E1 245: Online Prediction & Learning	**Fall 2015**
Lecture 10 — September 3	
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10.1 RECAP: Tracking regret in the actions game (continued)

Goal:

To build an efficient Learning algorithm for low regret w.r.t any sequence of actions ($\in \mathscr{E}_m$) with a bounded number of switches (m)

REXPWTS over $[N]^T$:

Initialize: set weight $w_1(i_1,....,i_T) \geq 0; \forall (i_1,....,i_T) \in [N]^T$.
At all time $t \geq 1$,

(1) Compute current weight of each compound expert $w'_t(i_1,....i_T) = w'_1(i_1,....,i_T)e^{-\eta \sum_{s=1}^{t-1} l(i_s,y_s)}$ $\forall (i_1,....,i_T) \in [N]^T$

(2) Play Action I_t with distribution:
$\Pr(I_t = i) = \frac{w'_{i,t}}{W'_t}$; where $w'_{i,t} = \sum_{(i_1,..i_{t-1},i_{t+1},..,i_T)} w'_t(i_1,..,i_{t-1},i,i_{t+1},..,i_T)$ & $W'_t = \sum_{j\in[N]} w'_{j,t}$

Consider the following algorithm that is efficient (maintain only N weights).

10.2 Fixed Share Algorithm [Herbster and Warmuth ('98)]

Algorithm Fixed Share Algorithm

1: **INPUT:**

Parameter $\eta > 0$; $0 \leq \alpha \leq 1$. Here α can be viewed as rate of shifting.

2: **INITIALIZE:**

set initial weights to all experts $w_{i,1} = \frac{1}{N}; \forall i \in [N]^T$

Algorithm Fixed Share Algorithm (continued)

3: At round t=1,2,3,4,....

Play action $I_t \in [N]$ according to the distribution $\{w_{i,t}\}_{i\in[N]}$

Observe y_t and compute $v_{i,t} = w_{i,t}e^{-\eta l(i,y_t)}$; $\forall i$

Update weight: $w_{i,t+1} = \alpha\frac{w_t}{N} + (1-\alpha)v_{i,t}$; here, $w_t = \sum\limits_{j\in[N]} v_{j,t}$

4: END

Note: $\alpha = 0 \Rightarrow$ FSA $\equiv$ REXPWTS algo over [N]

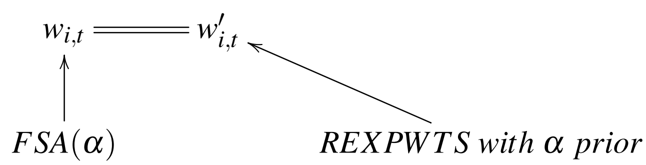
Next theorem states the equivalance of FSA(α) with REXPWTS over $[N]^T$.

Theorem 10.1. *For any $\alpha \in [0,1]$, any sequence of outcomes $(y_1,....,y_T; \text{any } 1 \le t \le T)$; the distribution of action I'_t played by REXPWTS with α prior is same as the distribution of action I_t played by FSA(α)*

Proof: Consider the REXPWTS Algorithm over $[N]^T$ with initial weight given by

$$w'_1(i_1,....,i_T) = \frac{1}{N}\left(\frac{\alpha}{N}\right)^{\#(i_1,....i_T)}\left(1-\alpha+\frac{\alpha}{N}\right)^{T-1-(\#(i_1,....,i_T))};$$

where $\#(i_1,....,i_T)$ = Total number of switches = $\sum\limits_{s=1}^{T-1} \mathbb{1}\{i_{s+1} \neq i_s\}$
It's enough to show $w_{i,t}$=$w'_{i,t}$ by induction over $t \ge 1$; $\forall i \in [N]$, and $\forall t \le T$, i.e.



Base Case: For $t = 1$, $w'_{i,1}$=$\frac{1}{N}$ = $w_{i,1}$, $\forall i \in [N]$. So base case is satisfied.
Induction Hypothesis: Assume that $w'_{i,s}$=$w_{i,s}$, $\forall i \in [N]$ and $\forall s < t$

$$\begin{aligned}
w'_{i,t} &= \sum_{i_1,..,i_{t-1},i_{t+1},..,i_T} w'_t(i_1,..,i_{t-1},i,i_{t+1},..,i_T) \\
&= \sum_{i_1,..,i_{t-1},i_{t+1},..,i_T} w'_1(i_1,..,i_{t-1},i,i_{t+1},..,i_T)e^{-\eta\sum\limits_{s=1}^{t-1} l(i_s,y_s)} \\
&= \sum_{i_1,..,i_{t-1}} w'_1(i_1,..,i_{t-1},i)e^{-\eta\sum\limits_{s=1}^{t-1} l(i_s,y_s)}
\end{aligned}$$

$$= \sum_{i_1,..,i_{t-1}} e^{-\eta \sum_{s=1}^{t-1} l(i_s,y_s)} \{w_1'(i_1,..,i_{t-1},i)\}$$

$$= \sum_{i_1,..,i_{t-1}} e^{-\eta \sum_{s=1}^{t-1} l(i_s,y_s)} w_1'(i_1,..,i_{t-1}) \left(\frac{\alpha}{N} + (1-\alpha)\, \mathbb{1}\{i_{s+1} = i_s\}\right)$$

$$= \left[\sum_{i_{t-1}} \left(\frac{\alpha}{N} + (1-\alpha)\, \mathbb{1}\{i_{s+1} = i_s\}\right) e^{-\eta l(i_{t-1},y_{t-1})}\right] \left[\sum_{i_1,..,i_{t-2}} w_1'(i_1,..,i_{t-1}) e^{-\eta \sum_{s=1}^{t-2} l(i_s,y_s)}\right]$$

$$= \left[\sum_{i_{t-1}} \left(\frac{\alpha}{N} + (1-\alpha)\, \mathbb{1}\{i_{s+1} = i_s\}\right) e^{-\eta l(i_{t-1},y_{t-1})}\right] w_{i_{t-1},t-1}$$

($w_{i_{t-1},t-1}$ = total weight of compound actions suggesting i_{t-1} action at time (t-1))

$$= \sum_{i_{t-1}} \left(\frac{\alpha}{N} + (1-\alpha)\, \mathbb{1}\{i_{s+1} = i_s\}\right) v_{i_{t-1},t-1}$$

$$= \frac{\alpha}{N} \sum_{i_{t-1}} v_{i_{t-1},t-1} + (1-\alpha) v_{i,t-1}$$

$$= w_{i,t} \qquad \text{(by definiion of FSA}(\alpha))$$

□

10.3 Tracking Regret Bound for FSA(α)

Lemma 10.2. *Consider running REXPWTS(η) over N experts with initial weights:* $w_{11}, w_{21}, ..., w_{N1} \geq 0$; $\sum_{i=1}^{N} w_{i,1} = 1$. *Assume that* $l : \mathscr{A} \times \mathscr{Y} \to [0,1]$, *Then*

$$\mathbb{E}\left[\sum_{t=1}^{T} l(i_t, y_t)\right] \leq \frac{1}{\eta} \log \frac{1}{W_{T+1}} + \frac{\eta T}{8}$$

where $W_{T+1} := \sum_{i=1}^{N} w_{i,1} e^{-\eta \sum_{t=1}^{T} l(i,y_t)}$

Proof: Home-work.

□

Theorem 10.3. *For every* $(i_1, ..., i_T) \in [N]^T$ *under FSA(α) with s=#$(i_1, ..., i_T)$= number of switches*

$$\mathbb{E}\left[\sum_{t=1}^{T} l(i_t, y_t)\right] - \sum_{t=1}^{T} l(i_t, y_t) \leq \frac{s+1}{\eta} \log N + \frac{1}{\eta} \log \frac{1}{\alpha^{(s)}(1-\alpha)^{(T-1-s)}} + \frac{\eta T}{8}$$

Proof: For FSA(α), FSA(α) $\equiv$ REXPWTS (α).
Using Lemma 10.2 we have

$$
\begin{aligned}
\mathbb{E}\left[\sum_{t=1}^{T} l(i_t^{FSA/REXPWTS}, y_t)\right] &\leq \frac{1}{\eta}\log\frac{1}{W_T'} + \frac{\eta T}{8} \\
&\leq \frac{1}{\eta}\log\frac{1}{W_T'(i_1,....,i_T)} + \frac{\eta T}{8} \\
&= \frac{1}{\eta}\log\frac{1}{\frac{1}{N}\left(\frac{\alpha}{N}\right)^{(s)}\left(1-\alpha+\frac{\alpha}{N}\right)^{(T-1-s)} e^{-\eta\sum_{t=1}^{T} l(i_t,y_t)}} + \frac{\eta T}{8} \\
&= \sum_{t=1}^{T} l(i_t,y_t) + \frac{s+1}{\eta}\log N + \frac{1}{\eta}\log\frac{1}{\alpha^{(s)}(1-\alpha)^{(T-1-s)}} + \frac{\eta T}{8}
\end{aligned}
$$

i.e.,

$$
\mathbb{E}\left[\sum_{t=1}^{T} l(i_t,y_t)\right] - \sum_{t=1}^{T} l(i_t,y_t) \leq \frac{s+1}{\eta}\log N + \frac{1}{\eta}\log\frac{1}{\alpha^{(s)}(1-\alpha)^{(T-1-s)}} + \frac{\eta T}{8}
$$

□

Corollary 10.4. *Let $m \leq \frac{T-1}{2}$ (infact much lower). Then, running FSA ($\alpha = \frac{m}{T-1}$) gives regret:*

$$
\mathbb{E}\left[\sum_t l(i_t^F SA, y_t)\right] - \min_{i_1,..,i_T\in\mathscr{E}(m)} \sum_t l(i_t,y_t) \leq \frac{m+1}{\eta}\log N + \frac{T-1}{\eta}H(\frac{m}{T-1}) + \frac{\eta T}{8}
$$

Proof: $\forall(i_1,....,i_T)$ with switches $\#(i_1,....,i_T) = s \leq m$, we know from theorem 10.3 that

$$
\begin{aligned}
Regret(i_1,....,i_T) &\leq \frac{s+1}{\eta}\log N + \frac{1}{\eta}\log\frac{1}{\alpha^{(s)}(1-\alpha)^{(T-1-s)}} + \frac{\eta T}{8} \\
&= \frac{s+1}{\eta}\log N + \frac{T-1}{\eta}\left[\frac{s}{T-1}\log\frac{1}{\alpha} + (1-\frac{s}{T-1})\log\frac{1}{1-\alpha}\right] + \frac{\eta T}{8} \\
&= \frac{m+1}{\eta}\log N + \frac{T-1}{\eta}\left[\frac{m}{T-1}\log\frac{1}{\alpha} + (1-\frac{m}{T-1})\log\frac{1}{1-\alpha}\right] + \frac{\eta T}{8} \\
&= \frac{m+1}{\eta}\log N + \frac{T-1}{\eta}\left[q\log\frac{1}{\alpha} + (1-q)\log\frac{1}{1-\alpha}\right] + \frac{\eta T}{8} \qquad (\text{q}=\tfrac{m}{T-1} \leq \tfrac{1}{2}) \\
&\leq \frac{m+1}{\eta}\log N + \frac{T-1}{\eta}H(\frac{m}{T-1}) + \frac{\eta T}{8} \qquad (\text{at } \alpha = q)
\end{aligned}
$$

□

Moreover, for optimal $\eta \Rightarrow Regret(\mathscr{E}(m)) \leq \sqrt{\frac{T}{2}\left[(m+1)\log N + (T-1)H(\frac{m}{T-1})\right]}$. In the last inequality of the proof of Corollary 10.4, we have used the below lemma.

Lemma 10.5. *If $q \leq \alpha \leq \frac{1}{2}$, Then $-[q\log\alpha + (1-q)\log(1-\alpha)] \leq H(\alpha)$ at $\alpha = q$.*

References

[1] Nicolo Cesa-Bianchi and Gabor Lugosi, “Prediction, Learning and Games”, Cambridge University Press, 2006.

[2] Gabor Bartok, David Pal, Csaba Szepesvari, and Istvan Szita, “Online learning - CMPUT 654”, Course Notes. 2011.