# Solutions: Homework 1

1. Let us generalize the majority algorithm as follows:
   - Let $\hat{N} = \{1, 2, \ldots, N\}$, $i = 1$.
   - Loop: Phase $i$:
     - Choose $\hat{y}_t$ according to the majority of experts in $\hat{N}$.
     - If $\hat{y}_t \neq y_t$, then remove the erring experts from $\hat{N}$.
     - Continue until $\hat{N} = \phi$.
     - Restore $\hat{N} = \{1, 2, \ldots, N\}$, $i = i + 1$.

   For each phase of the loop, the number of mistakes made is equal to $\lfloor \log_2 N \rfloor + 1$. Thus the total number of mistakes made,

   $$M_T \leq (m+1)(1 + \lfloor \log_2 N \rfloor) \leq 2(m+1)\log_2 N = O((m+1)\log_2 N)$$

   where the second inequality is true for all $N \geq 2$.

2. Substituting $\eta = \sqrt{\frac{\beta}{\gamma T}}$, we get the regret bound $R_T \leq 2\sqrt{\beta\gamma T}$. After the tweak, we get a regret of at most $\frac{\beta}{\eta_m} + \gamma\eta_m 2^m$ for every period, and we at most have $1 + \lfloor \log_2 T \rfloor$ periods. So

   $$\begin{aligned}
   R_T(\text{anytime algo}) &\leq \sum_{m=0}^{\lfloor \log_2 T \rfloor} \left( \frac{\beta}{\eta_m} + \gamma\eta_m 2^m \right) \\
   &= \sum_{m=0}^{\lfloor \log_2 T \rfloor} 2\sqrt{\beta\gamma 2^m} \\
   &= 2\sqrt{\beta\gamma}\frac{\sqrt{2}}{\sqrt{2}-1}(2^{0.5 \times \lfloor \log_2 T \rfloor} - 1) \\
   &\leq 2\sqrt{\beta\gamma T}\frac{\sqrt{2}}{\sqrt{2}-1} \\
   &= \frac{\sqrt{2}}{\sqrt{2}-1} \text{ (bound on } R_T)
   \end{aligned}$$

   where the first equality follows from substituting the value of $\eta_m$, the second equality by the geometric progression sum, and the second inequality from the fact that $2^{\log_2 x} = x$.

3. Using Bayes theorem, $P[I = i \mid Y_1 = y_1, \cdots, Y_{t-1} = y_{t-1}]$ can be written as,

   $$P[I = i \mid Y_1 = y_1, \cdots, Y_{t-1} = y_{t-1}] = \frac{P[Y_1 = y_1, \cdots, Y_{t-1} = y_{t-1} \mid I = i].P[I = i]}{\sum_{j=1}^{N} P[Y_1 = y_1, \cdots, Y_{t-1} = y_{t-1} \mid I = j].P[I = j]}.$$

   As we know that $P[I = i] = 1/N$, it suffices to prove $w_{i,t} \propto P[Y_1 = y_1, \cdots, Y_{t-1} = y_{t-1} \mid I = i]$. But

   $$\begin{aligned}
   P[Y_1 = y_1, \cdots, Y_{t-1} = y_{t-1} \mid I = i] &= \Pi_{j=1}^{t-1} P[Y_j = y_j \mid I = i] \\
   &= p^{\sum_{j=1}^{t-1} \mathbf{1}(y_j = f_{i,j})}(1-p)^{\sum_{j=1}^{t-1} \mathbf{1}(y_j \neq f_{i,j})} \\
   &= \frac{e^{-\eta \sum_{j=1}^{t-1} l(f_{i,j}, y_j)}}{(1 + e^{-\eta})^t} \propto w_{i,t}
   \end{aligned}$$

   where the first equality occurs since each $Y_t$ is an independent Bernoulli draw from $f_{i,t}$. Hence the result.

4. Let $f, g : [0, 1/2] \to \mathbb{R}$, and $f(x) = -\log(1-x) = \int_0^x \frac{1}{1-x}\, dx$, $g(x) = x + x^2 = \int_0^x (1+2x)\, dx$. We see that for every $x \in [0, 1/2]$,

$$x(1-2x) \geq 0 \Rightarrow 1 + x - 2x^2 \geq 1 \Rightarrow (1-x)(1+2x) \geq 1 \Rightarrow (1+2x) \geq \frac{1}{1-x}$$

Thus we have $\int_0^x \frac{1}{1-x}\, dx \leq \int_0^x (1+2x)\, dx$ for every $x \in [0, 1/2]$, implying $-\log(1-x) \leq x + x^2$ for every $x \in [0, 1/2]$.

5. (a) The quantity $e^{sX}$ can be expanded as,

$$e^{sX} = \sum_{k=0}^{\infty} \frac{(sX)^k}{k!} \leq \left(1 + \sum_{k=1}^{\infty} \frac{s^k X}{k!}\right) = 1 + X(e^s - 1),$$

where the inequality follows since $X^k \leq X, \forall X \in [0,1], k \geq 1$. Now we have,

$$\log \mathbb{E}(e^{sX}) = \log(1 + (e^s - 1)\mathbb{E}X) \leq (e^s - 1)\mathbb{E}X$$

where the inequality follows since $\log(1+x) \leq x$.

(b) Define $W_t = \sum_{i=1}^N w_{i,t}$. Then,

$$\log \frac{W_T}{W_0} = \log \left( \frac{\sum_{i=1}^N e^{-\eta L_{i,T}}}{N} \right) = \log \mathbb{E}_i e^{-\eta L_{i,T}} \leq (e^{-\eta} - 1)\mathbb{E}_i L_{i,T}. \quad (1)$$

$$\log \frac{W_T}{W_0} \geq -\eta \max_i L_{i,T} - \log N = -\eta L_T^* - \log N. \quad (2)$$

From (1) and (2), we have $\mathbb{E}_i L_{i,T} \leq \frac{\eta L_T^* + \log N}{1 - e^{-\eta}}$. Since $l(d, y)$ is convex in its first argument, we have $L_T = \sum_{t=1}^T l\left( \frac{\sum_{i=1}^N f_{i,T} e^{-\eta \sum_{t=1}^T l(f_{i,t}, y_t)}}{\sum_{i=1}^N e^{-\eta \sum_{t=1}^T l(f_{i,t}, y_t)}}, y_t \right) \leq L_{i,T}$, using Jensen's inequality. So, $L_T \leq \mathbb{E}_i L_{i,T} \leq \frac{\eta L_T^* + \log N}{1 - e^{-\eta}}$.

(c) Choosing $\eta = \log\left(1 + \sqrt{\frac{2 \log N}{L_T^*}}\right)$, we have

$$\begin{aligned} L_T \leq \log N \left( \frac{1}{1 - e^{-\eta}} \right) + \frac{\eta}{1 - e^{-\eta}} L_T^* &\leq \log N \left(1 + \sqrt{\frac{L_T^*}{2 \log N}}\right) + \frac{1 + e^\eta}{2} L_T^* \\ &= L_T^* + \sqrt{2 L_T^* \log N} + \log N \end{aligned}$$

where the second inequality occurs from $\eta \leq \frac{e^\eta - e^{-\eta}}{2} = \frac{(1 - e^{-\eta})(e^\eta + 1)}{2}$, and the others just by substituting the value of $\eta$. The regret bound $R_T = L_T - L_T^* \leq \sqrt{2 L_T^* \log N} + \log N$.

(d) Consider the algorithm:

- Initialize $m = 0$.
- Loop: Phase $m$:
  - Run a fresh copy of EXP-WTS with $\eta_m := \log(1 + \sqrt{(2 \log N / 2^m)})$ until the least cumulative loss of any expert in phase $m$ exceeds $2^m$.
  - Set $m = m + 1$.

The phase number at time $T \leq \lceil \log L_T^* \rceil$. In-phase regret incurred by the algorithm in phase $m \leq \sqrt{2(2^m) \log N} + \log N$, as the algorithm waits for the best expert to make a loss of $2^m$. Thus we have,

$$\begin{aligned} R_T(\text{anytime algo}) &\leq \sum_{m=0}^{\lceil \log_2 L_T^* \rceil} (\log N + \sqrt{2(2^m) \log N}) \\ &= \lceil \log L_T^* \rceil \log N + \sqrt{2 \log N} \left( \frac{\sqrt{2}^{\lceil \log L_T^* \rceil + 1} - 1}{\sqrt{2} - 1} \right) = O(\sqrt{2 L_T^* \log N}), \end{aligned}$$

where the last equality follows for large $L_T^*$, since $\log L_T^* = O(\sqrt{L_T^*})$ when $L_T^* \to \infty$.