E1 244: Detection and Estimation Theory (2019)
Homework 4

1. For the sequential binary hypotheses testing problem,
$$H_0 : Y_k \overset{\text{iid}}{\sim} \mathcal{N}(1,1), \quad k = 1, 2, \ldots$$
$$\text{vs.}$$
$$H_1 : Y_k \overset{\text{iid}}{\sim} \mathcal{N}(-1,1), \quad k = 1, 2, \ldots,$$
find the false alarm probability(α), missed detection probability(γ) and the expected sample size for the SPRT$(0.9, 1.1)$ under both the hypotheses, using Wald's approximations as discussed in class.

2. Investigate the Chernoff bound for testing between the two marginal densities
$$p_0(y) = \begin{cases} 1 & \text{if } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$
and,
$$p_1(y) = \begin{cases} 2y & \text{if } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$
for a sequence of i.i.d observations, $Y_1, Y_2, \ldots, Y_n$; i.e., compute the bounds on $P_F(\delta(\tau))$ and $P_M(\delta(\tau))$.

3. Consider a sequence of i.i.d Bernoulli observations, $Y_1, Y_2, \ldots$, with distribution
$$P(Y_k = 1) = 1 - P(Y_k = 0) = 1/3$$
under hypothesis H_0, and
$$P(Y_k = 1) = 1 - P(Y_k = 0) = 2/3$$
under hypothesis H_1.

   (a) Use Wald's approximation to suggest values of A and B so that the SPRT (A, B) has maximum error probability $p^* = \max(P_F, P_M)$ approximately equal to 0.01. Describe the resulting test in detail. Also, using Wald's approximations, give an approximation for the expected sample sizes $E\{N|H_0\}$ and $E\{N|H_1\}$.

(b) Find an integer n as small as you can so that the maximum error probability for the optimal test with fixed sample size n is no more than 0.01. Compare n to the expected sample sizes found in part (a) (Note: you can use a Chernoff bound estimate to find n, rather than finding the actual smallest possible n).

(c) Compute p^*, $E\{N|H_0\}$, and $E\{N|H_1\}$ exactly for the test you found in part (a), and compare with the approximate values you found in part (a). [Hint: use the fact that SPRT you found in part (a) is equivalent to SPRT (A', B') where A' and B' are integer powers of 2.]

4. Let $X_1, X_2, ..., X_n$ be gamma distributed i.i.d random variables with parameters $\alpha > 0$ and $\theta > 0$; the probability density function for each X_i is given by

$$f(x) = \frac{1}{\Gamma(\alpha)\,\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, \; x > 0.$$

Find the method of moments estimators for α and θ based on $X_1, \dots, X_n$. (You can find moment formulas for the gamma distribution on the Internet.)

5. Let $X_1, \dots, X_n \sim \text{Uniform}[a, b]$, where a, b are unknown parameters and $a < b$.

(a) Derive the method of moments estimator for (a, b).

(b) Derive the maximum likelihood estimator for (a, b).

6. Consider a Bernoulli random variable X with distribution,

$$P(X = 1) = 1 - P(X = 0) = p.$$

Find the maximum likelihood estimate (MLE) for p, given an iid random sample $X_1, X_2, \dots, X_n$.

7. Let $X_1, X_2, \dots, X_n$ be an iid random sample drawn from a Poisson distribution with parameter λ, $X_i \sim Poi(\lambda)$. Consider two unbiased estimators $\overline{X} = \frac{1}{n} \sum_{i=1}^{n} x_i$, and $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} x_i^2$. Which one has lower Mean-Square-Error (MSE)?