E1 244: Detection and Estimation Theory (2019)
Homework 5

1. Suppose $X_1, \ldots, X_n$ are iid samples from the Exponential distribution with parameter $\lambda > 0$. Find the Cramér-Rao lower bound[1] for unbiased estimation of λ. Does there exist an unbiased estimator of λ whose variance is equal to the lower bound?

2. Suppose $X_1, X_2, \ldots, X_n$ are iid uniform observations on the interval $(\theta, \theta + 1)$, $-\infty < \theta < \infty$.

   (a) Can you find a sufficient statistic for θ?

   (b) Can you find a minimal sufficient statistic for θ?

3. *Bayes estimators I*
   Suppose X is a sample from the binomial distribution with parameters $n \in \mathbb{N}$ (known) and $\theta \in [0, 1]$ (unknown).

   (a) Find the posterior mean estimate of θ, as a function of X, for the Beta(a, b) prior for $\theta \in [0, 1]$, i.e., $\pi(\theta) = \mathbb{1}\{0 \le \theta \le 1\}\, \theta^{a-1}(1-\theta)^{b-1}/B(a, b)$.
   (Search the Internet for properties of the Beta distribution.)

   (b) What is the mean square error of the estimator you found above?

4. *Bayes estimators II*
   Suppose $X_1, \ldots, X_n$ are iid Gaussian random variables with unknown mean $\theta \in \mathbb{R}$ and known variance σ^2.

   (a) Find the posterior distribution of θ given the observations $X_1, \ldots, X_n$, under the Gaussian prior distribution $\mathcal{N}(\mu, \tau^2)$ for θ.

   (b) Write down the posterior mean estimator under the same prior, as a function of $X_1, \ldots, X_n$, σ^2, μ, τ^2 and n. Is it unbiased for estimating θ?

   (c) Argue what happens to this estimator as n (the amount of data) tends to ∞.

5. *Cauchy Schwarz inequality*
   Prove: For two finite-variance random variables A and B on the same probability space, $E[AB]^2 \le E[A^2]\, E[B^2]$.

6. *Fisher information for iid samples*
   Suppose n iid observations $X_1, \ldots, X_n$, each from the density p_θ, define an observation X. Show that the Fisher information of X w.r.t. θ is n times the Fisher information of X_1 w.r.t. θ.

[1] Do not worry about verifying the CRLB technical conditions.

7. *Linear regression*
Suppose the random variables $X_1, X_2, \ldots, X_n$ satisfy

$$X_i = \alpha s_i + \eta_i, \quad i = 1, 2, \ldots, n$$

where $s_1, s_2, \ldots, s_n$ are known and fixed constants, $\eta_1, \eta_2, \ldots, \eta_n$ are iid $\mathcal{N}(0, \sigma^2)$, σ^2 unknown.

(a) Find a two-dimensional sufficient statistic for (α, σ^2).

(b) Find the MLE of α and find its bias. What is the distribution of the MLE?

(c) Show that $\frac{\sum X_i}{\sum s_i}$ is an unbiased estimator of α. Calculate its variance and compare it with the variance of the MLE of α.

(d) Show that $\sum(X_i/s_i)$ is also an unbiased estimator of α. Calculate its variance. Finally compare the MSE of all the above estimators of α, in particular can you write an ordering of the corresponding MSEs?

8. Suppose $X_1, \ldots, X_n$ are iid $\mathrm{Ber}(p)$, where $0 < p < 1$.

(a) Show that the variance of MLE of p attains the Cramer-Rao lower bound.

(b) For $n \geq 4$, show that the product $X_1 X_2 X_3 X_4$ is an unbiased estimator of p^4. Using this, can you find the best unbiased estimator of p^4? (*Hint: Use the Rao-Blackwell theorem to improve the estimator. Observe that the Binomial family is complete.*).

9. *Completeness of distributions*
Consider the family of exponential probability distributions $\{\mathrm{Exp}(\theta) : \theta > 0\}$.

(a) Is this family complete? Why/why not?

(b) For a single sample $X \sim \mathrm{Exp}(\theta)$, $\theta > 0$, can you find a sufficient statistic for θ?

(c) What is the Fisher information for a single sample? For n iid samples?

(d) Suppose $X_1, \ldots, X_n$ are iid samples from $\mathrm{Exp}(\theta)$, for some $\theta > 0$. Find the best unbiased estimator, as a function of $X_1, \ldots, X_n$, of the Fisher information of (a) a single sample, and (b) n iid samples.