E1 244: Detection and Estimation Theory (2019)
Homework 3

1. Consider the model $Y_k = \theta^{1/2} s_k R_k + N_k$, $k = 1, 2, \ldots, n$, where $s_1, s_2, \ldots, s_n$ is a known signal sequence , $\theta \geq 0$ is a constant, and $R_1, R_2, \ldots, R_n, N_1, N_2, \ldots, N_n$ are i.i.d. $\mathcal{N}(0, 1)$ random variables.
   (a) Consider the hypothesis pair $H_0 : \theta = 0$ versus $H_1 : \theta = A$, where A is a known positive constant. Describe the structure of the Neyman-Pearson detector.
   (b) Consider now the hypothesis pair $H_0 : \theta = 0$ versus $H_1 : \theta > 0$. Under what conditions on $s_1, s_2, \ldots, s_n$ does a UMP test exist?
   (c) For the hypothesis pair in part (b) with general $s_1, s_2, \ldots, s_n$, what is a locally optimum detector?

2. Consider an observed random n-vector $\underline{Y}$ that satisfies one of the two hypotheses:

$$H_0 : \underline{Y} = \underline{N},$$

$$versus$$

$$H_1 : \underline{Y} = \underline{N} + A\left[(1-\Theta)\underline{s}^{(0)} + \Theta \underline{s}^{(1)}\right],$$

   where $\underline{N} = \mathcal{N}(\underline{0}, \mathbf{I})$; the quantity A is a nonrandom positive scalar; the random parameter Θ is independent of $\underline{N}$ and takes on the values 0 and 1 with equal probabilities; and the signals $\underline{s}^{(0)}$ and $\underline{s}^{(1)}$ are known orthonormal signals.
   (a) Suppose the value of A is known. Find the likelihood ratio between the hypotheses H_0 and H_1.
   (b) Consider now the composite hypothesis-testing problem:

$$H_0 : A = 0,$$

$$versus$$

$$H_1 : A > 0.$$

   Find the locally most powerful test of level α. Draw the detector structure.

3. Consider the model

$$Y_k = \theta^{1/2} s_k R_k + N_k,\ \ k = 1, 2, \ldots, n \tag{1}$$

   where $s_1, s_2, \ldots, s_n$ is a known signal sequence, $\theta \geq 0$ is a constant, and $R_1, R_2, \ldots, R_n$, $N_1, N_2, \ldots, N_n$ are i.i.d. $\mathcal{N}(0, 1)$ random variables

(a) Consider the hypothesis pair,

$$H_0 : \theta = 0,$$
$$\textit{versus}$$
$$H_1 : \theta = A$$

where A is a known positive constant. Describe the structure of the Neyman-Pearson detector.

(b) Consider the hypothesis pair,

$$H_0 : \theta = 0,$$
$$\textit{versus}$$
$$H_1 : \theta > 0.$$

Under what conditions on $s_1, s_2, \ldots, s_n$ does a UMP exist?

(c) For the hypothesis pair of part (b) with $s_1, s_2, \ldots, s_n$ general, is there a locally optimum detector? If so find it. If not, describe the generalized likelihood ratio test.

4. Consider the following pair of hypotheses concerning a sequence $Y_1, Y_2, \ldots, Y_n$ of independent random variables,

$$H_0 : Y_k \sim \mathcal{N}(0, \sigma^2), \;\; k = 1, 2, \ldots, n$$
$$\textit{versus}$$
$$H_1 : Y_k \sim \mathcal{N}(\mu, \sigma^2), \;\; k = 1, 2, \ldots, n$$

where μ is a known constant and $\sigma > 0$ is unknown.

Does there exist a uniformly most powerful test. If so, find it and show that it is UMP. If not, show why and find the generalized likelihood ratio test.

5. A rank one signal is a random signal of length n whose mean is zero and whose covariance matrix has rank one, i.e., the covariance matrix Σ_S can be written as $\Sigma_S = \underline{u}\,\underline{u}^T$ where $\underline{u}$ is a $n \times 1$ vector.

(a) Show that the signal $S_k = Ah_k$, $k = 1, \ldots, n$, where $(h_k)_{k=1}^n$ is a deterministic sequence and A is a random variable with $\mathbb{E}[A] = 0$ and $\mathrm{var}(A) = \sigma_A^2$, is a rank one signal.

(b) Find the form of the Neyman-Pearson detector for a Gaussian rank one signal (a signal of the form defined in the preceding part where, additionally, A is Gaussian) embedded in i.i.d. Gaussian noise of variance σ^2.

6. (a) Consider the hypothesis pair

$$H_0 : Y = N,$$
$$\textit{versus}$$
$$H_1 : Y = N + S$$

where N and S are independent random variables each having pdf

$$p(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Find the likelihood ratio between H_0 and H_1.

(b) Find the threshold and detection probability for α-level Neyman-Pearson testing in (a).

(c) Consider the hypothesis pair

$$\begin{aligned} &H_0 : Y_k = N_k, && k = 1, \ldots, n \\ &\quad versus && \\ &H_1 : Y_k = N_k + S, && k = 1, \ldots, n \end{aligned}$$

where $n > 1$ and $N_1, \ldots, N_n$, and S are independent random variables each having the pdf given in (a). Find the likelihood ratio.

(d) Find the threshold for α-level Neyman-Pearson testing in (c).