E1 245: Online Prediction & Learning	Fall 2015
Lecture 12 — September 10	
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12.1 Recap

In the last lecture we defined σ-exp-concave function and gave theorem for regret of σ-exp-concave loss funtion. We started the problem of sequential investment in stock market as a case study. We defined market model and associated variables,

- m is number of stocks in the market
- $x_t := (x_{1,t}, x_{2,t}, x_{3,t} \ldots x_{m,t})$ where,

$$x_{i,t} = \frac{\text{closing price of stock } i \text{ at end of round } t}{\text{opening price of stock } i \text{ at end of round } t}$$

- $Q_t := (Q_{1,t}, Q_{2,t}, Q_{3,t}, \ldots Q_{i,t}) \in \triangle_m := \{\pi \in \mathbb{R}^m : \pi_i \geq 0 \forall i, \sum_{i=1}^{m} = 1\}$ where, $Q_{i,t}$ is fraction of current wealth to invest at time t in stock i
- $S_T(Q, x^T)$ is the final wealth after T rounds
- $\mathbb{Q}$ is a class of investment algorithms.

Two fundamental investment strategies can be,

1. Buy-and-Hold
2. Constantly Rebalancing Portfolios(CRP).

CRP strategies turn out to be quite non-trivial and optimal if $\{x_t\}_{t=1}^{T}$ is iid.

12.2 Cover's Universal Portfolio Algorithm(CUP)

12.2.1 Idea

Pick N CRPs that uniformly approximate all CRPs $\in \triangle_m$. In other words, distribute the wealth in all possible CRPs uniformly. Run EXPWTS algorithm with parameter, $\eta = \sigma = 1$ on all CRPs. So, regret is $o(\log N/\sigma)$ and all CRPs will get infinitesimal amount of wealth. At each t play

$$P_t = \frac{\sum_{i=1}^{N} b_i e^{-\sum_{s=1}^{t} \log \frac{1}{<x_s, b_i>}}}{\sum_{j=1}^{N} e^{-\sum_{s=1}^{t} \log \frac{1}{<x_s, b_j>}}}$$

12.2.2 Algorithm

- Initialize: $w_1(b) = 1 \ \ \forall b \in \triangle_m$.
- At each time $t \geq 1$, play

$$P_t = \int_{\triangle_m} \frac{w_t(b)}{\int_{\triangle_m} w_t(b')\mu(db')} b\mu(db)$$

where,

$$\begin{aligned} w_t(b) &= e^{-\overset{=1}{\eta} \sum_{s=1}^{t-1} \log \frac{1}{<x_s,b>}} \\ &= \prod_{s=1}^{t-1} < x_s, b > \\ &= S_{t-1}(b, x^{t-1}). \end{aligned}$$

Note

1. CUP, as stated, is highly inefficient. For instance, a naive implementation of CUP can be based on dividing $\triangle_m$ into grids/sub-cubes of side length δ each, then add-up contributions to approximate integral [$\approx \left(\frac{1}{\delta}\right)^m$ cubes to discretize].
2. Efficient (but approximate) implementation discovered later.
3. Other universal portfolios proposed later which are efficient but have worse regret.

Let's compute the wealth of CUP after T rounds:

$$\begin{aligned} S_T(CUP, x_T) &= \prod_{t=1}^{T} < P_t, x_t > \\ &= \prod_{t=1}^{T} \int_{\triangle_m} \frac{< x_t, b > S_{t-1}(b, x^{t-1})\mu(db)}{\int_{\triangle_m} S_{t-1}(b', x^{t-1})\mu(db')} \\ &= \prod_{t=1}^{T} \frac{\int_{\triangle_m} S_t(b, x^t)\mu(db)}{\int_{\triangle_m} S_{t-1}(b', x^{t-1})\mu(db')} \\ &= \int_{\triangle_m} S_t(b, x^t)\mu(db) \qquad \text{(Initial wealth is 1)}. \end{aligned}$$

So,

$$S_T(CUP, x_T) = \int_{\triangle_m} S_t(b, x^t)\mu(db), \tag{12.1}$$

which is equivalent to "BUY-&-HOLD over all CRPs".

Lemma 12.1. *Let*

$$Ball_\varepsilon(b^*) = \{(1-\varepsilon)b^* + \varepsilon b; b, b^* \in \triangle_m \forall 0 \le \varepsilon \le 1\}.$$

$$Vol(Ball_\varepsilon(b^*)) = \varepsilon^{m-1} Vol(\triangle_m).$$

Lemma 12.2.

$$\forall b \in Ball_\varepsilon(b^*), S_T(b, x^T) \ge S_T(b^*, x^T)(1-\varepsilon)^T.$$

Theorem 12.3. *The regret of CUP algorithm with respect to log loss and set of all Constantly Rebalancing Portfolios(CRPs) can be bounded as,*

$$\max_{x^T} Regret_T(CUP, x^T) \le (m-1)\log T + \underbrace{Const}_{o(1)}$$

Proof: Consider as market seq x^T. Let b^* be the best CRP for x^T,

$$b^* = \arg\max_{b \in \triangle_m} S_T(b, x^T)$$

Now,

$$\begin{aligned} S_T(CUP, x_T) &= \int_{\triangle_m} S_t(b, x^t)\mu(db) && \text{(Using 12.1)} \\ &\ge \int_{\text{Ball}_\varepsilon(b^*)} S_T(b, x^T)\mu(db) \\ &\ge (1-\varepsilon)^T S_T(b^*, x^T) \int_{\text{Ball}_\varepsilon(b^*)} \mu(db) && \text{(Using Lemma 12.2)} \\ &= (1-\varepsilon)^T \varepsilon^{m-1}.1.S_T(b^*, x^T). && \text{(Using Lemma 12.1)} \end{aligned}$$

$$\implies \text{Regret}(\text{CUP}, X_T) = \log \frac{S_T(b^*, x^T)}{S_T(\text{CUP}, x^T)} \tag{12.2}$$

$$\le (m-1)\log(\frac{1}{\varepsilon}) + T\log\left(\frac{1}{1-\varepsilon}\right). \tag{12.3}$$

Putting $\varepsilon = \frac{1}{T}$ in 12.2,

$$\text{Regret}(\text{CUP}, X_T) \le (m-1)\log T + T\log\left(\frac{1}{1-\frac{1}{T}}\right).$$

□

Bibliography

[1] Nicoló Cesa-Bianchi and Gábor Lugosi. *Prediction, Learning and Games*. Cambridge University Press. 2006

[2] Jacob Abernethy, *Prediction and Learning: It's Only a Game*, Scibed Lecture Notes, Fall 2013 http://web.eecs.umich.edu/ jabernet/eecs598course/fall2013/web/notes/lec13_102113.pdf