E1 245: Online Prediction & Learning	**Fall 2014**
Lecture 23 — October 30	
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23.1 Recap

In the last lecture we were analyzing the algorithm of Bandit Gradient Descent (B G D).The main idea is to construct an approximate measure of the gradient of the loss functions and work as in mirror descent. B G D is a method of online convex optimization with bandit feedback. The problem setting is:

The convex set is $K \subseteq \mathbb{R}^d$.
There is a sequence of loss functions $f_1, f_2, f_3.....f_T : K \to \mathbb{R}$. But, for simplicity, we have already assumed that the functions are bounded as $f_i \in [-c, c]$,$\forall i = 1,2,...T$ where$c \in \mathbb{R}_+$.
$\forall t = 1,2,...T$, algorithm picks an element $w_t \in K$.
At each time algorithm observes $f_t(w_t) \in \mathbb{R}$.
B G D Algorithm:
Parameters:$\alpha, \delta, \eta > 0$
There is a measure of inner and outer radius of K ie. $\exists\ r,R : r.B \leq K \leq R.B$ where
B =unit ball ($\mathbb{R}^d$) . ie. $B = \{x \in \mathbb{R}^d : ||x|| \leq 1\}$.
Initialize $y_1 = 0$.
$\forall t = 1,2,...T \ \ u_t \sim uniform(S)$ where S =unit sphere ($\mathbb{R}^d$) . ie. $S = \{x \in \mathbb{R}^d : ||x|| = 1\}$.
Play $x_t = y_t + \delta.u_t$.
Update as $y'_{t+1} = \prod_{(1-\alpha)K}(y_t - \eta.f_t(x_t).u_t)$ where $\prod_A(y) = argmin_{x\in A}||x-y||_2$.
Also we have established the result

$$E_{U_t \sim uniform(S)}[f_t(y_t + \delta.u_t).u_t.\frac{d}{\delta}] = \nabla f^*_{t,\delta}(y_t) \tag{23.1}$$

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where $f^*(x) = E_{V\sim uniform(B)}[f(x+\delta.v)]$ and d is the dimension of K.
Through a lemma we have the result that ,the optimum of $\sum_{t=1}^T f_t$
(ie.the minimizer of the sum of lossfunctions) over $(1-\alpha)K$ is near the optimum over K.

$$min_{x\in(1-\alpha)K} \sum_{t=1}^{T} f_t(x) \leq 2\alpha cT + min_{x\in K}\sum_{t=1}^{T} f_t(x) \tag{23.2}$$

23.2 Useful Lemmas to find the expected regret

Lemma 23.1. $\forall x \in (1-\alpha)K$,
$x + \alpha r.B \subseteq K$

Proof: Let $x \in (1-\alpha)K$, then
$(1-\alpha)K + \alpha r.B \subseteq (1-\alpha)K + \alpha.K \subseteq K$ $[\because r.B \subseteq K]$

$\square$

This lemma helps in saying that, whenever your perturbation u_t is controlled within a limit,your $x_t = y_t + \delta.U_t$ will be within K.

Lemma 23.2. *[Controlling how much f_t changes over points in $(1-\alpha)K$ and K]*
$\forall x \in (1-\alpha)K, y \in K,$
$|f_t(x) - f_t(y)| \leq (\frac{2c}{\alpha r}).||x-y||$

Proof: Let $x \in (1-\alpha)K, y \in K,$
$y - x = \triangle$
case1: If $||\triangle|| \geq \alpha.r$,then
$(\frac{||\triangle||}{\alpha r}.2c) \geq 2c \geq |f_t(x) - f_t(y)|$
case 2:If $||\triangle|| < \alpha r$,then
let us define $z := x + \frac{\triangle}{||\triangle||}\alpha r$ [ie. z is apoint in the direction of $(y-x)$ and beyond y.]
So y divides the length αr from x to z in the ratio $||\triangle|| : (\alpha r - ||\triangle||)$.
$\Rightarrow y = \frac{||\triangle||}{\alpha r}z + (1 - \frac{||\triangle||}{\alpha r})x$
But, by convexity of f_t ,
$f_t(y) \leq \frac{||\triangle||}{\alpha r} f_t(z) + (1 - \frac{||\triangle||}{\alpha r}) f_t(x) = f_t(x) + \frac{||\triangle||}{\alpha r}(f_t(z) - f_t(x))$
$\Rightarrow |f_t(x) - f_t(y)| \leq (\frac{2c}{\alpha r}).||x-y||$

$\square$

Lemma 23.3. *[Correctness of B G D]*
Whenever $\frac{\delta}{r} \leq \alpha \leq 1$,$x_t \in K, \forall t \geq 1$

Proof: By definition,$y_t \in (1-\alpha)K$.
$x_t \in y_t + \delta.S \subseteq y_t + \delta.B \subseteq y_t + \alpha r.B$ $[\because S \subseteq B$ and $\alpha r \geq \delta]$
From lemma 23.1 $x_t \in K$.

$\square$

23.3 Theorems

Theorem 23.4. *The expected regret of B G D inT rounds (with appropriate parameter values for α, δ, η) is*

$$O(T^{\frac{5}{6}}.c.\sqrt[3]{\frac{dR}{r}})$$

Proof: STEP 1: Analyze regret w.r.t the points $\{y_t\}$,loss functions $\{f_t^*\}$ and decision set $(1-\alpha)K$.
We know that $f^*(x) = E_{V\sim uniform(B)}[f(x+\delta.v)]$.
Define $g_t := f_t(y_t+\delta.u_t).u_t.\frac{d}{\delta}$
Through equation (23.1) we have
$E[g_t/y_t] = \nabla f_t^*(y_t)$
Consider **Stochastic Gradient Descent**.
Loss functions are $\{f_t^*\}$,decision set is $(1-\alpha)K$ and step size $\eta' := \eta\frac{\delta}{d}$.
Initialize $y_1 = 0$.
$\forall t > 1$, $y_{t+1} = \prod_{(1-\alpha)K}(y_t - \eta'.g_t)$ and $E[g_t/y_t] = \nabla f_t^*(y_t)$.

Theorem 23.5. *[Stochastic Gradient Descent (S G D)]*
Suppose $C_1, C_2, C_3... : K \to \mathbb{R}$ is a sequence of convex,differentiable functions on the convex set K.
Let $0 \subseteq K$,$w_1 = 0$ and $\forall t \geq 1$, $\eta \geq 0$.
$w_{t+1} := \prod_K(w_t - \eta.g_t)$ where $E[g_t/w_t] = \nabla C_t(w_t)$
$||g_t|| \leq G$ almost surely, $D := Sup_{x\in K}||x||$.
Then $E[\sum_{t=1}^T C_t(w_t) - min_{w\in K}\sum_{t=1}^T C_t(w)] \leq \frac{\eta}{2}TG^2 + \frac{D^2}{2\eta}$

We can apply the aboveteorem of S G D with $\{C_t := f_t^*\}$,$\{w_t := y_t\}$,$K \equiv (1-\alpha)K$,
$Sup_{x\in(1-\alpha)K}||g_t(x)|| = Sup_{x\in(1-\alpha)K}||\frac{d}{\delta}.f_t(y_t+\delta.u_t).u_t|| \leq \frac{d}{\delta}c =: G$ [$\because f_t$ is bounded and u_t is a unit vector]
and $Sup_{x\in(1-\alpha)K}||x||_2 \leq R =: D$ [Outer radius of K]
Then we get,
$E[\sum_{t=1}^T f_t^*(y_t) - min_{y\in(1-\alpha)K}\sum_{t=1}^T f_t^*(y)] \leq \frac{\eta'}{2}TG^2 + \frac{D^2}{2\eta'}$
On optimization,when we set
$\eta' = \frac{D}{G\sqrt{T}} = \frac{R}{\frac{dc}{\delta}\sqrt{T}} = \frac{R\delta}{dc\sqrt{T}} = \eta\frac{\delta}{d}$
$\Rightarrow \eta = \frac{R}{c\sqrt{T}}$

$$\therefore E[\sum_{t=1}^T f_t^*(y_t) - min_{y\in(1-\alpha)K}\sum_{t=1}^T f_t^*(y)] \leq \frac{Rdc}{\delta}\sqrt{T} \tag{23.3}$$

STEP 2: Connecting $f_t^*(y_t)$ to $f_t(y_t)$
Let us define a quantity called "EFFECTIVE LIPSCHITZ CONSTANT"-$L' := \frac{2c}{\alpha r}$

$$|f_t^*(y_t) - f_t(x_t)| \leq |f_t^*(y_t) - f_t(y_t)| + |f_t(y_t) - f_t(x_t)| \tag{23.4}$$

The LHS in the inequality can be controlled by controlling both terms on RHS of equation (23.4).
For any $z \in (1-\alpha)K$, $\frac{\delta}{r} \leq \alpha \leq 1$,we have
$|f_t^*(z) - f_t(z)| = |E_{V\sim uniform(B)}[f_t(z+\delta.v) - f_t(z)]| \leq E_{V\sim uniform(B)}[|f_t(z+\delta.v) - f_t(z)|]$
From Lemma 23.1 we know that $(z+\delta.v) \in K$
From Lemma 23.2 we know that this difference is bounded as

$|f_t^*(z) - f_t(z)| \leq E_{V\sim uniform(B)}[\frac{2c}{\alpha r}||\delta v||] \leq L'.\delta$ [$\because v$ is inside a unit ball]
The second term in RHS of equation (23.4) also can be bounded as above with the only difference that $x_t \in K$.
$\therefore |f_t^*(y_t) - f_t(x_t)| \leq L'.\delta + L'.\delta = 2L'.\delta$
Using this result on LHS of equation (23.3) appropriately gives

$$E[\sum_{t=1}^{T}(f_t(x_t) - 2L'.\delta) - min_{y\in(1-\alpha)K}\sum_{t=1}^{T}(f_t(y) + L'.\delta)] \leq \frac{Rdc}{\delta}\sqrt{T} \tag{23.5}$$

STEP 3: Connecting optimum over $(1-\alpha)K$ to optimum over K.
Applying the result in equation (23.2) to equation (23.5) gives
$E[\sum_{t=1}^{T} f_t(x_t)] - min_{x\in K}\sum_{t=1}^{T} f_t(x) \leq \frac{Rdc}{\delta}\sqrt{T} + 3L'.\delta T + 2\alpha cT$
$\therefore E[Regret_T^{BGD}] \leq \frac{Rdc}{\delta}\sqrt{T} + \frac{6\delta cT}{\alpha r} + 2\alpha cT$
The expression is like $\frac{x}{\delta} + \frac{y\delta}{\alpha} + z\alpha$
$\delta = \sqrt[3]{\frac{x^2}{y.z}} = \sqrt[3]{\frac{rR^2d^2}{12T}}$ and $\alpha = \sqrt[3]{\frac{x.y}{z^2}} = \sqrt[3]{\frac{3Rd}{2r\sqrt{T}}}$ will give
$E[Regret_T^{BGD}] \leq 3cT^{\frac{5}{6}}\sqrt[3]{\frac{dR}{r}}$

□

Theorem 23.6. *If each $\{f_t\}$ is L-Lipschitz functions,then*
$E[Regret_T^{BGD}] \leq 2T^{\frac{3}{4}}\sqrt[3]{3dRc(L+\frac{c}{r})}$

Proof: The proof follows from that of Theorem 23.4 with the difference that we can use the direct Lipschitz constant in step2.

□

23.4 Reshape K to avoid large $\frac{R}{r}$ ratio

We have got regret bound which is a dependent on $\frac{R}{r}$. This bound will become very bad when this ratio is large.

So as to avoid this, the convex set K (which is now $r.B \subseteq K \subseteq R.B$)can be put into "ISOTROPIC POSITION".
ie.$\exists$ an affine transformation$T : \mathbb{R}^d \to \mathbb{R}^d$ such that
$1.B \subseteq T(K) \subseteq d.B$
There are efficient algorithms to find such a transformation T (approximately).
eg.L.Lovasz and S.Vempala, in their work, guarantees a T such that
$B \subseteq T(K) \subseteq 1.01B$
So,before applying the B G D algorithm,we aill apply this transformation on K so that we will get
$r' = 1$ and $R' = 1.01d$
Let $f_t^{'} : T(K) \to \mathbb{R}$
In this new setting,$\forall y \in T(K), f_t^{'}(y) := f_t(T^{-1}(y))$

Lemma 23.7. *If f_t is L-Lipschitz over K and $R = Sup_{x \in K}||x||$,then $f_t^{'}$ is LR -Lipschitz over $T(K)$.*

After reshaping we will get $r' = 1$, $R' = 1.01d$ and $L' = LR$. Then
$E[Regret_T^{BGD}] \leq 6T^{\frac{3}{4}}d(\sqrt{cLR} + c)$ for L-Lipschitz and
$E[Regret_T^{BGD}] \leq 6T^{\frac{5}{6}}dc$ without the L-Lipschitz condition

23.5 References

1)Online convex optimization in the bandit setting: gradient descent without a gradient by Abraham D. Flaxman, Adam Tauman Kalai and H. Brendan McMahan.

http://people.cs.uchicago.edu/~kalai/papers/bandit/bandit.pdf