E1 244: Detection and Estimation Theory (2019)
Homework 6

1. Suppose $X_1, \ldots, X_n$ are iid samples from $Unif(0, \theta)$, for some $\theta > 0$. Show that the statistic $T(X) = \max_i X_i$ is complete.

2. Show that the family of Binomial(n, p) probability distributions, where $n \in \mathbb{N}$ is a fixed integer and p ranges from 0 to 1, is complete.

3. *Complete sufficient statistics and best unbiased estimation*
Suppose $T(X)$ is a complete and sufficient statistic, for θ, of the random sample $X \sim f_\theta$, $\theta \in \Theta$. Let $W(X)$ be an unbiased estimator for $g(\theta)$, where g is a given function. Is $E[W(X)|T(X)]$ a best unbiased estimator of $g(\theta)$? If yes/not, argue why/why not.

4. *Gaussian conditioning lemma, or the Gaussian 'swiss army knife'*
Prove the following. Suppose A and B are jointly Gaussian random vectors of dimensions m and n, respectively, with

$$\mathbb{E}\left[A\right] = \mu_A, \quad \mathbb{E}\left[B\right] = \mu_b,$$
$$\mathrm{Cov}[A] = \Sigma_A \text{ (invertible)}, \quad \mathrm{Cov}[B] = \Sigma_B, \text{and}$$
$$\mathrm{Cov}[A, B] = \mathbb{E}\left[(A - \mu_A)(B - \mu_B)^T\right] = \Sigma_{AB} = \Sigma_{BA}^T.$$

Then, the conditional distribution of B given A is Gaussian with mean

$$\mu_{B|A} = \mu_B + \Sigma_{BA}\,\Sigma_A^{-1}(A - \mu_A)$$

and covariance

$$\Sigma_{B|A} = \Sigma_B - \Sigma_{BA}\,\Sigma_A^{-1}\,\Sigma_{AB}.$$

5. *Kalman filtering*
Consider a sequence of binary random variables $\{X_k\}_{k=0}^\infty$ each taking on the values 0 or 1. Suppose that the probabilistic structure of the sequence is such that,

$$\begin{aligned} P(X_k = x_k | X_0 = x_0, X_1 = x_1, ..., X_{k-1} = x_{k-1}) &= P(X_k = x_k | X_{k-1} = x_{k-1}) \\ &\triangleq p_{x_{k-1}x_k} \end{aligned}$$

for all integers $k \geq 1$, and for all binary sequences $\{x_k\}_{k=0}^\infty$. Consider the observation model

$$Y_k = X_k + N_k, \quad k = 0, 1, 2, ...,$$

where $\{N_k\}_{k=0}^\infty$ is a sequence of independent and identically distributed random variables, independent of $\{X_k\}_{k=0}^\infty$ and having a marginal probability density function f. For each integer $t \geq 0$, let $\hat{X}_{t|t}$ denote the minimum mean squared error (MMSE) estimate of X_t, given measurements $Y_1, Y_2, .., Y_t$ and let $\hat{X}_{t|t-1}$ denote the MMSE estimate of X_t, given measurements $Y_1, Y_2, .., Y_{t-1}$, with $\hat{X}_{0|-1} \equiv E\{X_0\}$. Show that $\hat{X}_{t|t}$ and $\hat{X}_{t|t-1}$ satisfy the joint recursion

$$\hat{X}_{t|t} = \frac{\hat{X}_{t|t-1} f(y_t - 1)}{\hat{X}_{t|t-1} f(y_t - 1) + (1 - \hat{X}_{t|t-1}) f(y_t)}, \quad t \geq 0$$

and

$$\hat{X}_{t+1|t} = p_{1,1} \hat{X}_{t|t} + p_{1,0}(1 - \hat{X}_{t|t}), \quad t \geq 0$$

with initial condition $\hat{X}_{0|-1} = P(X_0 = 1)$. Can you generalize this result to non-binary sequences satisfying the above property?

6. *Recursive parameter estimation*
   Suppose X is a zero-mean Gaussian random variable with variance σ_0^2. At each time instant $k = 1, 2, \ldots$, let $Z_k = X + V_k$ denote the kth observation of X with independent Gaussian noise $V_k \sim \mathcal{N}(0, \sigma^2)$.
   (a) Find a recursive minimum mean-square error (MMSE) estimator $\hat{X}_k$ of X at each time k, as a function of the fresh observation Z_k and the MMSE estimator $\hat{X}_{k-1}$ at time $k - 1$.
   (b) What is the value of $\hat{X}_1$ when $\sigma^2 = 0$? Repeat for $\sigma^2 = \infty$.

7. Let X, Y_1 and Y_2 be random variables with finite mean and covariance, and denote $\sigma_{AB} = \text{Cov}[A, B]$ for each pair of random variables $A, B \in \{X, Y_1, Y_2\}$. Let $L_{12}^X(Y_1, Y_2)$ denote the best linear mean-squared-error (MSE) estimator of X given Y_1, Y_2; likewise, let $L_i^X(Y_i)$ denote the best linear MSE estimator of X given Y_i, $i = 1, 2$. Suppose $\sigma_{Y_1 Y_2} = \mathbb{E}[Y_1] = \mathbb{E}[Y_2] = 0$.
   (a) Assuming that the 1-step or individual best estimators $L_i^X(Y_i)$, $i = 1, 2$, along with the mean $\mathbb{E}[X]$, are known, can you express $L_{12}^X(Y_1, Y_2)$ in terms of them?
   (b) Can you express the MSE of $L_{12}^X(Y_1, Y_2)$ in terms of the given covariances?

8. *Kalman filtering without the Gaussianity assumption*
   Take the Gauss-Markov linear dynamical system considered in class, and replace the word 'independent' with 'uncorrelated' everywhere, e.g., the control sequence $\{U_t\}_{t \geq 0}$ is a sequence of uncorrelated random variables, X_0 is uncorrelated with each U_t and V_s, $s, t \geq 0$, etc.). In other words, we remove all Gaussianity assumptions. Show that each output $\hat{X}_{t|t}$, of the Kalman-Bucy filter, is still the best linear MMSE estimate of the true state X_t, for each time t.

9. *Kalman filtering*
   Consider the observation model:
   $$Y_k = N_k + \Theta s_k, \quad k = 1, 2, 3, ...,$$
   where $N_1, N_2, ...$ are i.i.d $\mathcal{N}(0, \sigma^2)$ random variables, $\Theta \sim \mathcal{N}(\mu, \nu^2)$ is independent of $N_1, N_2, ...$, and $s_1, s_2, ...$ is a known sequence. Let $\hat{\theta_n}$ denote the MMSE estimate of Θ given $Y_1, Y_2, ..., Y_n$. Find recursions for $\hat{\theta_n}$ and for the minimum-mean-squared error, $E\{(\hat{\theta_n} - \Theta)^2\}$, by recasting this problem as Kalman filtering problem.