.e., d) for each value of π_0.

Finally, using the analytical expression for $r_B(\delta)$ that you obtained in class, plot $r(\delta)$ (the optimal Bayes risk) vs d over the plots that you obtained via simulation in the previous parts, as a sanity check. (MATLAB has functions that directly return Gaussian cdf values.)

You need to submit a single MATLAB code file for the above, which will carry credit towards the evaluation of this homework. The direct execution of your script should result in a single plot with the empirical and the analytical Bayes risks plotted (vs d) in different colours, for the various values of π_0. **You must ensure that your script runs without errors on a MATLAB console!**