Index Coded PSK Modulation for Prioritized Receivers

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Divya U. S.

under the guidance of

Prof. B. Sundar Rajan

Department of Electrical Communication Engineering
Indian Institute of Science, Bangalore
Bangalore – 560 012 (INDIA)

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Abstract

We consider the index coding problem (ICP) where a server broadcasts coded messages over a noisy channel, to a set of receivers which knows some messages a priori, in such a way that all the receivers can decode its desired messages. It has been shown that the bandwidth required for transmitting the coded messages can be reduced by employing index coded PSK modulation (ICPM), where M-ary modulation is used instead of binary transmissions [14].

In this work, the message error performance of a receiver at high SNR is characterized by a parameter called *PSK index coding gain (PSK-ICG)*. It is shown that, for a given index code and mapping (of index code vectors to PSK signal points), the PSK-ICG of a receiver is determined by a metric called *minimum inter-set distance*. For a chosen index code and an arbitrary mapping (of broadcast vectors to PSK signal points), we derive a decision rule for the maximum likelihood (ML) decoder.

For a given ICP over a single-input single-output (SISO) AWGN broadcast channel *with a priori defined arbitrary order of priority among the receivers (prioritized receivers)*, and a chosen 2^N PSK constellation we propose an algorithm to find an *(index code, mapping)* pair (not necessarily unique), which gives the best performance in terms of PSK-ICG of the receivers in the following sense:

- No other pair of index code of length N (with 2^N index code vectors) and mapping can give a better PSK-ICG for the highest priority receiver.
- Also, given that the highest priority receiver achieves its best performance, the next highest priority receiver achieves its maximum gain possible and so on.

This algorithm is based on maximising the minimum inter-set distance where as the already available algorithm [14] is based on maximising the minimum Euclidean distance. With the simulation based studies it is shown that the minimum inter-set distance based algorithm performs much better than that based on the minimum Euclidean distance. The algorithm given in [14] considers a chosen index code alone and can be used only for a specific order of priority among the receivers. In the algorithm which we propose, all possible index codes of a chosen length are considered and the receivers can have any a priori defined order of priority.

An upper bound on the coding gain that can be achieved by each of the receivers, for the given ICP and a chosen 2^N PSK modulation is also obtained.

Next, we consider ICP over a multiple-input multiple-output (MIMO) Rayleigh fading channel. The receivers are equipped with single antenna and the server with two antennas. To obtain diversity gain along with coding gain, we propose a MIMO scheme which employs space time coding along with index coded PSK modulation. For a chosen index code, an arbitrary mapping (of broadcast vectors to PSK signal points) and a 2×1 MIMO system employing Alamouti code, we derive a decision rule for the maximum likelihood (ML) decoding. We show that for the best coding gain at high SNR, the mapping must maximize the minimum inter-set distance.

Finally, we discuss *optimal index codes for an ICP with prioritized receivers when ICPM is used over a SISO AWGN channel*. We have obtained an expression for the length of such an optimal code.

Publications

Journal

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Chapter 1

Introduction

Network coding technique has significantly improved the performance of communication networks, and has been studied extensively in the past two decades. Index coding problem (ICP) can be considered as a special case of network coding problem [1]. ICP has emerged as an important topic of recent research due to its applications in many of the practically relevant problems including that in satellite networks [2], topological interference management [3], wireless caching and cache enabled cloud radio access networks for 5G cellular systems [4].

1.1 Index Coding Problem over Noiseless Broadcast Channels

The noiseless index coding problem with side information was first studied in [5] as an Informed-Source Coding-On-Demand (ISCOD) problem, in which a central server (sender) wants to broadcast data blocks to a set of clients (receivers) which already has a proper subset of the data blocks. The problem is to minimize the data that must be broadcast, so that each receiver can derive its required data blocks. Consider the case of a sender with n messages denoted by the set $\mathcal{X} = \{x_1, x_2, ..., x_n\}$, $x_i \in \mathbb{F}_q$, $\mathbb{F}_q$ is a field with q elements, which it broadcasts as coded messages, to a set of m receivers, $\mathcal{R} = \{R_1, R_2, ..., R_m\}$. Each receiver $R_i \in \mathcal{R}$ wants a subset $\mathcal{W}_i$ of the messages, knows a priori a proper subset $\mathcal{K}_i$ of the messages, where $\mathcal{W}_i \cap \mathcal{K}_i = \phi$, and is identified by the pair $(\mathcal{W}_i, \mathcal{K}_i)$. The noiseless index coding problem is to find the smallest number of transmissions required and is specified by $(\mathcal{X}, \mathcal{R})$. The set $\mathcal{K}_i$ is referred to as the side information available to the receiver R_i.

Definition 1.1. An index code (IC) for a given ICP $(\mathcal{X}, \mathcal{R})$ is defined by an encoding function, $\mathcal{E} : \mathbb{F}_q^n \to \mathbb{F}_q^l$, and a set of m decoding functions $\mathcal{D}_i : \mathbb{F}_q^l \times \mathbb{F}_q^{|\mathcal{K}_i|} \to \mathbb{F}_q^{|\mathcal{W}_i|}$, $\forall i \in \{1, 2, ..., m\}$ corresponding to the m receivers, such that,

$$\mathcal{D}_i(\mathcal{E}(\mathbf{x}), \mathcal{K}_i) = \mathcal{W}_i, \forall \mathbf{x} \in \mathbb{F}_q^n, \forall i \in \{1, 2, ..., m\}.$$

In this report, we consider ICP over binary field ($q = 2$). The integer l, as defined above is called the length of the index code. An index code of minimum length is called an optimal index code for noiseless broadcast channels [6]. If the encoding function, $\mathcal{E}$ is linear, the index code is said to be linear and if all the decoding functions $\mathcal{D}_i$ are linear, the index code is said to be linearly decodable [7].

Ong and Ho [6] classified the index coding problems into different classes based on the side information set $\mathcal{K}_i$ and the want set $\mathcal{W}_i$ of receivers. Index coding problems where each receiver demands one unique message is called a single unicast index coding problem. For such problems, the number of receivers equals the number of messages (m = n). An ICP of this class can be characterised by a *side information graph* which is a simple directed graph with n vertices (representing the messages as well as the receivers) and a set of edges, where an edge from vertex i to vertex j, denoted by (i, j) exists if and only if the receiver R_i has side information x_j [7]. Bar-Yossef *et al.* identified that, for the class of single unicast ICP, the length of an optimal linear index code is given by the *minrank* of the side information graph, where the minrank is defined as follows: For a directed graph G with n vertices and no self loops, a 0-1 matrix $A = (a_{ij})$ *fits* G if $\forall i, j \in \{1, 2, ...n\}$, $a_{ii} = 1$ and $a_{ij} = 0$, if (i, j) is not an edge in G.

Definition 1.2.

$$\text{minrank}_2(G) \triangleq \min\{\text{rank}_2(A) : A \text{ fits } G\}$$

where $\text{rank}_2(.)$ denotes the rank over binary field, $\mathbb{F}_2$.

1.2 Index Coding Problems over Noisy Channels

So far in this chapter, we have considered index coding over noiseless broadcast channel. Even though this is interesting theoretically, index coding over noisy channels is more practical.

ICP over binary symmetric channel (BSC) was studied by Dau *et al.* [8]. The concept of linear error-correcting index code (ECIC) was introduced and the bounds on the optimal length of an ECIC was established. Error correcting decoding and syndrome decoding using ECIC was also discussed.

Noisy index coding over a wireless fading channel was considered in [9], [10]. Binary transmission of index coded bits was assumed and so a minimum length index code minimises the bandwidth consumed. It was found that, among the several optimal index codes, the one which minimises the maximum number of binary transmissions used by any receiver in decoding its desired messages, will result in the minimum error probability. For a special class of index coding problems, an algorithm to identify the optimal index codes which minimise maximum probability of error, among all the optimal index codes was also given. For this set up, the problem of identifying the number of optimal index codes possible for a given ICP is important and that was studied in [11], [12]. These studies considered binary transmissions and so these were not bandwidth optimal.

A special case of ICP over Gaussian broadcast channel, based on multidimensional QAM constellation with 2^n points, where every receiver demands all messages (which it does not have) from the source, was considered by Natarajan *et al.* [13]. A code design metric called *side information gain* was used to define the efficiency with which the index code exploits the side information.

The case of noisy index coding over AWGN broadcast channel, along with minimum Euclidean distance decoding, was studied in [14], [15], where the receivers demand a subset of messages as defined in [5]. The bandwidth required for transmitting the index codes can be reduced if M-ary modulation is used instead of binary transmissions and this scheme was referred to as index coded PSK modulation (ICPM). *PSK side information coding gain* was defined as the coding gain obtained by a receiver with side information, relative to a receiver with no side information. Receivers with side information, satisfying certain specified conditions, need to search only through a reduced number of signal points, which was referred to as the *effective signal set* seen by the receiver. An algorithm to map the broadcast vectors to PSK signal points so that the receiver with maximum side information gets maximum PSK side information coding gain, was also proposed. The algorithm assumes that an index code is given and is applicable only for one specific order of priority (in the non increasing order of amount of side information) among the receivers. Minimum Euclidean distance of the effective broadcast signal set seen by a receiver, was considered as the basic parameter which decides the message error probability of the receiver and the proposed algorithm tries to maximize the minimum Euclidean distance.

The problem of constructing space time codes for a special case of ICP where every receiver demands all messages (which it does not have) from the source, was considered in [16]. Even though the problem of broadcasting over a multiple-input multiple-output (MIMO) channel when the receivers have some side information was addressed, only a special case of ICP was considered.

In this report, first we consider noisy index coding over a single-input single-output (SISO) AWGN broadcast channel and discuss the maximum likelihood (ML) decoder for ICPM. We study the case in which only the length of the index code is specified for the ICP but not necessarily the index code. The receivers can have any a priori defined arbitrary order of priority among themselves (prioritized receivers). This ordering need not have any relation to the amount of side-information the receivers have which is the only case considered in [14]. For a chosen priority order, we consider all possible index codes, to obtain the mappings to appropriate PSK constellation which will result in the best message error performance in terms of PSK index coding gain (PSK-ICG, defined in Section 2.4) of the receivers, respecting the defined order of priority. Next, we consider the general case of a noisy index coding problem over a MIMO Rayleigh fading channel, where the receivers demand a subset of messages as defined in [5]. We propose a MIMO scheme, which employs the concept of ICPM, to provide both diversity gain and coding gain over a fading channel. Finally, we obtain an expression for the length of an optimal index code for ICP over a SISO AWGN channel when ICPM is used for prioritized receivers.

1.3 Our Contribution

First we consider a noisy index coding problem with n messages, over $\mathbb{F}_2$, which uses an AWGN broadcast channel for transmission. For the ICP $(\mathcal{X}, \mathcal{R})$, consider index codes of length N, $N < n$, which will generate 2^N broadcast vectors (elements of $\mathbb{F}_2^N$). The broadcast vectors are mapped to 2^N-PSK signal points, so that 2^N-PSK modulation can be used, to minimize the bandwidth requirement. Note that, transmitting one 2^N-PSK signal point instead of N BPSK signal points (as in noiseless index coding), results in $N/2$ fold saving in bandwidth.

Our contributions are summarized below:

- We derive a decision rule for maximum likelihood decoding which gives the best message error performance, for any receiver R_i, for a given index code and mapping.
- We show that, at very high SNR, the message error performance of the receiver employing ML decoder, depends on the minimum inter-set distance (defined in Section 2.4). The mapping which maximize the minimum inter-set distance is optimal for the best message error performance at high SNR.
- For the ICP $(\mathcal{X}, \mathcal{R})$, when the receivers are arranged in the decreasing order of priority, we propose an algorithm to find (index code, mapping) pairs, each of which gives the best message error performance for the receivers, for the given order of priority. Using

any one of the above (index code, mapping) pairs, the highest priority receiver achieves the maximum gain (PSK-ICG), that it can get using any IC and any mapping for 2^N-PSK constellation, at very high SNR. Given that the highest priority receiver achieves its best performance, the next highest priority receiver achieves its maximum gain possible and so on in the specified order of priority.

- An upper bound on the coding gain (PSK-ICG) that can be achieved by each receiver for the ICP $(\mathcal{X}, \mathcal{R})$ and a chosen 2^N PSK modulation, is obtained by considering each one of the receivers as the highest priority receiver.

Next, we extend the SISO case with AWGN broadcast channel to MIMO with Rayleigh fading channel. Our contributions in this extended case are summarized below:

- We propose a MIMO scheme employing Alamouti code over the 2^N-PSK signal set, for the noisy index coding problem, to achieve diversity gain.
- With the proposed scheme, for a chosen index code and 2^N-PSK signal set, we show that, for a receiver to attain its best coding gain at high SNR, the mapping (of broadcast vectors to 2^N-PSK signal points) must maximize the minimum inter-set distance.
- We derive a decision rule for maximum likelihood decoding which gives the best message error performance, for any receiver R_i, in the MIMO scheme, for a given index code and arbitrary mapping.

We have addressed the problem of finding the optimal index codes for a given ICP over a SISO AWGN broadcast channel with prioritized receivers employing ICPM. In this regard, we considered index codes of all possible lengths along with the appropriate constellation sizes.

- We have obtained an expression for the length of optimal index codes for an ICP with prioritized receivers when ICPM is used over an AWGN channel. Once the optimal length is known, we can find the best possible (index code, mapping) pairs across all possible index codes and mappings using the proposed algorithm.

1.4 Organisation of the Report

The main content of this report is organised into six chapters. Chapter 1 gives a background on the index coding problem (noiseless and noisy), recent works in this area and our specific contributions.

Chapter 2 explains the basic concepts of ICPM. The notations used through out this report are also provided. Then, a decision rule for the ML decoder of ICPM is derived and its high SNR approximation is obtained. The last part of this chapter explains a very important metric called inter-set distance and a parameter called PSK index coding gain.

Chapter 3 focuses on the case of ICP with prioritised receivers over a SISO AWGN channel. An algorithm for obtaining optimal (index code, mapping) pairs for a chosen constellation size (length of index code) is proposed. An upper bound on the coding gain that can be achieved by each of the receivers for a chosen 2^N PSK modulation is also obtained. The chapter concludes with the simulation results which further illustrates the efficiency of the proposed schemes.

In Chapter 4, a scheme with Alamouti code is proposed, to extend the concept of ICPM over a SISO AWGN channel to a Rayleigh faded MIMO channel. ML decoder decision rule, its high SNR approximation, diversity gain and coding gain of the proposed scheme, and simulation results for the MIMO scheme are included in this chapter.

Till chapter 4, the work primarily focuses on ICPM with a chosen constellation size (that means we considered only index codes of a chosen length). Chapter 5 addresses the problem of finding the length of optimal index codes for an ICP with prioritized receivers when ICPM is used over a SISO AWGN channel and an expression for the optimal length is obtained for the case where $|\mathcal{K}_1| < n - 1$.

The report is concluded in Chapter 6 by listing down several interesting problems for future work.

Chapter 2

Index Coded PSK Modulation and Inter-Set Distance

[1] In this chapter, the concept of ICPM over a SISO AWGN broadcast channel (here after referred to as SISO-AWGN-ICPM), where the sender and the receivers have one antenna each, and the related notions are included. We derive a decision rule for the ML decoder and discuss a high SNR approximation for this decoder. The concept of inter-set distance and PSK-ICG, which determines the message error performance of the receivers, is introduced.

2.1 Preliminaries and Notation

Let $[n] \triangleq \{1, 2, ..., n\}$. For a vector $\mathbf{z} = (z_1 z_2 ... z_n) \in \mathbb{F}_2^n$ and a subset $B = \{i_1, i_2, ..., i_b\}$ of $[n]$ (for any integer $b, 1 \leq b \leq n$), where $i_1 < i_2 < ... < i_b$, $\mathbf{z}_B$ denotes the vector $(z_{i_1} z_{i_2} ... z_{i_b})$.

Consider the noisy index coding problem over $\mathbb{F}_2$ with a single sender having a set of messages $\mathcal{X} = \{x_1, x_2, ..., x_n\}$, $x_i \in \mathbb{F}_2$, and a set of m receivers, $\mathcal{R} = \{R_1, R_2, ..., R_m\}$, where each

[1] A part of the content of this chapter appears in

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receiver R_i is identified by $(\mathcal{W}_i, \mathcal{K}_i)$, the *want* set and the *known* set. Let, $\mathcal{I}_i \triangleq \{j : x_j \in \mathcal{K}_i\}$ be the set of indices corresponding to the known set. It is sufficient to consider the case where each receiver demands only one message. If there is a receiver which demands more than one message, it can be considered as $|\mathcal{W}_i|$ equivalent receivers each demanding one message and having the same side information. Each R_i, $i \in [m]$ wants the message $x_{f(i)}$, where $f : [m] \to [n]$ and $x_{f(i)} \notin \mathcal{K}_i$, $\forall i \in [m]$.

For the given ICP, we consider scalar linear index codes of length N (not necessarily the minimum or optimum length), such that the set of all broadcast vectors gives $\mathbb{F}_2^N$. Let L be an $n \times N$ encoding matrix for one such index code, $\mathcal{C}$. Let $\mathbf{x} = (x_1 x_2 ... x_n)$ and $\mathbf{y} = (y_1 y_2 ... y_N)$ denote the message vector and the broadcast vector respectively, where $\mathbf{y} = \mathbf{x}L$.

Example 2.1. Consider the following ICP with $n = m = 5$ and $\mathcal{W}_i = x_i, \forall i \in \{1, 2, ..., 5\}$. The side information available with the receivers is as follows: $\mathcal{K}_1 = \{x_2, x_3\}$, $\mathcal{K}_2 = \{x_3, x_4, x_5\}$, $\mathcal{K}_3 = \{x_2, x_4, x_5\}$, $\mathcal{K}_4 = \{x_5\}$, $\mathcal{K}_5 = \{x_4\}$.

For this ICP we can choose a scalar linear index code of length $N = 3$, as given by the following encoding matrix L.

$$L = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

.

The index coded bits are given by

$(y_1 y_2 y_3) = (x_1 x_2 x_3 x_4 x_5)L$ as $y_1 = x_1 + x_4 + x_5$, $y_2 = x_1 + x_2 + x_3 + x_4 + x_5$, $y_3 = x_4 + x_5$.

Instead of using N BPSK transmissions, the N index coded bits of $\mathbf{y}$ are sent as a signal point from a 2^N-PSK signal set, over an AWGN channel, to save bandwidth [14]. We assume that for SISO-AWGN-ICPM, all the signal points are of unit energy (unless otherwise stated). In this report, we consider index coded 2^N-PSK modulation for a chosen N and so when we refer to index codes of length N, we consider only those index codes for which the set of all broadcast vectors is $\mathbb{F}_2^N$. Let the chosen 2^N-PSK signal set be denoted as $\mathcal{S} = \{s_1, s_2, ..., s_{2^N}\}$. Assume that for the index code $\mathcal{C}$ a mapping scheme specifies the mapping of $\mathbb{F}_2^N$ to the signal set $\mathcal{S}$. All receivers are assumed to know the encoding matrix, L for the index code $\mathcal{C}$.

Let $\mathbf{a}_i \in \mathbb{F}_2^{|\mathcal{K}_i|}$ be a realization of $\mathbf{x}_{\mathcal{I}_i}$. As each receiver R_i knows some messages (from its side information), R_i needs to consider only a subset of $\mathbb{F}_2^N$ for decoding and this subset is called the *effective broadcast vector set*.

Definition 2.1. For a chosen index code based on the encoding matrix L, the *effective broadcast vector set* seen by R_i for $\mathbf{x}_{\mathcal{I}_i} = \mathbf{a}_i$ is defined by,

$$\mathcal{C}_L(\mathbf{a}_i) \triangleq \{\mathbf{y} \in \mathbb{F}_2^N : \mathbf{y} = \mathbf{x}L, \mathbf{x}_{\mathcal{I}_i} = \mathbf{a}_i, x_{j'} \in \mathbb{F}_2, j' \in [n] \setminus \mathcal{I}_i\}.$$

The corresponding set of signal points in 2^N-PSK constellation is referred to as the *effective broadcast signal set* seen by R_i for $\mathbf{x}_{\mathcal{I}_i} = \mathbf{a}_i$ and is denoted by $\mathcal{S}_L(\mathbf{a}_i)$. For a chosen index code, all effective broadcast signal sets and effective broadcast vector sets seen by R_i are of the same size ($|\mathcal{S}_L(\mathbf{a}_i)| = |\mathcal{S}_L(\mathbf{a}'_i)| = |\mathcal{C}_L(\mathbf{a}_i)| = |\mathcal{C}_L(\mathbf{a}'_i)|$ where $\mathbf{a}_i, \mathbf{a}'_i \in \mathbb{F}_2^{|\mathcal{K}_i|}$).

Half the number of broadcast vectors in an effective broadcast vector set corresponds to $x_{f(i)} = 0$ and the remaining half corresponds to $x_{f(i)} = 1$. So, we can partition an effective broadcast vector set into two subsets as defined below.

Definition 2.2. The *0-effective broadcast vector set* seen by R_i for $\mathbf{x}_{\mathcal{I}_i} = \mathbf{a}_i$ is defined by,

$$\mathcal{C}_{L0}(\mathbf{a}_i) \triangleq \{\mathbf{y} \in \mathbb{F}_2^N : \mathbf{y} = \mathbf{x}L, \mathbf{x}_{\mathcal{I}_i} = \mathbf{a}_i, x_{f(i)} = 0, x_{j'} \in \mathbb{F}_2, j' \in [n] \setminus (\mathcal{I}_i \cup \{f(i)\})\}.$$

The corresponding set of signal points in 2^N-PSK constellation is referred to as the *0-effective broadcast signal set* seen by R_i for $\mathbf{x}_{\mathcal{I}_i} = \mathbf{a}_i$ and is denoted as $\mathcal{S}_{L0}(\mathbf{a}_i)$.

Definition 2.3. The *1-effective broadcast vector set* seen by R_i for $\mathbf{x}_{\mathcal{I}_i} = \mathbf{a}_i$ is defined by,

$$\mathcal{C}_{L1}(\mathbf{a}_i) \triangleq \{\mathbf{y} \in \mathbb{F}_2^N : \mathbf{y} = \mathbf{x}L, \mathbf{x}_{\mathcal{I}_i} = \mathbf{a}_i, x_{f(i)} = 1, x_{j'} \in \mathbb{F}_2, j' \in [n] \setminus (\mathcal{I}_i \cup \{f(i)\})\}.$$

The corresponding set of signal points in 2^N-PSK constellation is referred to as the *1-effective broadcast signal set* seen by R_i for $\mathbf{x}_{\mathcal{I}_i} = \mathbf{a}_i$ and is denoted as $\mathcal{S}_{L1}(\mathbf{a}_i)$.

The effective broadcast vector sets, 0-effective broadcast vector sets and 1-effective broadcast vector sets seen by R_2 for the IC in Example 2.1 is given in Table 2.1. It is clear that, two different realizations of $\mathbf{x}_{\mathcal{I}_i}$ may have the same effective broadcast vector set. However, 1-effective broadcast vector set for a particular realization of $\mathbf{x}_{\mathcal{I}_i}$ may become the 0-effective broadcast vector set of another realization of $\mathbf{x}_{\mathcal{I}_i}$ and vice versa. But the way in which the effective broadcast vector set gets partitioned will be the same. For example consider the case of $\mathcal{C}_L(011)$ and $\mathcal{C}_L(100)$ in Table 2.1.

Table 2.1: Effective broadcast vector sets and its partitions (seen by R_2) for the IC in Example 2.1.

$\mathbf{a}_2$	$\mathcal{C}_L(\mathbf{a}_2)$	$\mathcal{C}_{L0}(\mathbf{a}_2)$	$\mathcal{C}_{L1}(\mathbf{a}_2)$
(000)	$\{(000),(010),(110),(100)\}$	$\{(000),(110)\}$	$\{(010),(100)\}$
(001)	$\{(111),(101),(001),(011)\}$	$\{(111),(001)\}$	$\{(101),(011)\}$
(010)	$\{(111),(101),(001),(011)\}$	$\{(111),(001)\}$	$\{(101),(011)\}$
(011)	$\{(000),(010),(110),(100)\}$	$\{(000),(110)\}$	$\{(010),(100)\}$
(100)	$\{(000),(010),(110),(100)\}$	$\{(010),(100)\}$	$\{(000),(110)\}$
(101)	$\{(111),(101),(001),(011)\}$	$\{(101),(011)\}$	$\{(111),(001)\}$
(110)	$\{(111),(101),(001),(011)\}$	$\{(101),(011)\}$	$\{(111),(001)\}$
(111)	$\{(000),(010),(110),(100)\}$	$\{(010),(100)\}$	$\{(000),(110)\}$

Example 2.2. Consider the following ICP with $n = m = 6$ and $\mathcal{W}_i = x_i, \forall i \in \{1,2,...,6\}$. The side information available with the receivers is as follows: $\mathcal{K}_1 = \{x_2,x_3,x_4,x_5,x_6\}$, $\mathcal{K}_2 = \{x_1,x_3,x_4,x_5\}$, $\mathcal{K}_3 = \{x_2,x_4,x_6\}$, $\mathcal{K}_4 = \{x_1,x_6\}$, $\mathcal{K}_5 = \{x_3\}$, $\mathcal{K}_6 = \{\}$ (null set).

For this ICP we can choose a scalar linear index code of length $N = 4$, based on encoding matrix L, with $y_1 = x_1 + x_4$, $y_2 = x_2 + x_3$, $y_3 = x_5$, $y_4 = x_6$.
Then, the effective broadcast vector sets and its partitions as seen by R_2 are as given in Table 2.2.

Suppose an IC based on an encoding matrix L, and an effective broadcast vector set, $\mathcal{C}_L(\mathbf{a}_i)$ of R_i are given. $\mathcal{C}_L(\mathbf{a}_i)$ can be partitioned into 0-effective broadcast vector set and 1-effective broadcast vector set as follows:

Table 2.2: Effective broadcast vector sets and its partitions (seen by R_2) for the IC in Example 2.2.

$\mathbf{a}_2$	$\mathcal{C}_L(\mathbf{a}_2)$	$\mathcal{C}_{L0}(\mathbf{a}_2)$	$\mathcal{C}_{L1}(\mathbf{a}_2)$
(0000)	$\{(0000),(0100),(0001),(0101)\}$	$\{(0000),(0001)\}$	$\{(0100),(0101)\}$
(0001)	$\{(0010),(0011),(0110),(0111)\}$	$\{(0010),(0011)\}$	$\{(0110),(0111)\}$
(0010)	$\{(1000),(1001),(1100),(1101)\}$	$\{(1000),(1001)\}$	$\{(1100),(1101)\}$
(0011)	$\{(1010),(1011),(1110),(1111)\}$	$\{(1010),(1011)\}$	$\{(1110),(1111)\}$
(0100)	$\{(0000),(0100),(0001),(0101)\}$	$\{(0100),(0101)\}$	$\{(0000),(0001)\}$
(0101)	$\{(0010),(0011),(0110),(0111)\}$	$\{(0110),(0111)\}$	$\{(0010),(0011)\}$
(0110)	$\{(1000),(1001),(1100),(1101)\}$	$\{(1100),(1101)\}$	$\{(1000),(1001)\}$
(0111)	$\{(1010),(1011),(1110),(1111)\}$	$\{(1110),(1111)\}$	$\{(1010),(1011)\}$
(1000)	$\{(1000),(1001),(1100),(1101)\}$	$\{(1000),(1001)\}$	$\{(1100),(1101)\}$
(1001)	$\{(1010),(1011),(1110),(1111)\}$	$\{(1010),(1011)\}$	$\{(1110),(1111)\}$
(1010)	$\{(0000),(0100),(0001),(0101)\}$	$\{(0000),(0001)\}$	$\{(0100),(0101)\}$
(1011)	$\{(0010),(0011),(0110),(0111)\}$	$\{(0010),(0011)\}$	$\{(0110),(0111)\}$
(1100)	$\{(1000),(1001),(1100),(1101)\}$	$\{(1100),(1101)\}$	$\{(1000),(1001)\}$
(1101)	$\{(1010),(1011),(1110),(1111)\}$	$\{(1110),(1111)\}$	$\{(1010),(1011)\}$
(1110)	$\{(0000),(0100),(0001),(0101)\}$	$\{(0100),(0101)\}$	$\{(0000),(0001)\}$
(1111)	$\{(0010),(0011),(0110),(0111)\}$	$\{(0110),(0111)\}$	$\{(0010),(0011)\}$

- Identify an $\mathbf{x}$ such that $\mathbf{x}L \in \mathcal{C}_L(\mathbf{a}_i)$. Let the corresponding realization of $\mathbf{x}_{\mathcal{I}_i}$ be $\mathbf{a}_i$.
- For $\mathbf{a}_i$, partition $\mathcal{C}_L(\mathbf{a}_i)$ into $\mathcal{C}_{L0}(\mathbf{a}_i)$ and $\mathcal{C}_{L1}(\mathbf{a}_i)$

The partitioning of $\mathcal{C}_L(\mathbf{a}_i)$ can be illustrated with the ICP given in Example 2.2. Suppose the effective broadcast vector set, $\mathcal{C}_L(\mathbf{a}_2) = \{(0000),(0100),(0001),(0101)\}$ of R_2 needs to be partitioned into $\mathcal{C}_{L0}(\mathbf{a}_2)$ and $\mathcal{C}_{L1}(\mathbf{a}_2)$. Choose $\mathbf{x} = (110100)$ such that $\mathbf{y} = \mathbf{x}L = (0100) \in \mathcal{C}_L(\mathbf{a}_2)$, and then $\mathbf{a}_2 = (1010)$. Note that $y_2 = x_2 + x_3$, $x_3 \in \mathcal{K}_2$, R_2 wants x_2, and from $\mathbf{a}_2 = (1010)$, $x_3 = 0$. So $y_2 = x_2$ and only two broadcast vectors, (0000) and (0001) in $\mathcal{C}_L(1010)$ has $y_2 = 0$. So $\mathcal{C}_{L0}(1010) = \{(0000),(0001)\}$. Similarly, $\mathcal{C}_{L1}(1010) = \{(0100),(0101)\}$. It should be noted that for some other choice of $\mathbf{x}$ with $x_3 = 1$, we may get $\mathcal{C}_{L0}(\mathbf{a}_2) = \{(0100),(0101)\}$ and $\mathcal{C}_{L1}(\mathbf{a}_2) = \{(0000),(0001)\}$. We are only interested in partitioning the effective broadcast vector set into two subsets such that all broadcast vectors in each subset correspond to the same value of $x_{f(i)}$.

2.2 Maximum Likelihood Decoder

In this section we derive a decision rule for the maximum likelihood decoder for the receiver R_i. We follow an approach similar to the one used in [17] to obtain a decision rule for the ML decoder for the receiver R_i.

Let $\mathcal{M}$ be the map from $\mathbb{F}_2^N$ to the signal set $\mathcal{S}$. The received vector r is given by

$$r = \mathcal{M}(\mathbf{x}L) + w$$

where w is distributed as $\mathcal{CN}(0, N_0)$. The conditional probability density of r given that $\mathcal{M}(\mathbf{x}L)$ is transmitted (likelihood function) is

$$p(r|\mathcal{M}(\mathbf{x}L)) = \frac{1}{(\pi N_0)} \exp\left(-\frac{\|r - \mathcal{M}(\mathbf{x}L)\|^2}{N_0}\right). \tag{2.1}$$

Consider the decoder for a receiver R_i. The minimum error probability decoder should make a decision $x'_{f(i)}$ on the desired message $x_{f(i)}$ based on the received vector r and the side information $\mathbf{x}_{\mathcal{I}_i}$, minimizing the probability of error. Given $\mathbf{x}_{\mathcal{I}_i} = \mathbf{a}_i$, when $x_{f(i)} = 0$ the probability of error in this decision is $\mathbf{Pr}(x_{f(i)} = 1|\mathbf{x}_{\mathcal{I}_i} = \mathbf{a}_i, r)$ and that when $x_{f(i)} = 1$ is $\mathbf{Pr}(x_{f(i)} = 0|\mathbf{x}_{\mathcal{I}_i} = \mathbf{a}_i, r)$. To minimize the error probability, the decision $x'_{f(i)} = 0$ is taken if

$$\mathbf{Pr}(x_{f(i)} = 0|\mathbf{x}_{\mathcal{I}_i} = \mathbf{a}_i, r) > \mathbf{Pr}(x_{f(i)} = 1|\mathbf{x}_{\mathcal{I}_i} = \mathbf{a}_i, r) \tag{2.2}$$

and the decision $x'_{f(i)} = 1$ is taken if

$$\mathbf{Pr}(x_{f(i)} = 0|\mathbf{x}_{\mathcal{I}_i} = \mathbf{a}_i, r) < \mathbf{Pr}(x_{f(i)} = 1|\mathbf{x}_{\mathcal{I}_i} = \mathbf{a}_i, r). \tag{2.3}$$

Combining (2.2) and (2.3), and ignoring ties, the decision rule can be written as

$$\mathbf{Pr}(x_{f(i)} = 0|\mathbf{x}_{\mathcal{I}_i} = \mathbf{a}_i, r) \underset{0}{\overset{1}{\lessgtr}} \mathbf{Pr}(x_{f(i)} = 1|\mathbf{x}_{\mathcal{I}_i} = \mathbf{a}_i, r). \tag{2.4}$$

Using Bayes rule in (2.4), we obtain the decision rule in terms of the likelihood functions as

$$\frac{p(r|x_{f(i)}=0,\mathbf{x}_{\mathcal{I}_i}=\mathbf{a}_i)\mathbf{Pr}(x_{f(i)}=0)}{p(r)} \underset{0}{\overset{1}{\lessgtr}} \frac{p(r|x_{f(i)}=1,\mathbf{x}_{\mathcal{I}_i}=\mathbf{a}_i)\mathbf{Pr}(x_{f(i)}=1)}{p(r)},$$

which implies

$$p(r|x_{f(i)}=0,\mathbf{x}_{\mathcal{I}_i}=\mathbf{a}_i)\mathbf{Pr}(x_{f(i)}=0) \underset{0}{\overset{1}{\lessgtr}} p(r|x_{f(i)}=1,\mathbf{x}_{\mathcal{I}_i}=\mathbf{a}_i)\mathbf{Pr}(x_{f(i)}=1). \tag{2.5}$$

$\mathcal{S}_{L0}(\mathbf{a}_i)$, the 0-effective broadcast signal set seen by R_i (for $\mathbf{a}_i$), is the set of all signal points corresponding to broadcast vectors with $x_{f(i)}=0$ and $\mathbf{x}_{\mathcal{I}_i}=\mathbf{a}_i$. Therefore,

$$p(r|x_{f(i)}=0,\mathbf{x}_{\mathcal{I}_i}=\mathbf{a}_i)=p(r|\mathcal{S}_{L0}(\mathbf{a}_i)). \tag{2.6}$$

Similarly,

$$p(r|x_{f(i)}=1,\mathbf{x}_{\mathcal{I}_i}=\mathbf{a}_i)=p(r|\mathcal{S}_{L1}(\mathbf{a}_i)). \tag{2.7}$$

Assuming that all the messages take values 0 or 1 with equal probability, from (2.5), (2.6) and (2.7) we obtain the decision rule as

$$\sum_{k:s_k\in\mathcal{S}_{L0}(\mathbf{a}_i)} p(r|s_k) \underset{0}{\overset{1}{\lessgtr}} \sum_{k:s_k\in\mathcal{S}_{L1}(\mathbf{a}_i)} p(r|s_k). \tag{2.8}$$

From (2.1) and (2.8),

$$\sum_{k:s_k\in\mathcal{S}_{L0}(\mathbf{a}_i)} \left(\frac{1}{(\pi N_0)}\exp\left(-\frac{\|r-s_k\|^2}{N_0}\right)\right) \underset{0}{\overset{1}{\lessgtr}} \sum_{k:s_k\in\mathcal{S}_{L1}(\mathbf{a}_i)} \left(\frac{1}{(\pi N_0)}\exp\left(-\frac{\|r-s_k\|^2}{N_0}\right)\right).$$

Thus we obtain the ML decision rule as,

$$\sum_{k:s_k\in\mathcal{S}_{L0}(\mathbf{a}_i)} \left(\exp\left(-\frac{\|r-s_k\|^2}{N_0}\right)\right) \underset{0}{\overset{1}{\lessgtr}} \sum_{k:s_k\in\mathcal{S}_{L1}(\mathbf{a}_i)} \left(\exp\left(-\frac{\|r-s_k\|^2}{N_0}\right)\right). \tag{2.9}$$

2.3 High SNR Approximation for ML Decoder

At high SNR, we can approximate the ML decision rule by considering only the dominant terms on both the sides (assuming that the SNR is high enough that the contribution of the remaining terms will not make any change in the decision) of (2.9). In such a case, the decoding is same as the minimum Euclidean distance decoding. The decoder will find the signal point in the effective broadcast signal set, which is closest to r. If the decoded signal point belongs to 0-effective broadcast signal set the desired message is decoded as 0, else 1. Simulation results (Section 3.4.2, Figure 3.7 and Figure 3.8) also indicate that this approximation is valid at high SNR. So, even though the minimum Euclidean distance decoder is not theoretically optimal, for practical purposes, at high SNR, the minimum Euclidean distance decoder can be used instead of ML decoder.

2.4 Inter-Set Distance and PSK Index Coding Gain

It is clear that the ML decoder decision (2.9) is based on the Euclidean distance of all signal points in 0-effective broadcast signal set to the received vector r relative to that of the signal points in 1-effective broadcast signal set. This indicates that, to reduce the message error probability, the signal points in 0-effective broadcast signal set and 1-effective broadcast signal set must be as separated as possible in terms of Euclidean distance.

Definition 2.4. *Inter-set distance* of an effective broadcast signal set seen by a receiver R_i is the minimum among the Euclidean distances between a signal point in the 0-effective broadcast signal set and a signal point in the 1-effective broadcast signal set.

$$d_{IS}(\mathcal{S}_L(\mathbf{a}_i)) \triangleq \min\{d(s_a, s_b) : s_a \in \mathcal{S}_{L0}(\mathbf{a}_i), s_b \in \mathcal{S}_{L1}(\mathbf{a}_i)\}$$

where $d(s_a, s_b)$ denotes the Euclidean distance between 2^N-PSK signal points, s_a and s_b.

Consider R_1 in Example 2.1. Assume that a mapping as shown in Figure 2.1(a) is chosen.

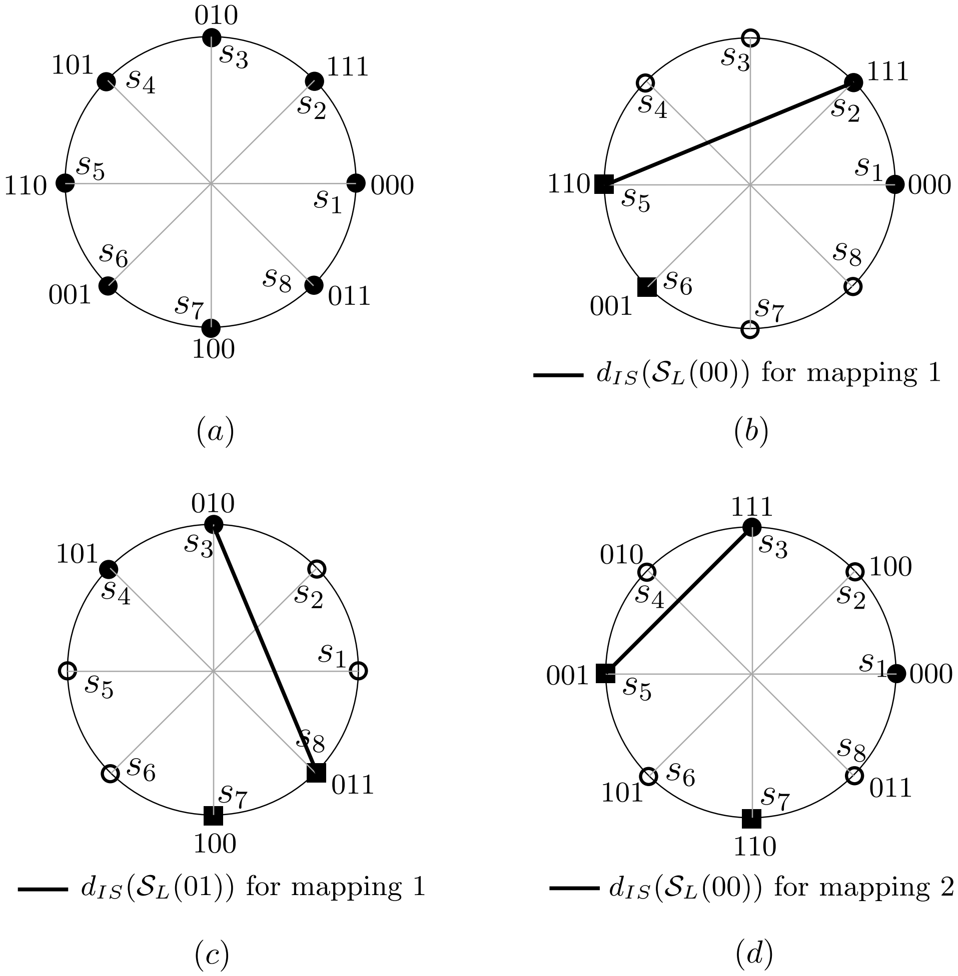

Figure 2.1: 8-PSK mapping and inter-set distance for R_1 in Example 2.1.

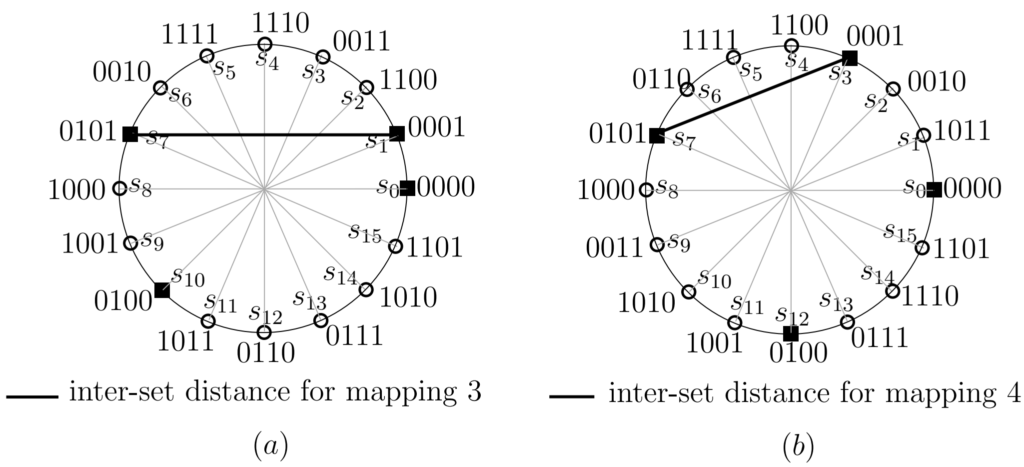

Figure 2.2: Inter-set distance ($d_{IS}(\mathcal{S}_L(0000))$) of R_2 for the ICP in Example 2.2 for two different mappings.

Let the side information available with R_1 be $\mathbf{x}_{\mathcal{I}_1} = (00)$. R_1 can find the effective broadcast vector set as $\mathcal{C}_L(00) = \{(000), (111), (001), (110)\}$ and its partitions as $\mathcal{C}_{L0}(00) = \{(000), (111)\}$ and $\mathcal{C}_{L1}(00) = \{(110), (001)\}$. From the mapping, R_1 finds $\mathcal{S}_L(00) = \{s_1, s_2, s_5, s_6\}$, $\mathcal{S}_{L0}(00) = \{s_1, s_2\}$ and $\mathcal{S}_{L1}(00) = \{s_5, s_6\}$. The inter-set distance of the effective broadcast signal set seen by R_1 for this case is shown in Figure 2.1(b). Next consider the case when the side information available with R_1 is $\mathbf{x}_{\mathcal{I}_i} = (01)$. Then, the effective broadcast vector set is $\mathcal{C}_L(01) = \{(010), (101), (100), (011)\}$ and its partitions can be obtained as $\mathcal{C}_{L0}(01) = \{(010), (101)\}$ and $\mathcal{C}_{L1}(01) = \{(100), (011)\}$. From the mapping, $\mathcal{S}_L(01) = \{s_3, s_4, s_7, s_8\}$, $\mathcal{S}_{L0}(00) = \{s_3, s_4\}$ and $\mathcal{S}_{L1}(00) = \{s_7, s_8\}$. The inter-set distance of the effective broadcast signal set seen by R_1 for this case with, $\mathbf{x}_{\mathcal{I}_1} = (01)$ is shown in Figure 2.1(c).

For the ICP given in Example 2.2 for receiver R_2, $\mathcal{C}_L(0000) = \{(0000), (0001), (0100), (0101)\}$. The inter-set distance of $\mathcal{S}_L(0000)$ seen by R_2 for two different mappings is given in Figure 2.2. The inter-set distance is more for mapping 3 (Figure 2.2(a)) than for mapping 4 (Figure 2.2(b)).

Definition 2.5. For a given index code and mapping, the minimum inter-set distance for a receiver R_i, denoted by $d^{(i)}_{IS,min}$, is defined as the minimum of the inter-set distances among all the effective broadcast signal sets seen by R_i.

$$d^{(i)}_{IS,min} \triangleq \min\{d_{IS}(\mathcal{S}_L(\mathbf{a}_i)) : \mathbf{a}_i \in \mathbb{F}_2^{|\mathcal{K}_i|}\}$$

In the case of Example 2.1 with the mapping as shown in 2.1(a), the inter-set distance of both the effective broadcast signal sets seen by the receiver R_1 is the same and so the minimum inter-set distance is same as the inter-set distance of any one of the effective broadcast signal sets and is as shown in Figure 2.1(b) or 2.1(c). The minimum inter-set distance for R_1 for another mapping is shown in 2.1(d). Clearly, the mapping shown in Figure 2.1(a) has a larger minimum inter-set distance for R_1 than the one in Figure 2.1(d). The minimum inter-set distance for all the receivers is given in Table 2.3.

Similarly, the minimum inter-set for the receivers in the ICP given in Example 2.2 is given in

Table 2.3: Minimum inter-set distance for the receivers in the ICP in Example 2.1.

Mapping	$d^{(1)}_{IS,min}$	$d^{(2)}_{IS,min}$	$d^{(3)}_{IS,min}$	$d^{(4)}_{IS,min}$	$d^{(5)}_{IS,min}$
Mapping 1	1.8477	1.4142	1.4142	0.7653	0.7653
Mapping 2	1.4142	0.7653	0.7653	0.7653	0.7653

Table 2.4: Minimum inter-set distance for the receivers in the ICP in Example 2.2.

Mapping	$d^{(1)}_{IS,min}$	$d^{(2)}_{IS,min}$	$d^{(3)}_{IS,min}$	$d^{(4)}_{IS,min}$	$d^{(5)}_{IS,min}$	$d^{(6)}_{IS,min}$
Mapping 3	2	1.8477	0.7653	0.7653	0.3901	0.3901
Mapping 4	2	1.1111	0.7653	0.7653	0.3901	0.3901

Table 2.4 for the mappings given in Figure 2.2.

Definition 2.6. The PSK Index Coding Gain (PSK-ICG) of a receiver R_i, for a given IC and mapping is defined as

$$g_i \triangleq 20 \times log\left(\frac{d^{(i)}_{IS,min}}{d_{min,n}}\right)$$

where $d^{(i)}_{IS,min}$ is the minimum inter-set distance for R_i and $d_{min,n}$ is the minimum Euclidean distance between any two signal points in a 2^n-PSK constellation.

The PSK-ICG of the receivers for the ICP discussed in Example 2.1 is given in Table 2.5.

Table 2.5: PSK-ICG (in dB) of the receivers for the ICP considered in Example 2.1.

Mapping	g_1	g_2	g_3	g_4	g_5
Mapping 1	19.48	17.16	17.16	11.83	11.83
Mapping 2	17.16	11.83	11.83	11.83	11.83

Similarly, the PSK-ICG of the receivers for the ICP considered in Example 2.2, is given in Table 2.6.

Table 2.6: PSK-ICG (in dB) of the receivers for the ICP discussed in Example 2.2.

Mapping	g_1	g_2	g_3	g_4	g_5	g_6
Mapping 3	26.18	25.49	17.84	17.84	11.98	11.98
Mapping 4	26.18	21.07	17.84	17.84	11.98	11.98

Chapter 3

SISO Index Coded PSK Modulation for Prioritized Receivers

[1] In this section we consider SISO-AWGN-ICPM where the receivers are prioritized. For a given ICP and a chosen 2^N PSK constellation, we propose an algorithm to find (index code, mapping) pairs, each of which gives the best message error performance for the receivers, for the given order of priority. An upper bound on the coding gain that can be achieved by each of the receivers for the given ICP and a chosen 2^N PSK modulation is also obtained.

3.1 Concept of Prioritized Receivers

In many applications there can be a priori defined arbitrary order of priority among the receivers. One such application is in wireless sensor data acquisition. If we assume that a sensor data is encoded as the message vector, we may have highest priority for the most significant bit (MSB), as an error in this bit is more critical than that in the least significant bit (LSB). In this

[1] A part of the content of this chapter appears in

- Divya U. Sudhakaran and B. Sundar Rajan, "Maximum likelihood decoder for index coded PSK modulation for priority ordered receivers," accepted for 2017 IEEE 86th Vehicular Technology Conference: VTC2017-Fall 24-27 September 2017, Toronto, Canada.
- Divya U. Sudhakaran and B. Sundar Rajan, "Maximum likelihood decoder for index coded PSK modulation for priority ordered receivers," arXiv: 1703.03222v1 [cs.IT] 9 Mar 2017.

case, it is desirable to improve the performance of the highest priority receiver even if it results in a degradation in performance of a low priority receiver. By prioritized receivers, we mean that,

- There is a priori defined arbitrary order of priority among the receivers.
- The PSK-ICG of the highest priority receiver must be maximised even if it reduces the PSK-ICG of lower priority receivers.
- Assume that the priority order is $(R_1, R_2, ..., R_m)$. Given that R_1 gets it best possible PSK-ICG, the PSK-ICG that can be achieved by R_2 must be maximised. Further, given that R_1 and R_2 achieves their best possible performance in this way, PSK-ICG that can be achieved by R_3 must be maximised and so on in the order of priority.

3.2 Mapping based on Inter-Set Distances

In the ML decision rule as given in (2.9), the decoder makes an error at high SNR, if the broadcasted signal point is in 0-effective broadcast signal set but r is closest to a signal point in 1-effective broadcast signal set or vice versa. The probability of this event is more when the minimum inter-set distance is less. At high SNR, this error is dominant and so an optimal mapping for the best message error performance must maximize the minimum inter-set distance. Among the mappings which has the same minimum inter-set distance, the one which has less multiplicity (of the pairs which result in the minimum inter-set distance) will perform better and among the mappings which has the same minimum inter-set distance and multiplicity, the one which has more second minimum inter-set distance will perform better and so on.

Eventhough, at high SNR, the minimum Euclidean distance decoder can be used instead of ML decoder, the labeling needs to be done in such a way that the minimum inter-set distance is maximized and not the minimum Euclidean distance of the effective broadcast signal sets.

The simulation results in Section 3.4.1 illustrates this important point which is a significant contribution of this work. For example, consider the mappings given in Figure 2.1(a) and Figure 2.1(d). With the mapping shown in Figure 2.1(a), the minimum inter-set distance for R_1 is more but the minimum Euclidean distance of its effective broadcast signal sets is less, compared to that with the mapping shown in Figure 2.1(d). The simulation results (discussed in Section 3.4.1) show that R_1 performs better with the mapping in Figure 2.1(a) than with the mapping in Figure 2.1(d).

For the given ICP and 2^N-PSK constellation, when the receivers are arranged in the decreasing order of priority, we propose an algorithm which maximizes the minimum inter-set distance, to find (index code, mapping) pairs, each of which gives the optimal message error performance for the receivers, for the given order of priority. Assume that the decreasing order of priority for the receivers is $(R_1, R_2, ..., R_m)$. Here optimality is based on minimum inter-set distance and is in the following sense:

- No other mapping of 2^N-PSK constellation for any index code, can give PSK-ICG $> g_1$ for R_1.
- Any mapping for any index code which gives the PSK-ICG g_i for receiver R_i, $\forall i \in \{1, 2, ..., j-1\}$ cannot give a PSK-ICG $> g_j$ for R_j, $j \leq m$.

It may so turn out that maximizing the gain of a receiver R_i, minimizes the gain that can be achieved by a lower priority receiver R_j. With this algorithm it is not necessary that a higher priority receiver will get higher PSK-ICG compared to that of the lower priority receivers. The PSK-ICG achieved by a receiver R_j depends on its priority, $\mathcal{W}_j$, $\mathcal{K}_j$, $\mathcal{W}_i$ and $\mathcal{K}_i$ $\forall i$ such that R_i is a higher priority receiver than R_j. This is further explained with an example and simulation results in Section 3.4.1.

In the following subsections, we explain the mapping algorithm and then illustrate it with examples.

3.2.1 Mapping Algorithm

Without loss of generality, assume that the decreasing order of priority among the receivers is $(R_1, R_2, ..., R_m)$. For a given index code based on encoding matrix L, an optimal mapping for a receiver R_i is obtained as follows:

1. Find all effective broadcast vector sets for $\mathbf{a}_i \in \mathbb{F}_2^{|\mathcal{K}_i|}$. These sets partition $\mathbb{F}_2^N$.
2. Consider an effective broadcast vector set, $\mathcal{C}_L(\mathbf{a}_i)$.
3. Partition the effective broadcast vector set into 0-effective broadcast vector set ($\mathcal{C}_{L0}(\mathbf{a}_i)$) and 1-effective broadcast vector set ($\mathcal{C}_{L1}(\mathbf{a}_i)$).
4. All the broadcast vectors in $\mathcal{C}_{L0}(\mathbf{a}_i)$ must be mapped to adjacent signal points. Let the set of signal points corresponding to $\mathcal{C}_{L0}(\mathbf{a}_i)$ be $\mathcal{S}_{L0}(\mathbf{a}_i)$.
5. All the broadcast vectors in $\mathcal{C}_{L1}(\mathbf{a}_i)$ must be mapped to signal points diametrically opposite to signal points in $\mathcal{S}_{L0}(\mathbf{a}_i)$. This will result in a mapping with broadcast vectors in $\mathcal{C}_{L1}(\mathbf{a}_i)$ mapped to adjacent signal points.
6. Repeat steps 3 to 5 by considering the remaining effective broadcast vector sets one by one.

In the case of Example 2.1, for the chosen index code, there are two effective broadcast vector sets which partitions $\mathbb{F}_2^3$ for R_1. Consider any one of the effective broadcast vector sets say $\mathcal{C}_L(00)$ and its partitions $\mathcal{C}_{L0}(00)$ and $\mathcal{C}_{L1}(00)$. To obtain an optimal mapping for the receiver R_1, map all the broadcast vectors in $\mathcal{C}_{L0}(00)$ to adjacent signal points. Then map all the broadcast vectors in $\mathcal{C}_{L1}(00)$ to signal points diametrically opposite to signal points in $\mathcal{S}_{L0}(00)$. One such mapping is shown in Figure 2.1(b). Then consider the next effective broadcast vector set, $\mathcal{C}_L(01)$ and its partitions. As in the case of $\mathcal{C}_L(00)$, map all the broadcast vectors in $\mathcal{C}_{L0}(01)$ and $\mathcal{C}_{L1}(01)$. One possible way of mapping is given in Figure 2.1(c). Thus an optimal mapping for R_1 is obtained as shown in Figure 2.1(a).

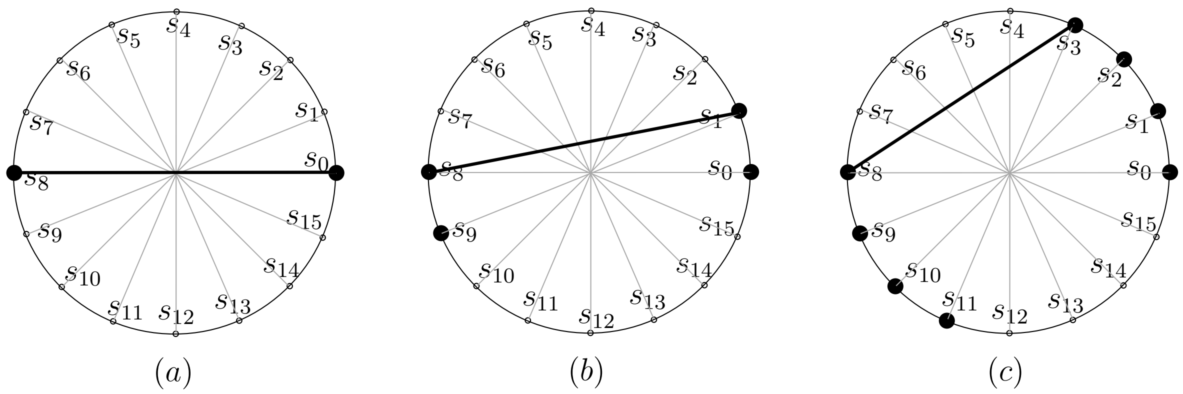

Figure 3.1: Variation of the minimum inter-set distance with $|\mathcal{C}_L(\mathbf{a}_i)|$.

For a receiver R_i, when we compare the optimal mappings for two different index codes, the code which has less $|\mathcal{C}_L(\mathbf{a}_i)|$ will perform better as the minimum inter-set distance will be more (note that for both the index codes we do optimal mapping). For example consider an ICP and three possible index codes based on encoding matrices L_1, L_2 and L_3 with $N = 4$. Assume that the corresponding effective broadcast vector sets have cardinality as follows: $|\mathcal{C}_{L_1}(\mathbf{a}_1)| = 2$, $|\mathcal{C}_{L_2}(\mathbf{a}_1)| = 4$ and $|\mathcal{C}_{L_3}(\mathbf{a}_1)| = 8$. The minimum inter-set distance seen by R_1 for index codes based on L_1, L_2 and L_3 is shown in Figure 3.1((a), (b) and (c) respectively). It is clear from the figure that the code which has lesser $|\mathcal{C}_L(\mathbf{a}_i)|$ will result in better (more) minimum inter-set distance with the optimal mapping as explained above.

The mapping algorithm is explained below. Index codes are identified using the corresponding encoding matrices.

1. The algorithm starts by considering $\mathbb{L}_N$, the set of all index codes of length N, for the given ICP. For R_i define,

$$\eta_i \triangleq \min_{L \in \mathbb{L}_N} |\mathcal{C}_L(\mathbf{a}_i)|$$

2. Find η_1. If $\eta_1 < 2^N$ proceed to step 4 with $i = 1$.

3. If $\eta_1 = 2^N$, R_1 sees the full 2^N-PSK constellation as the effective broadcast signal set. In such a case all mappings for all the index codes will give same PSK-ICG for R_1 with

$d_{IS,min}^{(1)}$ same as the minimum Euclidean distance between any two points of 2^N-PSK constellation. In such a case, any mapping for any index code is optimum for R_1. Then consider the next highest priority receiver, R_2 and continue until a receiver R_i for which $\eta_i < 2^N$ is found. If $\eta_i = 2^N$ for all receivers, do an arbitrary mapping and exit. Now consider the case where a receiver R_i for which $\eta_i < 2^N$ is found.

4. Let $\{L : |\mathcal{C}_L(\mathbf{a}_i)| = \eta_i\}$ be $\{L_1, L_2, ..., L_{n_{L,i}}\}$. For each $L_j, j \in \{1, 2, ..., n_{L,i}\}$, find optimal mappings for R_i. Let there be $n_{\mathcal{M},i}$ optimal mappings for each index code and denote the mappings corresponding to index code L_j as $\mathcal{M}_{j1}, \mathcal{M}_{j2}, ..., \mathcal{M}_{jn_{\mathcal{M},i}}$. Define $\mathcal{O}$, the set of ordered pairs as,

$$\mathcal{O} \triangleq \{(L_1, \mathcal{M}_{11}), (L_1, \mathcal{M}_{12}), ..., (L_1, \mathcal{M}_{1n_{\mathcal{M},i}}), (L_2, \mathcal{M}_{21}), (L_2, \mathcal{M}_{22}), ..., (L_2, \mathcal{M}_{2n_{\mathcal{M},i}}), ...,$$
$$(L_{n_{L,i}}, \mathcal{M}_{n_{L,i}1}), (L_{n_{L,i}}, \mathcal{M}_{n_{L,i}2}), ..., (L_{n_{L,i}}, \mathcal{M}_{n_{L,i}n_{\mathcal{M},i}})\}.$$

The set $\mathcal{O}$ contains all the (index code, mapping) pairs which give the maximum gain possible for R_i. Now from this set, identify the pairs which give maximum gain for R_{i+1}. For this, choose the pairs which have maximum $d_{IS,min}^{(i+1)}$. Now consider these pairs as set $\mathcal{O}$ and continue until the last receiver R_m is considered and the pairs which have maximum $d_{IS,min}^{(m)}$ is obtained. These are the (index code, mapping) pairs which are optimal.

3.2.2 Illustration of Mapping Algorithm

We illustrate the mapping algorithm given as Algorithm 1 with an example. Consider the ICP given in Example 2.1. Assume that the decreasing order of priority is $(R_1, R_2, R_3, R_4, R_5)$. Let $N = 3$ which is also the length of the optimal index code in this case. Since this is a single unicast ICP, we can find all index codes by considering fitting matrices [7] of rank 3. There are a total of 32 such matrices. For each of these 32 matrices, choose any 3 independent rows as a basis for the row space. So we obtain 32 row spaces (which represents 32 index codes for the given ICP). Of these, only six row spaces are distinct. From Corollary 1 in [12], the number of index codes possible with the optimal length c for a single-unicast IC problem is given by

$\frac{\mu}{c!}\prod_{i=0}^{c-1}(2^c - 2^i)$ where μ is the number of distinct row spaces of c-ranked fitting matrices.

For the example under consideration, there are a total of 168 index codes (28 index codes for each distinct row space). So, $\mathbb{L}_3$ contains 168 index codes. $\eta_1 = \min_{L \in \mathbb{L}_3} |\mathcal{C}_L(\mathbf{a}_1)| = 4$. There are 84 index codes with $\eta_1 = 4$ and 32 optimal mappings for R_1, for each of these index codes. The set $\mathcal{O}$ has $32 * 84 = 2688$ (index code, mapping) pairs which are optimal for R_1. One such $(L, \mathcal{M})$ pair has the index code as given in Example 2.1 and mapping as given in Figure 2.1(a). Consider R_2. After all pairs in $\mathcal{O}$ are considered, the maximum value possible for $d^{(2)}_{IS,min} = 1.414$ and there are 336 pairs which are optimal for R_2. Now consider R_3. All 336 pairs give the same $d^{(3)}_{IS,min} = 1.414$. For R_4 and R_5 all pairs have same minimum inter-set distance and these 336 pairs give the (index code, mapping) pairs which are optimal for the ICP considered. For illustration, four such $(L, \mathcal{M})$ pairs are given below and the complete list is given in Appendix A. Index code based on encoding matrix L is given in the form of (y_1, y_2, y_3). $\mathcal{M}$ is given as an ordered list of eight integers, representing the decimal equivalent of the 3-tuple, which is mapped to $(s_1, s_2, ..., s_8)$ where $(s_1, s_2, ..., s_8)$ are 8-PSK signal points as shown in Figure 2.1.

- $((x_1, x_2 + x_3, x_4 + x_5), (0, 1, 2, 3, 4, 5, 6, 7))$
- $((x_1, x_2 + x_3, x_4 + x_5), (0, 1, 6, 7, 4, 5, 2, 3))$
- $((x_1, x_2 + x_3, x_1 + x_4 + x_5), (0, 1, 2, 3, 5, 4, 7, 6))$
- $((x_1, x_1 + x_2 + x_3, x_4 + x_5), (0, 1, 2, 3, 6, 7, 4, 5))$

Claim 1: Algorithm 1 guarantees that for a given ICP, no other mapping of 2^N-PSK constellation for any index code of length N, can give PSK-ICG $> g_1$ for R_1.

Proof. The coding gain (PSK-ICG) achieved by a receiver is maximized when the minimum inter-set distance is maximum. Consider an index code of length N. Using Algorithm 1, for each of the effective broadcast signal sets of the highest priority receiver, the broadcast vectors

Algorithm 1 Algorithm to find optimal (index code, mapping) pairs for a given ICP.

1: $i \leftarrow 1$
2: Find $\eta_i = \min_{L \in \mathbb{L}_N} |\mathcal{C}_L(\mathbf{a}_i)|$
3: **if** ($\eta_i = 2^N$) **then**
4: $\quad i \leftarrow i + 1$
5: $\quad$ **if** $(i > m)$ **then**
6: $\quad\quad$ Do an arbitrary mapping and Exit.
7: $\quad$ **else**
8: $\quad\quad$ Goto 2
9: **else**

- Consider the set of index codes $\{L_1, L_2, ..., L_{n_{L,i}}\} = \{L : |\mathcal{C}_L(\mathbf{a}_i)| = \eta_i\}$
- Find $\mathcal{O}$, the set of all (index code, optimal mapping) pairs for R_i.

10: $\quad i \leftarrow i + 1$
11: $\quad$ **if** $(i > m)$ **then**
12: $\quad\quad$ Output $\mathcal{O}$ and Exit.
13: $\quad$ **else**
14: $\quad\quad$ Choose any $(L, \mathcal{M}) \in \mathcal{O}$
15: $\quad\quad$ $\mathcal{O}^i \leftarrow \{(L, \mathcal{M})\}$. Find $\delta = d_{IS,min}^{(i)}$.
16: $\quad\quad$ $\mathcal{O} \leftarrow \mathcal{O} \setminus \{(L, \mathcal{M})\}$
17: $\quad\quad$ **if** $(\mathcal{O} = \{\})$ **then**
18: $\quad\quad\quad$ $\mathcal{O} \leftarrow \mathcal{O}^i$
19: $\quad\quad\quad$ Goto 10
20: $\quad\quad$ **else**
21: $\quad\quad\quad$ Consider any $(L, \mathcal{M}) \in \mathcal{O}$. Find $d_{IS,min}^{(i)}$.
22: $\quad\quad\quad$ **if** ($d_{IS,min}^{(i)} > \delta$) **then**
23: $\quad\quad\quad\quad$ $\mathcal{O}^i \leftarrow \{(L, \mathcal{M})\}$, $\delta = d_{IS,min}^{(i)}$. Goto 16.
24: $\quad\quad\quad$ **else**
25: $\quad\quad\quad\quad$ **if** ($d_{IS,min}^{(i)} = \delta$) **then**
26: $\quad\quad\quad\quad\quad$ $\mathcal{O}^i \leftarrow \mathcal{O}^i \cup \{(L, \mathcal{M})\}$. Goto 16.
27: $\quad\quad\quad\quad$ **else**
28: $\quad\quad\quad\quad\quad$ Goto 16.

in 0-effective broadcast vector set are always mapped to adjacent points. Similarly, the broadcast vectors in 1-effective broadcast vector set are always mapped to adjacent points. These sets of points are placed diametrically opposite to each other. Thus, the minimum inter-set distance is maximized for the chosen index code and the mapping is optimal.

When we compare the message error performance of R_1 with respect to different possible index codes, the code which has less $|\mathcal{C}_L(\mathbf{a}_i)|$ performs better. Index codes with minimum $|\mathcal{C}_L(\mathbf{a}_i)|$ are only considered for mapping in Algorithm 1. So, the pairs considered by Algorithm 1 has index codes with minimum $|\mathcal{C}_L(\mathbf{a}_i)|$ and mappings which are optimal. No other mapping of 2^N-PSK constellation for any index code of length N, can give PSK-ICG $> g_1$ for R_1. □

Claim 2: Algorithm 1 guarantees that, any mapping for any index code which gives the PSK-ICG g_i for receiver R_i, $\forall i \in \{1, 2, ..., j-1\}$ cannot give a PSK-ICG $> g_j$ for R_j, $j \leq m$.

Proof. Algorithm 1 finds all (index code, mapping) pairs which are optimal for R_1. In the next step, among these pairs, which ever gives the maximum gain for R_2 are chosen. So, given that R_1 has the same PSK-ICG, it is not possible to find another pair for which R_2 performs better. Same argument extends to other receivers as well. □

Algorithm 1 can also be used to obtain optimal (index code, mapping) pairs for a given set of index codes of length N. In this case the algorithm must be run by considering the given set of index codes instead of all possible index codes of length N. This can be illustrated using the ICP given in Example 2.2. Assume that the decreasing order of priority is $(R_1, R_2, R_3, R_4, R_5, R_6)$. Let $N = 4$ and assume that only one index code as given in Example 2.2 need to be considered (the given set of index codes is a singleton set). Consider the highest priority receiver R_1. Obtain the effective broadcast vector sets seen by R_1 for $\mathbf{a}_1 \in \mathbb{F}_2^5$ and partition these sets. The effective broadcast vector sets and its partitions for R_1 are given in Table 3.1. For any realization of $\mathbf{x}_{\mathcal{I}_1} = \mathbf{a}_1$ which is not listed in Table 3.1, the effective broadcast vector set is same as one of the effective broadcast vector sets given in the table.

Table 3.1: Effective broadcast vector sets and its partitions (seen by R_1) for the IC in Example 2.2.

$\mathbf{a}_1$	$\mathcal{C}_L(\mathbf{a}_1)$	$\mathcal{C}_{L0}(\mathbf{a}_1)$	$\mathcal{C}_{L1}(\mathbf{a}_1)$
(00000)	$\{(0000),(1000)\}$	$\{(0000)\}$	$\{(1000)\}$
(00001)	$\{(0001),(1001)\}$	$\{(0001)\}$	$\{(1001)\}$
(00010)	$\{(0010),(1010)\}$	$\{(0010)\}$	$\{(1010)\}$
(00011)	$\{(0011),(1011)\}$	$\{(0011)\}$	$\{(1011)\}$
(01000)	$\{(0100),(1100)\}$	$\{(0100)\}$	$\{(1100)\}$
(01001)	$\{(0101),(1101)\}$	$\{(0101)\}$	$\{(1101)\}$
(01010)	$\{(0110),(1110)\}$	$\{(0110)\}$	$\{(1110)\}$
(01011)	$\{(0111),(1111)\}$	$\{(0111)\}$	$\{(1111)\}$

There are 645120 optimal mappings for R_1. The set $\mathcal{O}$ has 645120 (index code, mapping) pairs which are optimal for R_1, with the index code being the same for all the pairs. Consider R_2. After all pairs in $\mathcal{O}$ are considered, the maximum value possible for $d^{(2)}_{IS,min} = 1.847$ and there are 128 pairs which are optimal for R_2. Now consider R_3. There are 24 pairs which are optimal with $d^{(3)}_{IS,min} = 0.765$. For R_4 there are 16 optimal pairs with minimum inter-set distance $d^{(4)}_{IS,min} = 0.765$. For R_5 and R_6 all these pairs give the same minimum inter-set distance. These 16 pairs are the optimal mappings for the IC considered and is given below as an ordered list of sixteen integers, representing the decimal equivalent of the 4-tuple, which is mapped to $(s_1, s_2, ..., s_{16})$ where $(s_1, s_2, ..., s_{16})$ are 16-PSK signal points as shown in Figure 3.2.

1. $(0, 1, 12, 15, 2, 3, 14, 5, 8, 9, 4, 7, 10, 11, 6, 13)$

2. $(0, 1, 12, 7, 10, 11, 6, 5, 8, 9, 4, 15, 2, 3, 14, 13)$

3. $(0, 1, 12, 3, 14, 15, 2, 5, 8, 9, 4, 11, 6, 7, 10, 13)$

4. $(0, 1, 12, 11, 6, 7, 10, 5, 8, 9, 4, 3, 14, 15, 2, 13)$

5. $(0, 13, 12, 1, 10, 7, 6, 11, 8, 5, 4, 9, 2, 15, 14, 3)$

6. $(0, 13, 12, 1, 6, 11, 10, 7, 8, 5, 4, 9, 14, 3, 2, 15)$

7. $(0, 13, 12, 1, 14, 3, 2, 15, 8, 5, 4, 9, 6, 11, 10, 7)$

8. $(0, 13, 12, 1, 2, 15, 14, 3, 8, 5, 4, 9, 10, 7, 6, 11)$

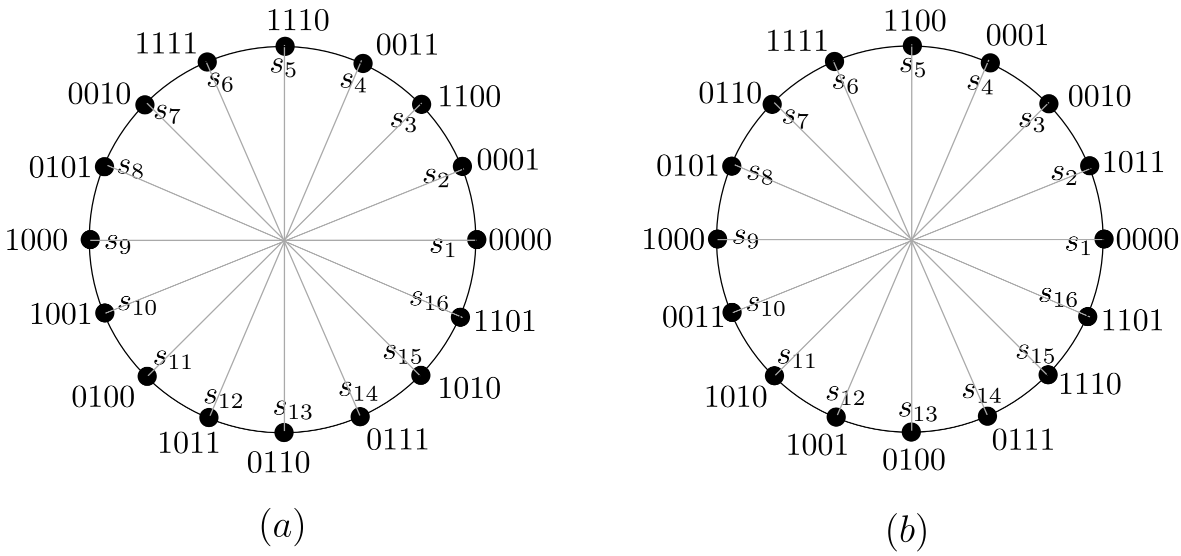

Figure 3.2: Two 16-PSK mappings for the IC in Example 2.2.

9. $(0, 3, 15, 14, 2, 9, 4, 5, 8, 11, 6, 7, 10, 1, 12, 13)$

10. $(0, 11, 6, 7, 10, 9, 4, 5, 8, 3, 14, 15, 2, 1, 12, 13)$

11. $(0, 15, 2, 3, 14, 9, 4, 5, 8, 7, 10, 11, 6, 1, 12, 13)$

12. $(0, 7, 10, 11, 6, 9, 4, 5, 8, 15, 2, 3, 14, 1, 12, 13)$

13. $(0, 13, 14, 3, 2, 15, 4, 9, 8, 5, 6, 11, 10, 7, 12, 1)$

14. $(0, 13, 6, 11, 10, 7, 4, 9, 8, 5, 14, 3, 2, 15, 12, 1)$

15. $(0, 13, 2, 15, 14, 3, 4, 9, 8, 5, 10, 7, 6, 11, 12, 1)$

16. $(0, 13, 10, 7, 6, 11, 4, 9, 8, 5, 2, 15, 14, 3, 12, 1)$

As an example, the third mapping is shown in Figure 3.2(a).

3.3 An Upper Bound on the Coding Gain

Using Algorithm 1, the highest priority receiver always gets its maximum possible coding gain (PSK-ICG). Thus the coding gain achieved by it is an upper bound on the coding gain that it can get using any other mapping or index code, for any order of priority. Now, if we consider another receiver as the highest priority receiver, we can use the algorithm to find the upper bound on the coding gain that it can achieve. Proceeding in a similar way, we can find the upper bound on the coding gain for each of the receivers, for the given ICP and a chosen 2^N PSK modulation.

Consider the ICP given in Example 2.2 and assume that the index code is also specified as given in the example.

Table 3.2: Upper bound on the PSK-ICG that can be achieved by the receivers in Example 2.2.

Parameter	R_1	R_2	R_3	R_4	R_5	R_6
$d_{IS,min}^{(i)}$	2	1.9615	1.6629	1.6629	0.3901	0.3901
PSK-ICG (in dB)	26.18	26.01	24.58	24.58	11.98	11.98

By considering each one of the receivers as the highest priority receiver, mapping is done as per the proposed algorithm and the upper bound on the PSK-ICG that can be achieved by each one of these receivers for the chosen index code is given in Table 3.2.

3.4 Simulation Results

3.4.1 Comparison of Mapping Algorithms

We have considered the ICP given in Example 2.1 over a SISO AWGN channel and used Algorithm 1 to obtain all optimal (index code, mapping) pairs. One such pair, $(L_1, \mathcal{M}_1)$ has the index code as given in Example 2.1 and mapping as given in Figure 2.1(a). We compared this optimal mapping with another mapping $\mathcal{M}_2$ (shown in Figure 2.1(d)) which is not optimal for the same index code, L_1. The pair $(L_1, \mathcal{M}_2) \notin \mathcal{O}$, the output set obtained from the execution of the algorithm. The second mapping ($\mathcal{M}_2$) used an algorithm based on maximizing the minimum Euclidean distances [14]. By simulation, we have obtained the message error probability of the receivers for the two different mappings. Simulation results are given in Figure 3.3.

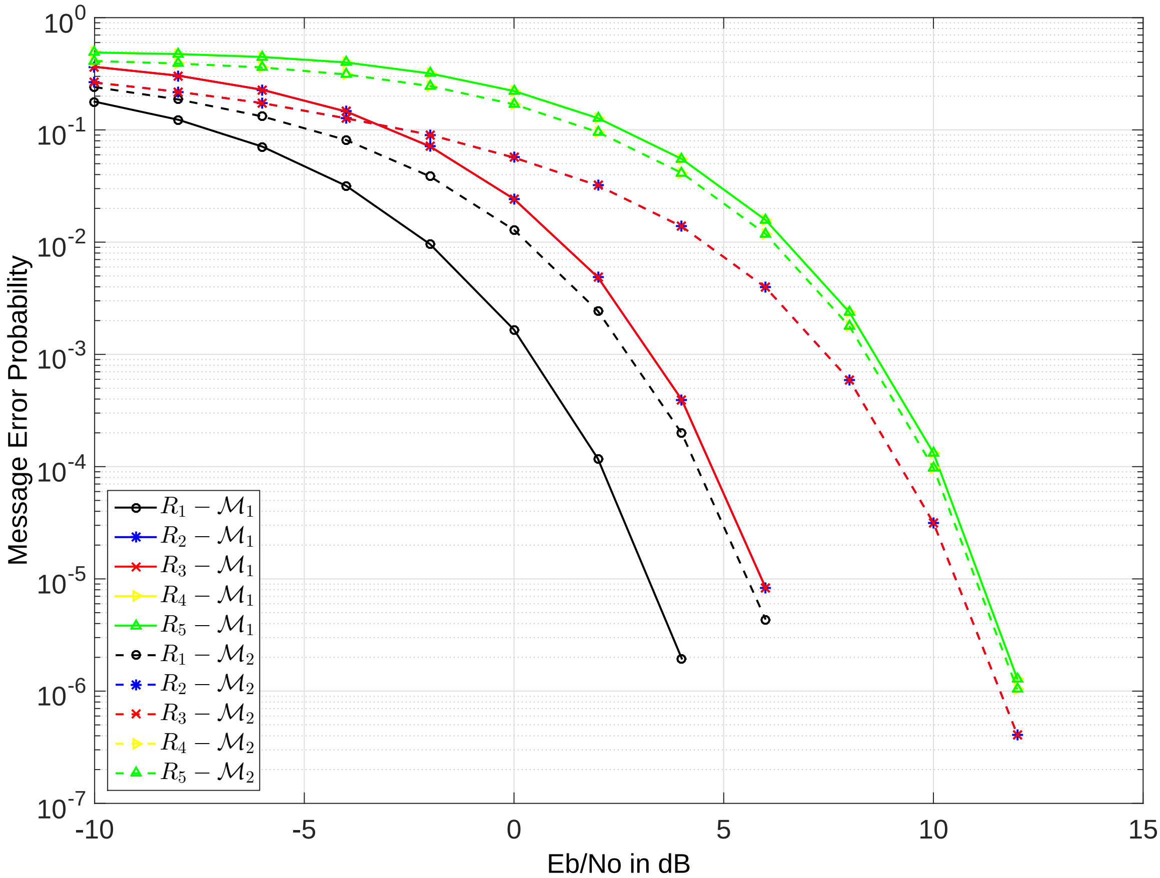

Figure 3.3: Simulation results comparing the performance of receivers for two different mappings for the ICP in Example 2.1

The performance of receivers R_1, R_2 and R_3 is significantly better with $\mathcal{M}_1$ than with $\mathcal{M}_2$ at high SNR. The minimum inter-set distances are more for $\mathcal{M}_1$ (Figure2.1(a)) than for $\mathcal{M}_2$ (Figure2.1(d)). For receivers R_4 and R_5, the minimum inter-set distances are same for both the mappings.

We have carried out simulation based studies to compare the performance of the receivers for the ICP and the IC given in Example 2.2 for two different mappings as given in Figure 3.2. The mapping ($\mathcal{M}_3$) given in Figure 3.2(a) used Algorithm 1 and the mapping ($\mathcal{M}_4$) given in

Figure 3.2(b) used the algorithm based on maximizing the minimum Euclidean distances [14]. Simulation results are given in Figure 3.4.

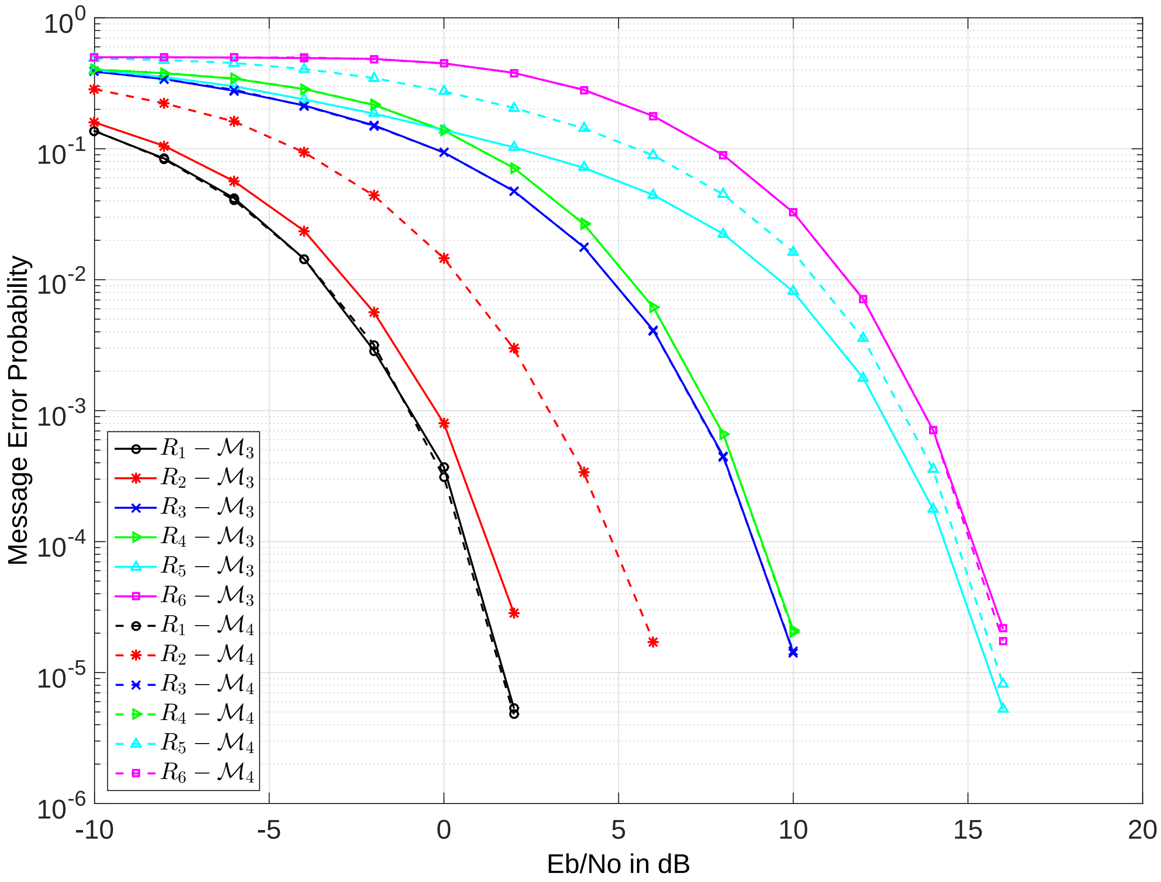

Figure 3.4: Simulation results comparing the performance of receivers for two different mappings for the ICP in Example 2.2

For R_1, R_3, R_4, R_5 and R_6, the minimum inter-set distances and hence the performances are the same for both the mappings. But the performance of receiver R_2 is significantly better with $\mathcal{M}_3$ than with $\mathcal{M}_4$ at high SNR.

The simulation results indicate the effectiveness of the algorithm based on minimum inter-set distances (Algorithm 1) for mapping the broadcast vectors to PSK signal points.

It should be noted that Algorithm 1 does not guarantee that all the receivers will perform better or as good as that with any other algorithm. It is possible that, a mapping based on some algorithm (say, Algorithm 2) gives a better performance to a receiver R_j than that with

Algorithm 1. But then there will be a receiver R_i which performs better with Algorithm 1 than with Algorithm 2, where R_i is a higher priority receiver than R_j. In other words, Algorithm 1 attempts to maximize the gain achieved by the receivers by considering the receivers in the given order of priority. This is illustrated in Example 3.1.

Example 3.1. Consider the following ICP with $n = m = 5$ and $\mathcal{W}_i = x_i, \forall i \in \{1, 2, ..., 5\}$. The side information available with the receivers is as follows: $\mathcal{K}_1 = \{x_2, x_3, x_4, x_5\}$, $\mathcal{K}_2 = \{x_1, x_4, x_5\}$, $\mathcal{K}_3 = \{x_1, x_4\}$, $\mathcal{K}_4 = \{x_2\}$, $\mathcal{K}_5 = \{\}$.

For this ICP a scalar linear index code of length $N = 4$ (not optimal), is specified as $y_1 = x_1 + x_2$, $y_2 = x_3$, $y_3 = x_4$, $y_4 = x_5$. Assume that the decreasing order of priority is given as $(R_1, R_2, R_3, R_4, R_5)$.

Using Algorithm 1, optimal mappings for the specified IC is obtained, of which one mapping ($\mathcal{M}_5$) is given in Figure 3.5(a). Another mapping $\mathcal{M}_6$, is found by using the algorithm based on maximizing the minimum Euclidean distances [14] and is given in Figure 3.5(b). Simulation results comparing the performance of the receivers for these two mappings are given in Figure 3.6.

It is clear from Figure 3.6 that, R_2 performs better with $\mathcal{M}_5$ than with $\mathcal{M}_6$. But R_3, which is of lower priority than R_2, has better performance with $\mathcal{M}_6$.

3.4.2 High SNR Approximation of ML Decoder

Simulations were carried out to verify the high SNR approximation of ML decoder. For the ICP and the index code given in Example 2.1, we first considered the mapping $\mathcal{M}_1$ (Figure 2.1(a)), which maximizes the minimum inter-set distance of the receivers. We carried out ML decoding and minimum Euclidean distance decoding for the SISO-AWGN-ICPM. The simulation results are given in Figure 3.7.

The simulation results given in Figure 3.8 compares the performance of ML decoding with minimum Euclidean distance decoding for mapping $\mathcal{M}_2$ (Figure 2.1(d)).

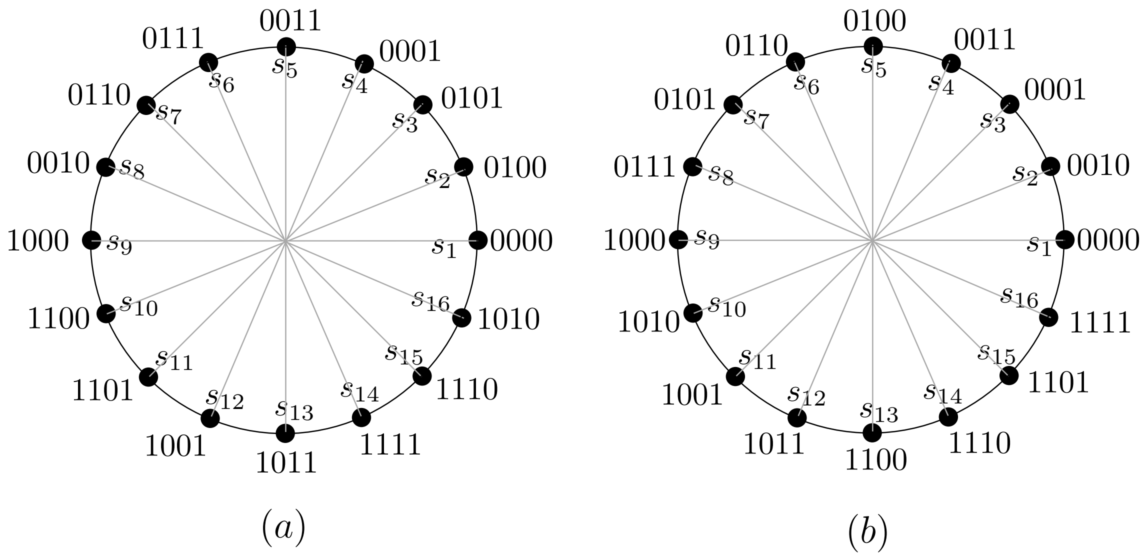

Figure 3.5: Two 16-PSK mappings for the IC in Example 3.1.

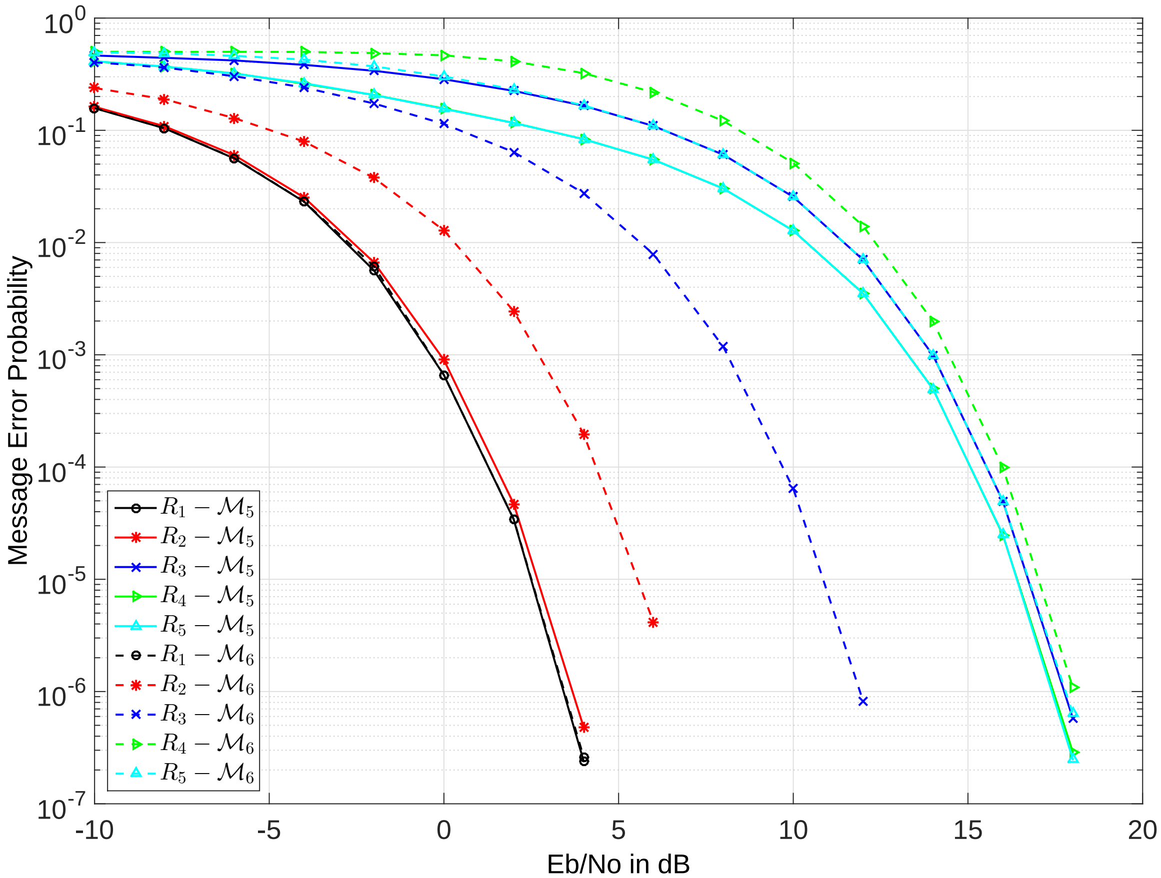

Figure 3.6: Simulation results comparing the performance of receivers for two different mappings (Example 3.1).

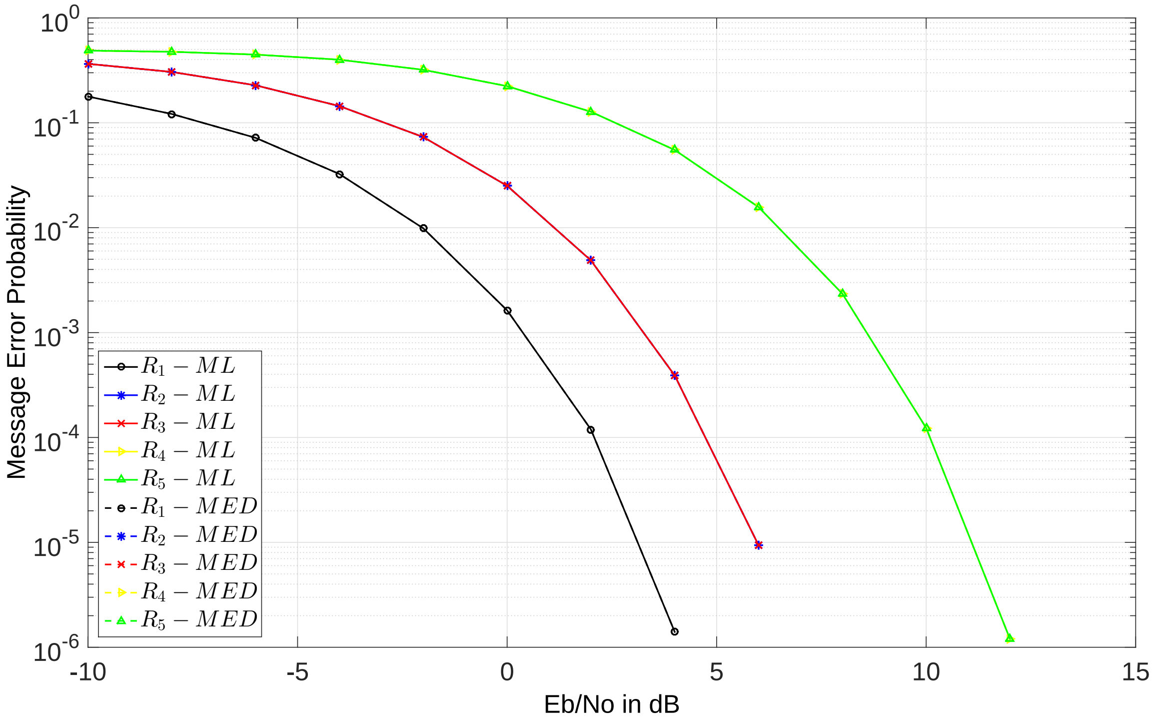

Figure 3.7: Simulation results comparing the performance of ML decoder (ML) and minimum Euclidean distance decoder (MED) with mapping $\mathcal{M}_1$ for SISO-AWGN-ICPM.

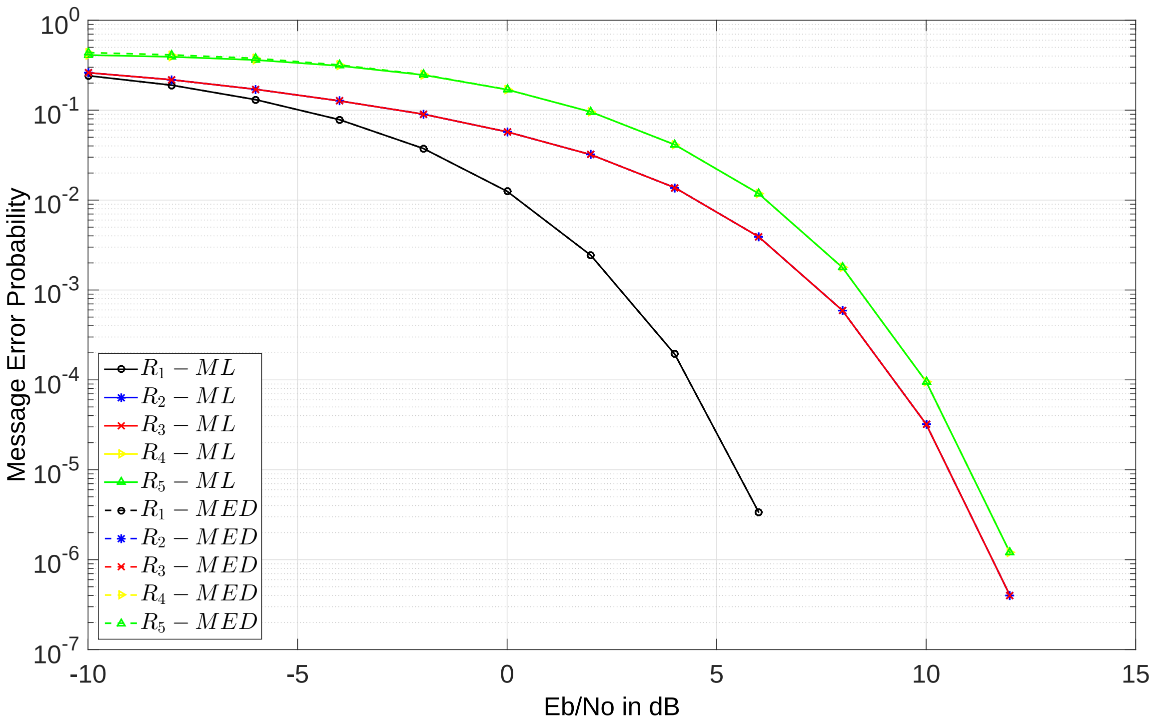

Figure 3.8: Simulation results comparing the performance of ML decoder (ML) and minimum Euclidean distance decoder (MED) with mapping $\mathcal{M}_2$ for SISO-AWGN-ICPM.

With mapping $\mathcal{M}_1$, the performance of ML decoder and minimum Euclidean distance decoder is very similar for any receiver R_i. In the case of mapping $\mathcal{M}_2$, ML decoder performs slightly better than minimum Euclidean distance decoder for receivers R_4 and R_5 at low SNR. But at high SNR, the performance of both the decoders is the same for any receiver.

3.4.3 Upper Bound on the Coding Gain

For the ICP and the IC in Example 2.2, we have found the upper bound on the PSK-ICG that can be achieved by the receivers as given in Table 3.2.

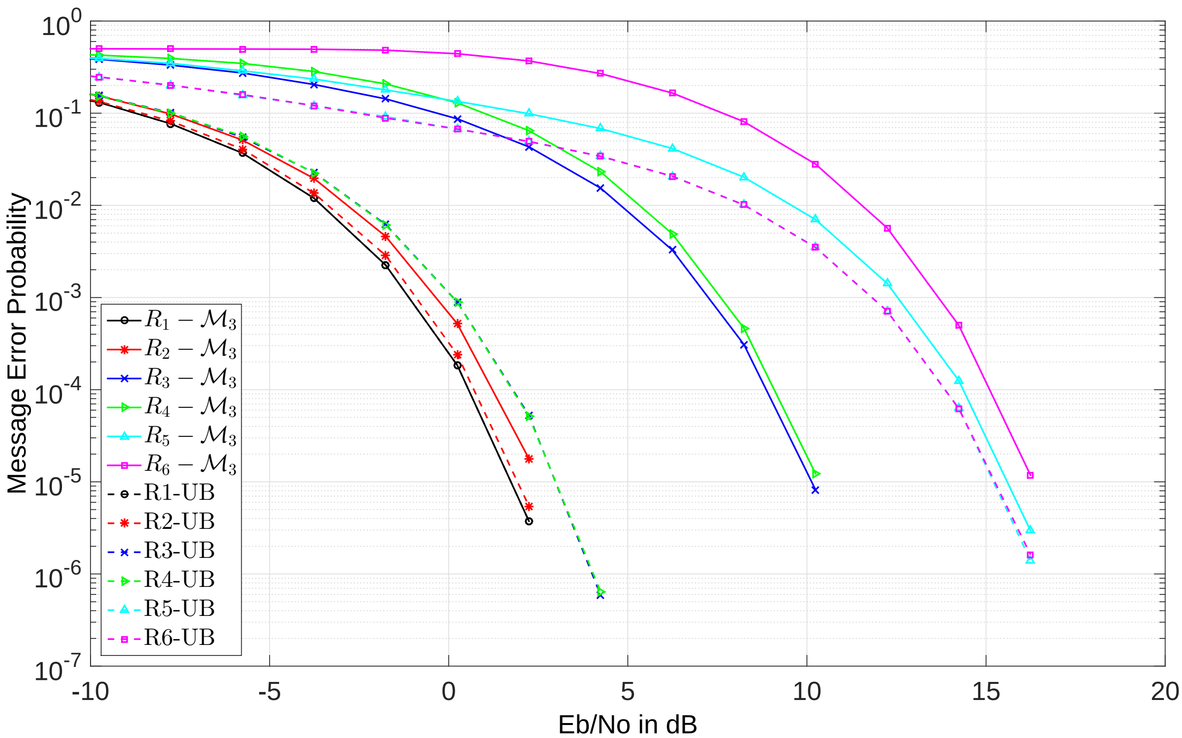

Figure 3.9: Message error performance with mappings which achieve the upper bound on the PSK-ICG for each one of the receivers, and the mapping given in Figure 3.2(a).

With the mappings which achieve the upper bound on PSK-ICG for each of the receivers, the message error performance is studied using simulations (with MED) and is given in Figure 3.9. The message error performance using the mapping given in Figure 3.2(a) is also shown.

Chapter 4

MIMO Index Coded Modulation

[1] In this chapter we consider noisy index coding problem over a Rayleigh fading channel. We propose a MIMO scheme which provides diversity gain and coding gain to the receivers. We derive a decision rule for the maximum likelihood decoder for the receivers when Alamouti code is employed but the derivation is extendable for any space-time code obtained from orthogonal designs [18]. The diversity gain and the coding gain of the proposed scheme are also discussed.

4.1 Proposed MIMO Scheme

We explain MIMO scheme using 2×1 MIMO Rayleigh fading channel and the Alamouti code. The sender has two transmit antennas and each of the receivers has one receive antenna. In each symbol time, the sender has n messages ($\mathbf{x} = (x_1 x_2 ... x_n)$) to be transmitted to the receivers. Each of the receivers wants one message and knows some messages. Let $\mathbf{x}[1]$ and $\mathbf{x}[2]$ be the message vectors corresponding to symbol time $t = 1$ and $t = 2$. Let $\mathbf{y}[1]$ and $\mathbf{y}[2]$ denote the corresponding broadcast vectors, and, $s[1] = \mathcal{M}(\mathbf{x}[1]L)$ and $s[2] = \mathcal{M}(\mathbf{x}[2]L)$ be the corresponding 2^N-PSK signal points (where $\mathcal{M}$ is the map from $\mathbb{F}_2^N$ to the signal set $\mathcal{S}$). After

[1] A part of the content of this chapter appears in:
Divya U. Sudhakaran and B. Sundar Rajan, "Alamouti-index-coded PSK modulation for priority ordered receivers," Communicated to IEEE GLOBECOM 2017, 04-08 December 2017, Singapore.

mapping the broadcast vectors to 2^N-PSK signal points the sender employs 2×2 Alamouti code as follows to obtain transmit diversity.

$$\begin{bmatrix} s[1] & -s^*[2] \\ s[2] & s^*[1] \end{bmatrix}.$$

Consider a receiver, R_i. The channel is assumed to be quasi static and the channel coefficients h_1 (between the first transmit antenna and the receive antenna) and h_2 (between the second transmit antenna and the receive antenna) are independent and identically distributed (i.i.d) as $\mathcal{CN}(0,1)$. Assume that the perfect channel state information is available at the receiver. The signal received at $t = 1$ and $t = 2$ can be written as

$$\begin{aligned} r[1] =& h_1 s[1] + h_2 s[2] + w[1] \\ r[2] =& -h_1 s^*[2] + h_2 s^*[1] + w[2] \end{aligned} \tag{4.1}$$

respectively, where the additive noises, $w[1]$ and $w[1]$, are i.i.d $\mathcal{CN}(0, N_0)$ random variables. The average energy spent by the sender in each symbol time is unity and the total energy spent for each message vector (across two symbol times) is also unity.

4.2 Maximum Likelihood Decoder

In this section we derive a decision rule for the maximum likelihood decoder, for the receiver R_i. From (4.1), we obtain,

$$h_1^* r[1] + h_2 r^*[2] = (|h_1|^2 + |h_2|^2) s[1] + h_1^* w[1] + h_2 w^*[2] \tag{4.2}$$

$$h_2^* r[1] - h_1 r^*[2] = (|h_1|^2 + |h_2|^2) s[2] + h_2^* w[1] - h_1 w^*[2] \tag{4.3}$$

Equations (4.2) and (4.3) are of the form

$$r = (|h_1|^2 + |h_2|^2)\mathcal{M}(\mathbf{x}L) + w$$

where w is distributed as $\mathcal{CN}(0, N_0(|h_1|^2 + |h_2|^2))$. Consider the decoding of messages at $t = 1$ (corresponding to message vector $\mathbf{x}[1]$). Then, $r = h_1^* r[1] + h_2 r^*[2]$ and $\mathcal{M}(\mathbf{x}L) = s[1]$. The receiver R_i wants $x_{f(i)}$. The conditional probability density of r given that $\mathcal{M}(\mathbf{x}L)$ is transmitted and the channel coefficients are known to the receiver, is

$$p(r|\mathcal{M}(\mathbf{x}L), h_1, h_2) = \frac{\exp\left(-\frac{|r-(|h_1|^2+|h_2|^2)\mathcal{M}(\mathbf{x}L)|^2}{N_0(|h_1|^2+|h_2|^2)}\right)}{(\pi N_0(|h_1|^2 + |h_2|^2))}. \tag{4.4}$$

Following an approach similar to the one used in the case of SISO-AWGN-ICPM (Section 2.2), we obtain the ML decision rule as,

$$\sum_{k:s_k \in \mathcal{S}_{L0}(\mathbf{a}_i)} \left(\exp\left(-\frac{|r - (|h_1|^2 + |h_2|^2)s_k|^2}{N_0(|h_1|^2 + |h_2|^2)}\right)\right) \underset{0}{\overset{1}{\lessgtr}} \sum_{k:s_k \in \mathcal{S}_{L1}(\mathbf{a}_i)} \left(\exp\left(-\frac{|r - (|h_1|^2 + |h_2|^2)s_k|^2}{N_0(|h_1|^2 + |h_2|^2)}\right)\right). \tag{4.5}$$

The ML decoder decision is based on the Euclidean distance of all signal points scaled by $(|h_1|^2 + |h_2|^2)$, in 0-effective broadcast signal set to the received vector r relative to that of the scaled signal points in 1-effective broadcast signal set. To reduce the message error probability, the minimum inter-set distance must be maximised. In the MIMO scheme also, the ML decoding can be approximated by minimum Euclidean distance decoding at high SNR (simulation results are provided in Section 4.4.2, Figure 4.3 and Figure 4.4).

4.3 Diversity Gain and Coding Gain

In this section, we derive an upper bound on the pairwise error probability for the proposed MIMO scheme. We make use of the approach in [19] for this derivation.

To find the pairwise error probability we consider the minimum Euclidean distance decoding which is a high SNR approximation of the ML decoder. Consider a receiver R_i and assume that its required message, $x_{f(i)} = 0$. The decoder will find the signal point in $\mathcal{S}_L(\mathbf{a}_i)$ which is closest to r (after scaling). If this signal point belongs to $\mathcal{S}_{L0}(\mathbf{a}_i)$ the decoding is error free (as all signal points in $\mathcal{S}_{L0}(\mathbf{a}_i)$ will give $x_{f(i)} = 0$). So when a signal point $s_k \in \mathcal{S}_{L0}(\mathbf{a}_i)$ is transmitted (that means $x_{f(i)} = 0$), the decoder makes an error if and only if it decodes to a signal point $s_{k'} \in \mathcal{S}_{L1}(\mathbf{a}_i)$. In such a case,

$$|r - (|h_1|^2 + |h_2|^2)s_{k'}|^2 < |r - (|h_1|^2 + |h_2|^2)s_k|^2. \tag{4.6}$$

But when ever, $s_k \in \mathcal{S}_{L0}(\mathbf{a}_i)$ and $s_{k'} \in \mathcal{S}_{L1}(\mathbf{a}_i)$ satisfies (4.6), an error may not occur, as there can be a signal point $s_{k''} \in \mathcal{S}_{L0}(\mathbf{a}_i)$ which satisfies,

$$|r - (|h_1|^2 + |h_2|^2)s_{k''}|^2 < |r - (|h_1|^2 + |h_2|^2)s_{k'}|^2. \tag{4.7}$$

We can write an upper bound on the pairwise error probability when $s_k \in \mathcal{S}_{L0}(\mathbf{a}_i)$ is transmitted and the decoder decodes to $s_{k'} \in \mathcal{S}_{L1}(\mathbf{a}_i)$ as,

$$\mathbf{Pr}\{s_k \to s_{k'}|h_1, h_2\} \leq \mathcal{Q}\left(\frac{|(|h_1|^2 + |h_2|^2)(s_k - s_{k'})|}{2\sqrt{N_0(|h_1|^2 + |h_2|^2)/2}}\right) \tag{4.8}$$

Assuming normalized signal power, we have $SNR = 1/N_0$ and (4.8) implies,

$$\mathbf{Pr}\{s_k \to s_{k'}|h_1, h_2\} \leq \mathcal{Q}\left(\sqrt{\frac{SNR(|h_1|^2 + |h_2|^2)|(s_k - s_{k'})|^2}{2}}\right). \tag{4.9}$$

Since $\mathcal{Q}(x)$ is upper bounded by $\exp(-x^2/2)$, we have,

$$\mathbf{Pr}\{s_k \to s_{k'}|h_1, h_2\} \leq \exp\left(\frac{-SNR(|h_1|^2 + |h_2|^2)|(s_k - s_{k'})|^2}{4}\right). \tag{4.10}$$

Averaging (4.10) with respect to h_1 and h_2 we get,

$$\mathbf{Pr}\{s_k \to s_{k'}\} \leq \mathbb{E}_{h_1,h_2}\left[\exp\left(\frac{-SNR(|h_1|^2 + |h_2|^2)|(s_k - s_{k'})|^2}{4}\right)\right].$$

Under the independent Rayleigh fading assumption and making use of the fact that the moment generating function for a unit mean exponential random variable X is $\mathbb{E}(e^{sX}) = 1/(1-s)$, the upper bound can be obtained as,

$$\mathbf{Pr}\{s_k \to s_{k'}\} \leq \left(\frac{1}{1 + \frac{SNR|s_k - s_{k'}|^2}{4}} \right)^2.$$

So, we have the upper bound on the pairwise error probability as,

$$\mathbf{Pr}\{s_k \to s_{k'}\} \leq \frac{16}{SNR^2 |s_k - s_{k'}|^4}, \tag{4.11}$$

where $s_k \in \mathcal{S}_{L0}(\mathbf{a}_i)$ and $s_{k'} \in \mathcal{S}_{L1}(\mathbf{a}_i)$. From (4.11) it is clear that the proposed scheme gives a diversity of two. It is interesting that, unlike the classical MIMO case, the coding gain of the MIMO scheme for noisy index coding problem over fading channel is decided by the inter-set distance. The mapping of broadcast vectors, $\mathbb{F}_2^N$ to the signal set $\mathcal{S}$ is very crucial in deciding the coding gain. The mapping must maximize the minimum inter-set distance to obtain best coding gain. Among the mappings which has the same minimum inter-set distance, the one which has less multiplicity (of the pairs which result in the minimum inter-set distance) will perform better. The Algorithm 1 can be used to obtain a mapping which maximizes the minimum inter-set distance, for prioritized receivers.

4.4 Simulation Results

4.4.1 Diversity Gain and Coding Gain

Similar to the case of SISO-AWGN-ICPM, we have considered the ICP and the index code given in Example 2.1 for the proposed MIMO scheme in the case of prioritized receivers. Assume that the decreasing order of priority is $(R_1, R_2, R_3, R_4, R_5)$. For $N = 3$, we have used the mapping $\mathcal{M}_1$ (Figure 2.1(a)), which maximizes the minimum inter-set distance of the receivers, considered in the given order of priority.

We studied the message error performance of receivers for the proposed MIMO scheme with the 2×2 Alamouti code by simulation and compared it with that of a SISO scheme (for the

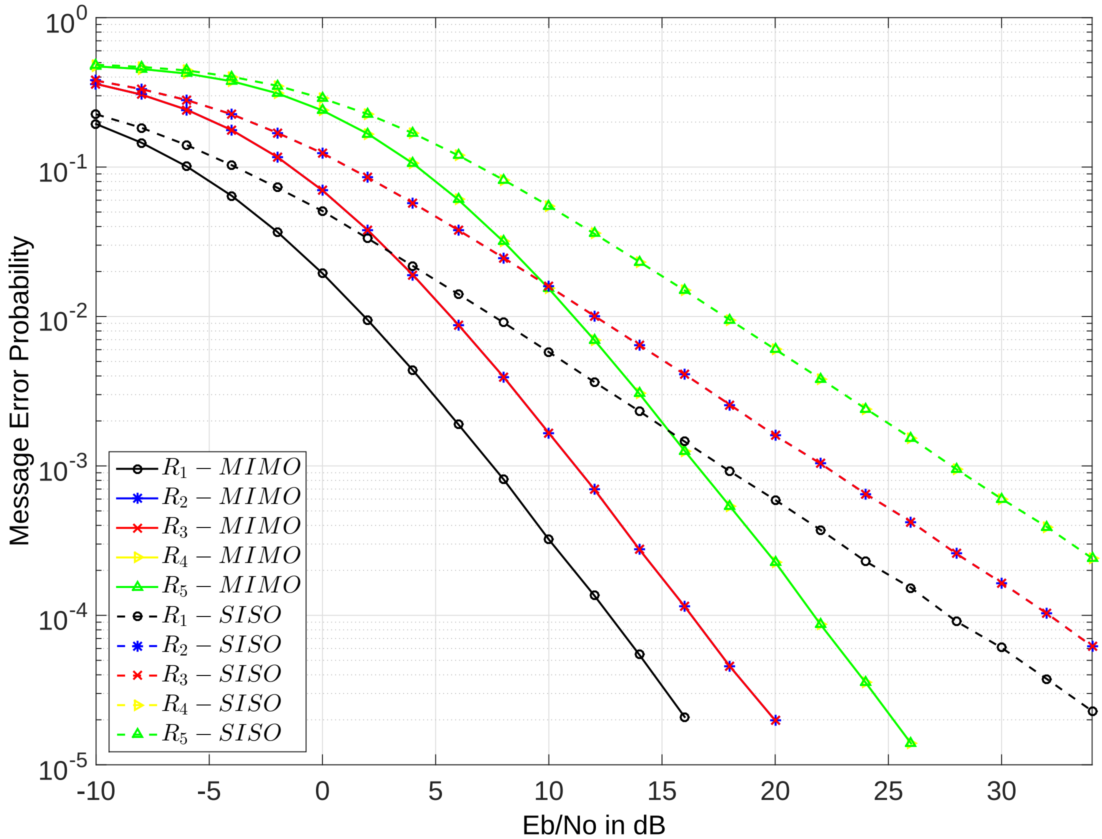

Figure 4.1: Simulation results comparing the performance of proposed MIMO scheme with a SISO scheme.

same index code, mapping and over Rayleigh fading channel), where the sender has only one transmit antenna. ML decoding was performed in both the cases. The simulation results are given in Figure 4.1.

The variation in the performance of the receivers in the MIMO scheme is due to the difference in the inter-set distance distribution of the receivers. When we focus on the performance of a receiver R_i, it is very clear that the proposed MIMO scheme gives a diversity gain of 2 compared to the diversity gain of 1 for the SISO scheme.

Let us move on to consider the coding gain. For the same ICP and index code, the performance of receivers for the proposed MIMO scheme with mapping $\mathcal{M}_1$ is compared with that of the mapping $\mathcal{M}_2$ (Figure 2.1(d)). The simulation results are given in Figure 4.2. In both the

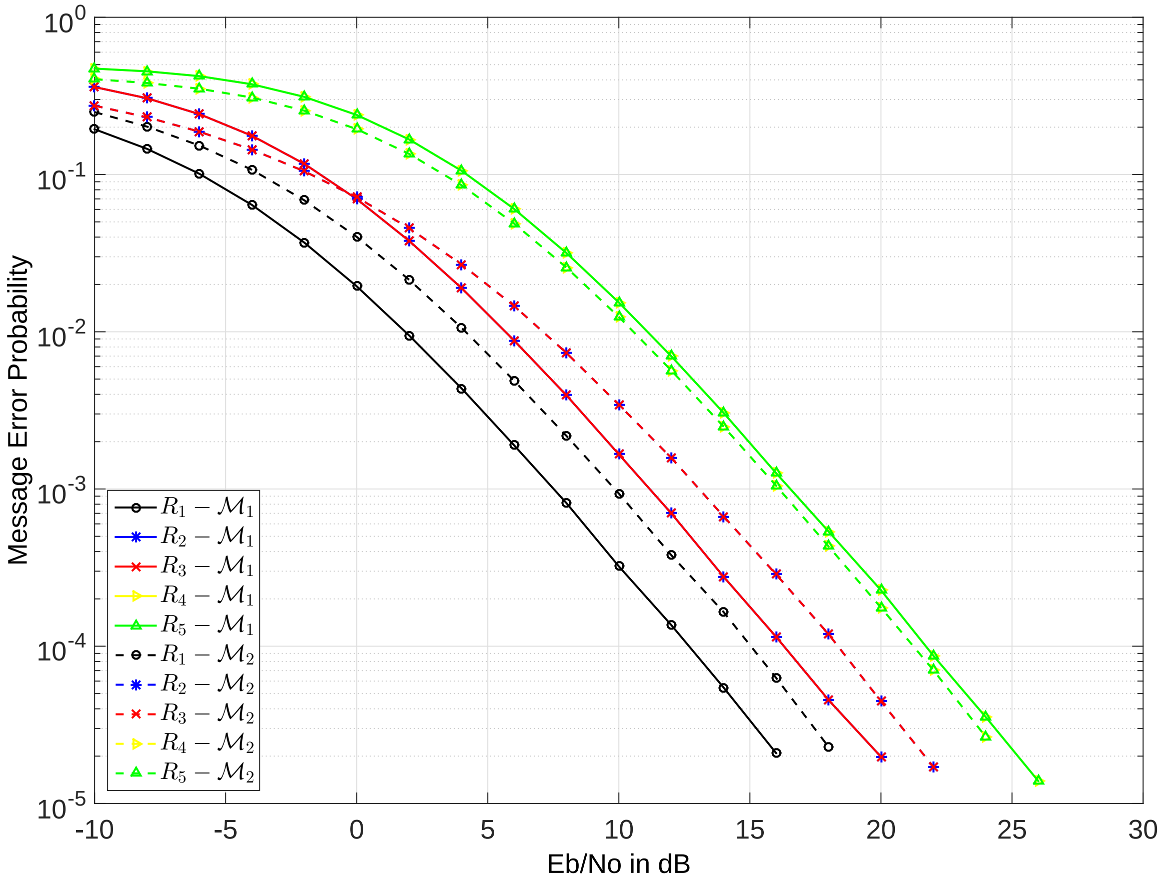

Figure 4.2: Simulation results comparing the performance of proposed MIMO scheme for two different mappings.

cases we have used ML decoding.

As in the case of SISO AWGN channel, the performance of receivers R_1, R_2 and R_3 is significantly better with $\mathcal{M}_1$ than with $\mathcal{M}_2$ at high SNR. For receivers R_4 and R_5, the minimum inter-set distances are same for both the mappings. But the multiplicity is more for $\mathcal{M}_1$ than for $\mathcal{M}_2$ and this resulted in the better performance of the receivers, with $\mathcal{M}_2$ than with $\mathcal{M}_1$. In the case of SISO AWGN channel, this difference is hardly noticeable as the probability of error has an inverse exponential relation with inter-set distance and the effect of multiplicity is not as significant as in the case of the fading channel (where the probability of error has inverse polynomial relation with inter-set distance).

The simulation results validate our claim that the proposed MIMO scheme gives a diversity gain of two and the coding gain is controlled by the minimum inter-set distance distribution.

4.4.2 High SNR Approximation of ML Decoder

As in the case of SISO-AWGN-ICPM, simulations were carried out to verify the high SNR approximation of ML decoder. For the ICP and the index code given in Example 2.1, we first considered the mapping $\mathcal{M}_1$, which maximizes the minimum inter-set distance of the receivers. We carried out ML decoding and minimum Euclidean distance decoding for the proposed MIMO scheme. The simulation results are given in Figure 4.3.

We repeated the simulation with mapping $\mathcal{M}_2$, which maximizes the minimum Euclidean distance, and compared the performance of ML decoding with minimum Euclidean distance decoding. The simulation results are given in Figure 4.4.

Similar to the case of the case of SISO-AWGN-ICPM, with mapping $\mathcal{M}_1$, the performance of ML decoder and minimum Euclidean distance decoder is very similar for any receiver R_i. In the case of mapping $\mathcal{M}_2$, ML decoder performs slightly better than minimum Euclidean distance decoder for receivers R_4 and R_5 at low SNR. But at high SNR, the performance of both the decoders is the same for any receiver.

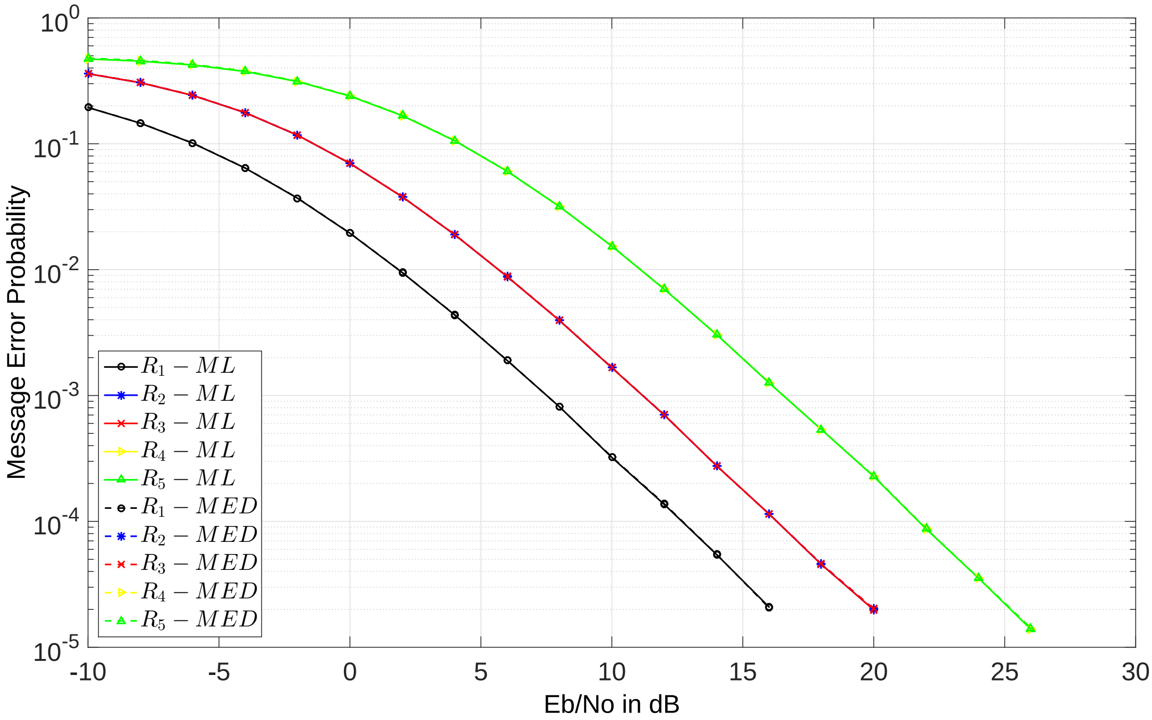

Figure 4.3: Simulation results comparing the performance of ML decoder (ML) and minimum Euclidean distance decoder (MED) with mapping $\mathcal{M}_1$ for the proposed MIMO scheme.

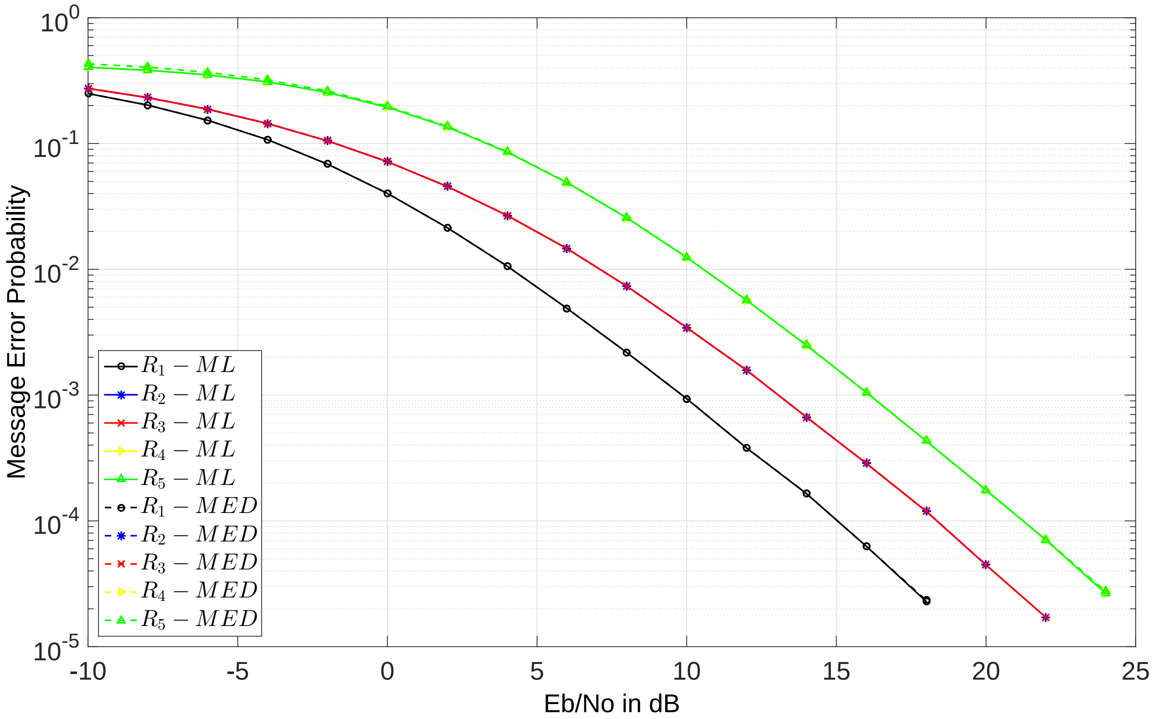

Figure 4.4: Simulation results comparing the performance of ML decoder (ML) and minimum Euclidean distance decoder (MED) with mapping $\mathcal{M}_2$ for the proposed MIMO scheme.

Chapter 5

Optimal Index Codes for ICPM for Prioritized Receivers

Consider an ICP $(\mathcal{X}, \mathcal{R})$, over a SISO AWGN broadcast channel with n messages and m receivers where each receiver demands one message. Let N_o be the length of the optimal IC for the ICP when the broadcast channel is noiseless. So far we have discussed the optimal mapping for a given ICP, chosen N and a defined order of priority among the receivers. In this chapter, we address the case where N is varied from N_o to n, to find the length of optimal index codes for the ICP when ICPM is employed over an AWGN broadcast channel. For this optimal length, we can use the algorithm proposed in Chapter 3 (Algorithm 1) to obtain optimal (index code, mapping) pairs across all possible mappings for all possible index codes (of any length).

For comparing the performance of a receiver with varying constellation size, the average energy of the signal points is assumed to be the same (unity) irrespective of the constellation size. Consider N such that $N_o < N \leq n$. Since the index code length is a variable and the PSK-ICG depends on the index code length, let us denote the PSK-ICG obtained by the highest priority receiver R_1, when an optimal (index code, mapping) pair and 2^N PSK modulation is used, by $g_{1,N}$ in this chapter. As illustrated in Example 5.1, $g_{1,N}$ can be greater than g_{1,N_o} and this implies that in the case of noisy index coding with PSK modulation, an index code of length $N \neq N_o$ used along with 2^N PSK constellation may perform better than an index code of length

N_o used along with 2^{N_o} PSK constellation for the given order of priority among the receivers.

Example 5.1. Consider the ICP given in Example 2.1. Let $\mathcal{C}_1$ be an optimal index code (for a noiseless channel) of length $N = N_o = 3$ with $y_1 = x_1 + x_4 + x_5$, $y_2 = x_1 + x_2 + x_3 + x_4 + x_5$ and $y_3 = x_4 + x_5$. An optimal mapping corresponding to this index code obtained using Algorithm 1 is shown in Figure 5.1(a). Let $\mathcal{C}_2$ be a non-optimal index code (over a noiseless channel) of length $N = 4$ with $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_3$ and $y_4 = x_4 + x_5$. For $\mathcal{C}_2$ an optimal mapping for R_1 is shown in Figure 5.1(b). It is clear from the figure that $d^{(1)}_{IS,min}$ with $N = 4$ is more than that with $N = N_o = 3$. So for the given order of priority, choosing non-optimal index codes (of noiseless ICP) along with optimal mappings for appropriate PSK constellations, gives more PSK-ICG. The improved message error performance of R_1 is due to the following reasons:

- The size of the effective broadcast vector set remains the same even though the total number of broadcast vectors has increased.
- Since the constellation size has increased, the broadcast vectors in the 0-effective broadcast vector set are mapped to signal points which are closer. Similarly, the broadcast vectors in the 1-effective broadcast vector set are also mapped to signal points which are closer and diametrically opposite to that of the broadcast vectors in the 0-effective broadcast vector set. This implies that the minimum inter-set distance has increased.

However, the performance of the lower priority receivers can be adversely affected when the index code length increases. For example, the PSK-ICG obtained by R_4 and R_5 which has $d^{(4)}_{IS,min} = d^{(5)}_{IS,min} = d_{min,N}$ decreases as N increases from 3 to 4.

It is worth noting that the improvement in the performance of the highest priority receiver can be obtained only if we use the algorithm which maximises the minimum inter-set distance. If we employ an algorithm which maximises the minimum Euclidean distances of the effective signal sets [14], the performance of R_1 will be the same for $\mathcal{C}_1$ and $\mathcal{C}_2$, and the PSK-ICG will be the same, but less than that obtained using Algorithm 1 for $\mathcal{C}_1$ or $\mathcal{C}_2$.

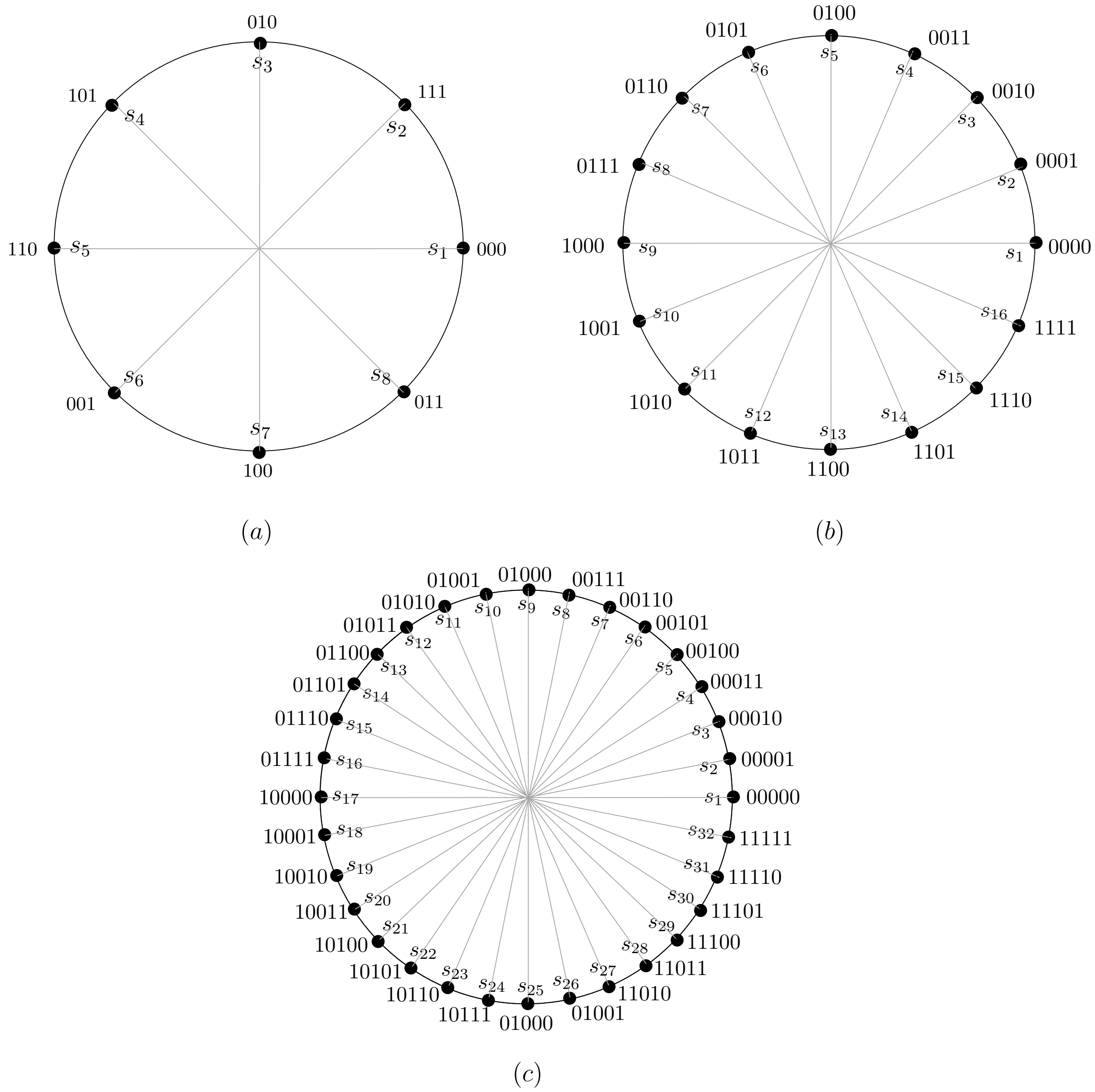

Figure 5.1: Optimal mappings for R_1 with three different constellation sizes as in Example 5.1

Now consider the IC, $\mathcal{C}_3$ with $N = 5$ where the index coded bits are same as the message bits. An optimal mapping for R_1 is shown in Figure 5.1(c). N has increased from 4 to 5 and the size of the effective signal set seen by R_1 has increased from 4 to 8. In this case, even though the constellation size has increased, $d_{IS,min}^{(1)}$ is less than that with $N = 4$ but higher than that with $N = 3$.

So for this example, executing Algorithm 1 with $N = 4$ will give optimal (index code, mapping) pairs across all possible constellation sizes, 2^N with $3 \leq N \leq 5$. In other words, for this ICP, an optimal (index code, mapping) pair obtained from Algorithm 1 with $N = 4$, will give the best message error performance for the receivers, among all possible index codes of any length and any mapping for PSK modulation, with the given order of priority.

It is clear that when ICPM is used, an optimal index code for a noiseless channel need not provide the optimum performance for the prioritized receivers over a noisy channel. In the subsequent part of this chapter, we obtain an expression for the length of an optimal index code for ICPM with prioritized receivers, provided $|\mathcal{K}_1| < n-1$.

Definition 5.1. An index code based on an encoding matrix L_{opt} is said to be an *optimal index code for the ICP with prioritized receivers when ICPM is used over a SISO AWGN broadcast channel*, if there exists a mapping $\mathcal{M}_{opt}$ such that the pair, $(L_{opt}, \mathcal{M}_{opt})$ gives the best PSK-ICG for the prioritized receivers among all possible pairs $(L, \mathcal{M})$, where L corresponds to an index code of length N, $N_o \leq N \leq n$ and $\mathcal{M}$ is a mapping from $\mathbb{F}_2^N$ to 2^N PSK constellation. Let the length of the index code based on L_{opt} be denoted as N_{opt}.

It must be noted that all index codes of length N_{opt} need not be optimal index codes for the ICP with prioritized receivers when ICPM is employed.

For the given ICP $(\mathcal{X}, \mathcal{R})$ with R_i demanding $x_{f(i)}$ and the decreasing order of priority among the receivers as $(R_1, R_2, ..., R_m)$, we can always obtain a reduced ICP $(\mathcal{X}_{red}, \mathcal{R}_{red})$ as follows:

- Remove all receivers which demand messages in $\mathcal{K}_1$. That is, $\mathcal{R}_{red} = \{R_j : x_{f(j)} \notin \mathcal{K}_1\}$.
- Remove all messages that are present as side information of R_1 from the problem. So we have, $\mathcal{X}_{red} = \mathcal{X} \setminus \mathcal{K}_1$. Let the receivers in $\mathcal{R}_{red}$ be specified as $(\mathcal{W}_{i,red}, \mathcal{K}_{i,red})$ where $\mathcal{W}_{i,red} = \mathcal{W}_i$ and $\mathcal{K}_{i,red} = \mathcal{K}_i \setminus \mathcal{K}_1 \quad \forall i$ such that $R_i \in \mathcal{R}_{red}$.

The length of the optimal index code for the reduced ICP $(\mathcal{X}_{red}, \mathcal{R}_{red})$ over a noiseless broadcast channel is denoted as $N_{o,red}$.

Theorem 5.1. The length of an optimal index code for an ICP with prioritized receivers when index coded PSK modulation is employed over a SISO AWGN broadcast channel is given by

$$N_{opt} = |\mathcal{K}_1| + N_{o,red} \tag{5.1}$$

provided $|\mathcal{K}_1| < n-1$.

Proof. Let the number of independent binary linear combinations of the encoded bits which are known to receiver R_1 be k. Then the size of the effective broadcast vector set seen by R_1 is $2^{(N-k)}$. For clarity, let us denote the minimum inter-set distance for receiver R_1 by $d^{(1)}_{IS,min}(N,k)$ for a chosen index code of length N.

Claim 3: In an optimal index code for an ICP with prioritized receivers using ICPM over an AWGN broadcast channel, $k = |\mathcal{K}_1|$, provided $|\mathcal{K}_1| < n-1$.

Proof. We prove the claim by contradiction. Suppose $N_{opt} = N'$ and $k = k' < |\mathcal{K}_1|$. Assuming the signal points to be of unit energy we have,

$$(d^{(1)}_{IS,min}(N',k'))^2 = 2\left(1 - \cos\left(\pi - \frac{2\pi}{2^{N'}}\left(2^{(N'-k'-1)} - 1\right)\right)\right). \tag{5.2}$$

But in this case, we can always find another index code of length $N'' = N' + (|\mathcal{K}_1| - k')$ such that $k = k'' = |\mathcal{K}_1|$.

$$(d^{(1)}_{IS,min}(N'',k''))^2 = 2\left(1 - \cos\left(\pi - \frac{2\pi}{2^{N''}}\left(2^{(N'-k'-1)} - 1\right)\right)\right). \tag{5.3}$$

From (5.2) and (5.3),

$$(d^{(1)}_{IS,min}(N'',k''))^2 > (d^{(1)}_{IS,min}(N',k'))^2 \tag{5.4}$$

provided $k' \neq N'-1$. It is easy to verify that $k' = N'-1$ if and only if $|\mathcal{K}_1| = n-1$. For R_1 to obtain maximum PSK-ICG, the minimum inter-set distance must be maximised. From (5.4) and the fact that $k \leq |\mathcal{K}_1|$ the claim follows. □

Let the length of an optimal index code based on L_{opt} for a noisy ICP as being discussed be N_{opt}. Since $k = |\mathcal{K}_1|$ for the IC based on L_{opt}, we can always find another IC based on L'_{opt} of length N_{opt} such that k encoded bits are the messages available as side information of R_1. It suffices to find the length of the IC based on L'_{opt}. The k encoded bits (messages available as side information of R_1) satisfies the requirements of all the receivers (and only those receivers) which demand messages belonging to $\mathcal{K}_1$. The remaining encoded bits must satisfy the demands of all other receivers. Consider two index codes with lengths N' and N'' such that $k = k' = k'' = |\mathcal{K}_1|$ and $N'' > N'$. Then,

$$\begin{aligned}(d^{(1)}_{IS,min}(N',k'))^2 =& 2\left(1-\cos\left(\pi - \frac{2\pi}{2^{N'}}\left(2^{(N'-k'-1)}-1\right)\right)\right)\\ =& 2\left(1-\cos\left(\pi - \frac{2\pi}{2^{(k+1)}}+\frac{2\pi}{2^{N'}}\right)\right),\end{aligned}$$

$$\begin{aligned}(d^{(1)}_{IS,min}(N'',k''))^2 =& 2\left(1-\cos\left(\pi - \frac{2\pi}{2^{N''}}\left(2^{(N''-k''-1)}-1\right)\right)\right)\\ =& 2\left(1-\cos\left(\pi - \frac{2\pi}{2^{(k+1)}}+\frac{2\pi}{2^{N''}}\right)\right)\end{aligned}$$

and

$$(d^{(1)}_{IS,min}(N'',k''))^2 < (d^{(1)}_{IS,min}(N',k'))^2. \tag{5.5}$$

From (5.5), it is clear that to maximise the minimum inter-set distance for R_1, N must be as small as possible. So for an optimal index code for a noisy ICP as being discussed, $k = |\mathcal{K}_1|$ and N must be as small as possible. The $k = |\mathcal{K}_1|$ encoded bits can meet the demands of all the receivers and only those receivers which want messages belonging to $\mathcal{K}_1$. To obtain as small an N as possible, we consider the reduced ICP $(\mathcal{X}_{red}, \mathcal{R}_{red})$ (as already defined in this chapter) over a noiseless broadcast channel. The encoded bits corresponding to, an optimal index code of $(\mathcal{X}_{red}, \mathcal{R}_{red})$ ($N_{o,red}$ bits), and $\mathcal{K}_1$ ($|\mathcal{K}_1|$ bits), gives an optimal index code for an ICP with prioritized receivers using index coded PSK modulation over a noisy broadcast channel. □

5.1 Simulation Results

Consider the ICP given in Example 2.1 over a SISO AWGN channel. The reduced ICP $(\mathcal{X}_{red}, \mathcal{R}_{red})$ has $\mathcal{R}_{red} = \{R_1, R_4, R_5\}$, $\mathcal{X}_{red} = \{x_1, x_4, x_5\}$, $\mathcal{W}_i = x_i, i \in \{1,4,5\}$, $\mathcal{K}_1 = \{\}$, $\mathcal{K}_4 = \{x_5\}$ and $\mathcal{K}_5 = \{x_4\}$. In this case $N_{o,red} = 2$ and $N_{opt} = |\mathcal{K}_1| + 2 = 4$. We have simulated the message error performance of the receivers for three constellation sizes ($N_o < N \leq n$) using the index codes $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ as given in Example 5.1 and mappings as shown in Figure 5.1(a) ($N = 3$),

Figure 5.1(b) ($N = 4$) and Figure 5.1(c) ($N = 5$). The simulation results as given in Figure 5.2 and Figure 5.3, show that the message error performance of R_1 is optimum when $N = 4$. Any other value of N will result in a lower PSK-ICG for R_1. Since we are considering prioritized receivers, we can conclude that $N = 4$ is the optimum length for this ICP over AWGN channel.

However, it should be noted that a slight improvement in performance of R_1 is achieved at the cost of considerable degradation of performance of low priority receivers (like R_5 and R_3).

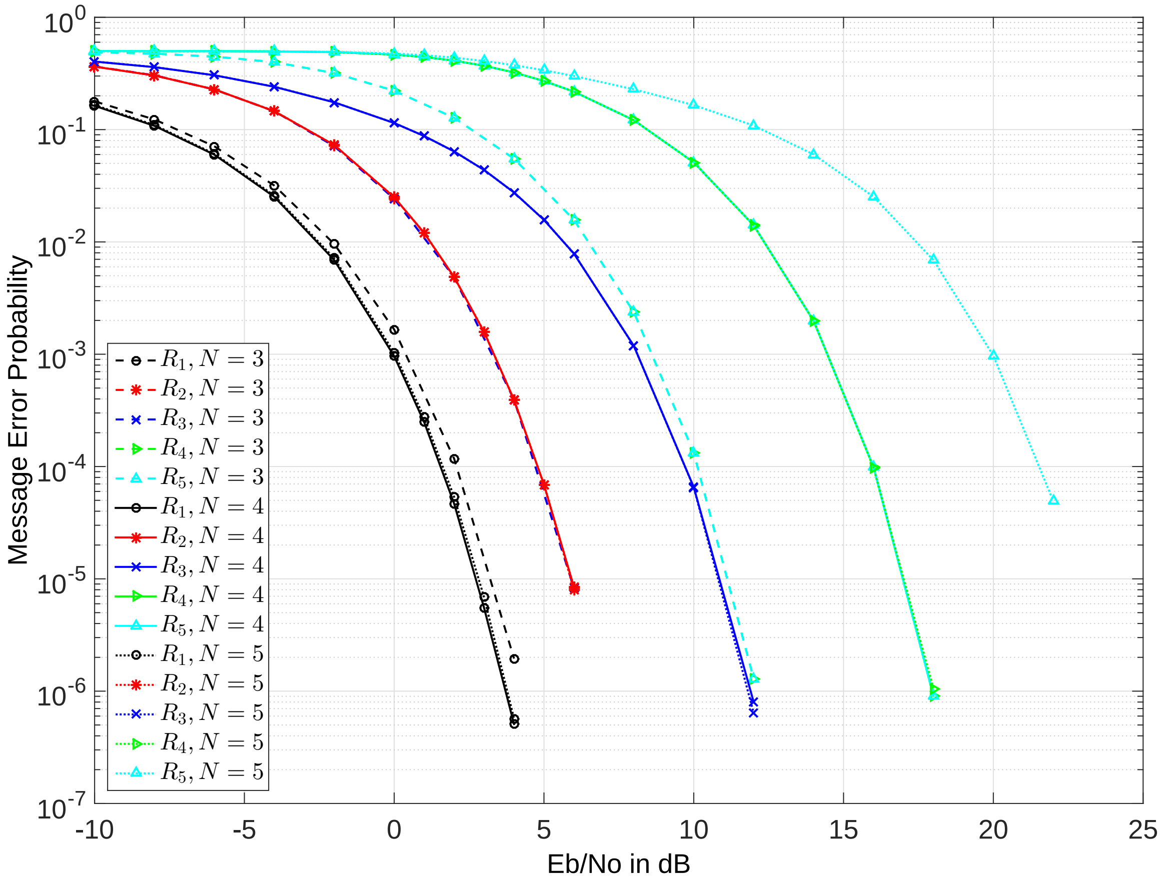

Figure 5.2: Performance comparison of receivers with three different constellation sizes as in Example 5.1

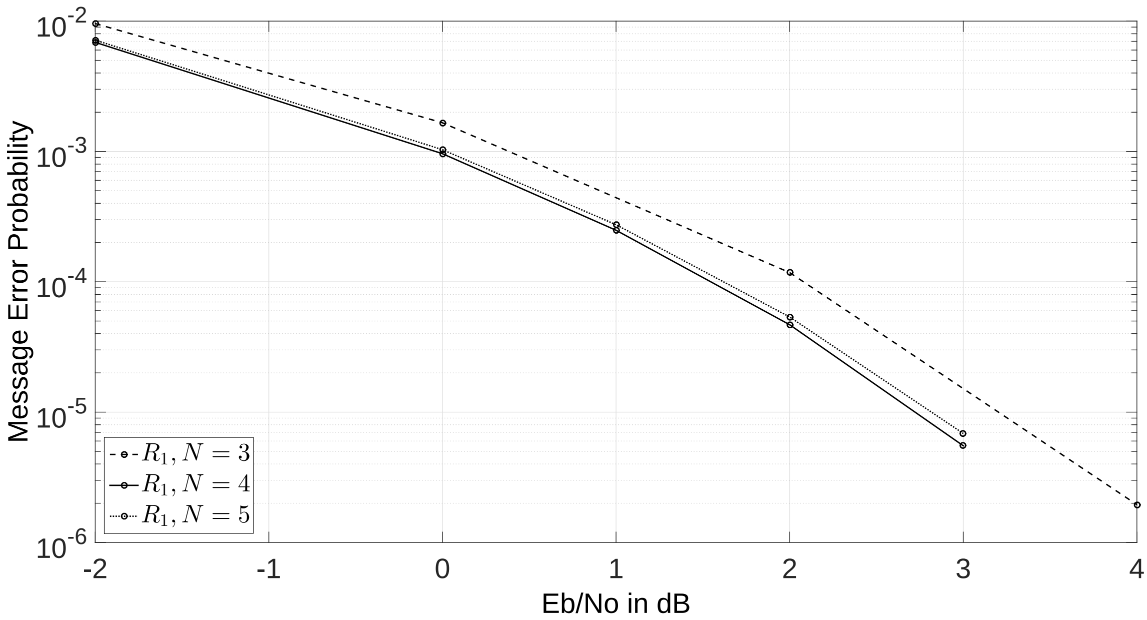

Figure 5.3: Performance comparison of R_1 with three different constellation sizes at high SNR

Chapter 6

Conclusion

6.1 Summary of Results

We have shown that, when ICPM is used over a SISO AWGN channel, for a given index code and mapping, the performance of a receiver at high SNR is determined primarily by the minimum inter-set distance. A decision rule for the ML decoder is obtained for SISO-AWGN-ICPM. In the case of ICP with prioritized receivers, we have considered the problem of finding optimal (index code, mapping) pairs across all possible mappings for all possible index codes of a chosen length. This problem was not addressed so far in literature. For any given ICP with $|\mathcal{K}_1| < n - 1$ over an AWGN channel, optimal length of an index code can be found using the proposed theorem (Theorem 5.1) and then the proposed algorithm can be executed to obtain optimal (index code, mapping) pairs across all possible mappings for all possible index codes, each of which gives the best PSK-ICG for the receivers, for any given order of priority.

Finding all index codes of a chosen length (greater than or equal to the optimal length) for a given ICP is in general NP hard. If it is too complex to find all the index codes, the algorithm can be executed by considering a chosen set of index codes. But the complexity of the proposed algorithm increases exponentially with the length of the index code.

Then we considered noisy index coding problem, where the receivers demand a subset of messages, over Rayleigh fading channel. We have proposed a MIMO scheme and studied the performance over a 2×1 MIMO channel, which was not considered so far in the literature. We have also shown that in the case of a noisy index coding problem the mapping of broadcast vectors to constellation points is very crucial and the coding gain is decided by the inter-set distance distribution.

6.2 Scope for Future Work

Some interesting directions for future research are as follows.

- We have considered ICP where each receiver demands only one message. If a receiver wants $|\mathcal{W}_i|$ messages, it is considered as $|\mathcal{W}_i|$ equivalent receivers each demanding one message and having the same side information. An interesting future work would be to study whether an improvement in the message error performance can be obtained if the receivers demanding more than one message are considered without splitting.
- The proposed algorithm to find the set of optimal (index code, mapping) pairs is based on minimum inter-set distance. It may be useful to study whether the receiver performance can be improved by considering the inter-set distance distribution, especially at low SNR.
- It would be interesting to compare the performance of index coded PSK modulation with index coded QAM modulation or other similar schemes, when an algorithm based on inter-set distances is employed.
- Another open problem is to consider arbitrary number of transmit and receive antennas with space-time codes that are not obtained from orthogonal designs.

Appendix A

Optimal (index code, mapping) pairs $(\mathcal{C}, \mathcal{M})$, for Example 2.1 is given in Table A.1. $\mathcal{C}$ is given in the form of (y_1, y_2, y_3). $\mathcal{M}$ is given as an ordered list of eight integers, representing the decimal equivalent of the 3-tuple, in the order of $(\mathbf{s_1}, \mathbf{s_2}, ...\mathbf{s_8})$ where $(\mathbf{s_1}, \mathbf{s_2}, ...\mathbf{s_8})$ are signal points as shown in Figure 2.1.

Sl. No.	$\mathcal{C}$	$\mathcal{M}$
1	$(x_1, x_2 + x_3, x_4 + x_5)$	$(0, 1, 2, 3, 4, 5, 6, 7)$
2	$(x_1, x_2 + x_3, x_4 + x_5)$	$(0, 1, 6, 7, 4, 5, 2, 3)$
3	$(x_1, x_2 + x_3, x_4 + x_5)$	$(0, 3, 2, 5, 4, 7, 6, 1)$
4	$(x_1, x_2 + x_3, x_4 + x_5)$	$(0, 7, 6, 5, 4, 3, 2, 1)$
5	$(x_1, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 1, 3, 2, 4, 5, 7, 6)$
6	$(x_1, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 1, 7, 6, 4, 5, 3, 2)$
7	$(x_1, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 2, 3, 5, 4, 6, 7, 1)$
8	$(x_1, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 6, 7, 5, 4, 2, 3, 1)$
9	$(x_1, x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 1, 2, 3, 5, 4, 7, 6)$
10	$(x_1, x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 1, 7, 6, 5, 4, 2, 3)$
11	$(x_1, x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 3, 2, 4, 5, 6, 7, 1)$
12	$(x_1, x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 6, 7, 4, 5, 3, 2, 1)$
13	$(x_1, x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 1, 3, 2, 5, 4, 6, 7)$
14	$(x_1, x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 1, 6, 7, 5, 4, 3, 2)$
15	$(x_1, x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 2, 3, 4, 5, 7, 6, 1)$
16	$(x_1, x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 7, 6, 4, 5, 2, 3, 1)$
17	$(x_1, x_1 + x_2 + x_3, x_4 + x_5)$	$(0, 1, 2, 3, 6, 7, 4, 5)$

18	$(x_1, x_1 + x_2 + x_3, x_4 + x_5)$	$(0, 1, 4, 5, 6, 7, 2, 3)$
19	$(x_1, x_1 + x_2 + x_3, x_4 + x_5)$	$(0, 3, 2, 7, 6, 5, 4, 1)$
20	$(x_1, x_1 + x_2 + x_3, x_4 + x_5)$	$(0, 5, 4, 7, 6, 3, 2, 1)$
21	$(x_1, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 3, 2, 1, 4, 7, 6, 5)$
22	$(x_1, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 3, 6, 5, 4, 7, 2, 1)$
23	$(x_1, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 1, 2, 7, 4, 5, 6, 3)$
24	$(x_1, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 5, 6, 7, 4, 1, 2, 3)$
25	$(x_1, x_1 + x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 3, 2, 1, 6, 5, 4, 7)$
26	$(x_1, x_1 + x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 3, 4, 7, 6, 5, 2, 1)$
27	$(x_1, x_1 + x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 1, 2, 5, 6, 7, 4, 3)$
28	$(x_1, x_1 + x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 7, 4, 5, 6, 1, 2, 3)$
29	$(x_1, x_1 + x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 1, 3, 2, 6, 7, 5, 4)$
30	$(x_1, x_1 + x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 1, 5, 4, 6, 7, 3, 2)$
31	$(x_1, x_1 + x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 2, 3, 7, 6, 4, 5, 1)$
32	$(x_1, x_1 + x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 4, 5, 7, 6, 2, 3, 1)$
33	$(x_1, x_1 + x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 1, 2, 3, 7, 6, 5, 4)$
34	$(x_1, x_1 + x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 1, 5, 4, 7, 6, 2, 3)$
35	$(x_1, x_1 + x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 3, 2, 6, 7, 4, 5, 1)$
36	$(x_1, x_1 + x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 4, 5, 6, 7, 3, 2, 1)$
37	$(x_1, x_1 + x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 1, 3, 2, 7, 6, 4, 5)$
38	$(x_1, x_1 + x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 1, 4, 5, 7, 6, 3, 2)$
39	$(x_1, x_1 + x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 2, 3, 6, 7, 5, 4, 1)$
40	$(x_1, x_1 + x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 5, 4, 6, 7, 2, 3, 1)$
41	$(x_1, x_2 + x_3 + x_4 + x_5, x_1 + x_4 + x_5)$	$(0, 3, 2, 1, 5, 6, 7, 4)$
42	$(x_1, x_2 + x_3 + x_4 + x_5, x_1 + x_4 + x_5)$	$(0, 3, 7, 4, 5, 6, 2, 1)$
43	$(x_1, x_2 + x_3 + x_4 + x_5, x_1 + x_4 + x_5)$	$(0, 1, 2, 6, 5, 4, 7, 3)$
44	$(x_1, x_2 + x_3 + x_4 + x_5, x_1 + x_4 + x_5)$	$(0, 4, 7, 6, 5, 1, 2, 3)$
45	$(x_1, x_1 + x_4 + x_5, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 3, 1, 2, 7, 4, 6, 5)$
46	$(x_1, x_1 + x_4 + x_5, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 3, 6, 5, 7, 4, 1, 2)$
47	$(x_1, x_1 + x_4 + x_5, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 2, 1, 4, 7, 5, 6, 3)$
48	$(x_1, x_1 + x_4 + x_5, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 5, 6, 4, 7, 2, 1, 3)$

49	$(x_1+x_2+x_3, x_2+x_3, x_4+x_5)$	$(0,1,2,3,4,5,6,7)$
50	$(x_1+x_2+x_3, x_2+x_3, x_4+x_5)$	$(0,1,6,7,4,5,2,3)$
51	$(x_1+x_2+x_3, x_2+x_3, x_4+x_5)$	$(0,3,2,5,4,7,6,1)$
52	$(x_1+x_2+x_3, x_2+x_3, x_4+x_5)$	$(0,7,6,5,4,3,2,1)$
53	$(x_1+x_4+x_5, x_2+x_3, x_4+x_5)$	$(0,5,2,7,4,1,6,3)$
54	$(x_1+x_4+x_5, x_2+x_3, x_4+x_5)$	$(0,5,6,3,4,1,2,7)$
55	$(x_1+x_4+x_5, x_2+x_3, x_4+x_5)$	$(0,7,2,1,4,3,6,5)$
56	$(x_1+x_4+x_5, x_2+x_3, x_4+x_5)$	$(0,3,6,1,4,7,2,5)$
57	$(x_1+x_2+x_3+x_4+x_5, x_2+x_3, x_4+x_5)$	$(0,5,2,7,4,1,6,3)$
58	$(x_1+x_2+x_3+x_4+x_5, x_2+x_3, x_4+x_5)$	$(0,5,6,3,4,1,2,7)$
59	$(x_1+x_2+x_3+x_4+x_5, x_2+x_3, x_4+x_5)$	$(0,7,2,1,4,3,6,5)$
60	$(x_1+x_2+x_3+x_4+x_5, x_2+x_3, x_4+x_5)$	$(0,3,6,1,4,7,2,5)$
61	$(x_1+x_2+x_3, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,1,3,2,4,5,7,6)$
62	$(x_1+x_2+x_3, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,1,7,6,4,5,3,2)$
63	$(x_1+x_2+x_3, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,2,3,5,4,6,7,1)$
64	$(x_1+x_2+x_3, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,6,7,5,4,2,3,1)$
65	$(x_1+x_2+x_3, x_2+x_3, x_1+x_4+x_5)$	$(0,1,3,2,5,4,6,7)$
66	$(x_1+x_2+x_3, x_2+x_3, x_1+x_4+x_5)$	$(0,1,6,7,5,4,3,2)$
67	$(x_1+x_2+x_3, x_2+x_3, x_1+x_4+x_5)$	$(0,2,3,4,5,7,6,1)$
68	$(x_1+x_2+x_3, x_2+x_3, x_1+x_4+x_5)$	$(0,7,6,4,5,2,3,1)$
69	$(x_1+x_2+x_3, x_2+x_3, x_1+x_2+x_3+x_4+x_5)$	$(0,1,2,3,5,4,7,6)$
70	$(x_1+x_2+x_3, x_2+x_3, x_1+x_2+x_3+x_4+x_5)$	$(0,1,7,6,5,4,2,3)$
71	$(x_1+x_2+x_3, x_2+x_3, x_1+x_2+x_3+x_4+x_5)$	$(0,3,2,4,5,6,7,1)$
72	$(x_1+x_2+x_3, x_2+x_3, x_1+x_2+x_3+x_4+x_5)$	$(0,6,7,4,5,3,2,1)$
73	$(x_1+x_4+x_5, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,5,7,2,4,1,3,6)$
74	$(x_1+x_4+x_5, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,5,3,6,4,1,7,2)$
75	$(x_1+x_4+x_5, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,2,7,1,4,6,3,5)$
76	$(x_1+x_4+x_5, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,6,3,1,4,2,7,5)$
77	$(x_1+x_2+x_3+x_4+x_5, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,5,2,7,4,1,6,3)$
78	$(x_1+x_2+x_3+x_4+x_5, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,5,6,3,4,1,2,7)$
79	$(x_1+x_2+x_3+x_4+x_5, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,7,2,1,4,3,6,5)$

80	$(x_1 + x_2 + x_3 + x_4 + x_5, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 3, 6, 1, 4, 7, 2, 5)$
81	$(x_1 + x_2 + x_3, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 3, 2, 1, 4, 7, 6, 5)$
82	$(x_1 + x_2 + x_3, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 3, 6, 5, 4, 7, 2, 1)$
83	$(x_1 + x_2 + x_3, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 1, 2, 7, 4, 5, 6, 3)$
84	$(x_1 + x_2 + x_3, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 5, 6, 7, 4, 1, 2, 3)$
85	$(x_1 + x_2 + x_3, x_1 + x_4 + x_5, x_4 + x_5)$	$(0, 3, 2, 1, 6, 5, 4, 7)$
86	$(x_1 + x_2 + x_3, x_1 + x_4 + x_5, x_4 + x_5)$	$(0, 3, 4, 7, 6, 5, 2, 1)$
87	$(x_1 + x_2 + x_3, x_1 + x_4 + x_5, x_4 + x_5)$	$(0, 1, 2, 5, 6, 7, 4, 3)$
88	$(x_1 + x_2 + x_3, x_1 + x_4 + x_5, x_4 + x_5)$	$(0, 7, 4, 5, 6, 1, 2, 3)$
89	$(x_1 + x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5, x_2 + x_3 + x_4 + x_5)$	$(0, 3, 2, 1, 5, 6, 7, 4)$
90	$(x_1 + x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5, x_2 + x_3 + x_4 + x_5)$	$(0, 3, 7, 4, 5, 6, 2, 1)$
91	$(x_1 + x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5, x_2 + x_3 + x_4 + x_5)$	$(0, 1, 2, 6, 5, 4, 7, 3)$
92	$(x_1 + x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5, x_2 + x_3 + x_4 + x_5)$	$(0, 4, 7, 6, 5, 1, 2, 3)$
93	$(x_1 + x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5, x_1 + x_4 + x_5)$	$(0, 3, 1, 2, 7, 4, 6, 5)$
94	$(x_1 + x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5, x_1 + x_4 + x_5)$	$(0, 3, 6, 5, 7, 4, 1, 2)$
95	$(x_1 + x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5, x_1 + x_4 + x_5)$	$(0, 2, 1, 4, 7, 5, 6, 3)$
96	$(x_1 + x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5, x_1 + x_4 + x_5)$	$(0, 5, 6, 4, 7, 2, 1, 3)$
97	$(x_1 + x_2 + x_3 + x_4 + x_5, x_2 + x_3 + x_4 + x_5, x_1 + x_4 + x_5)$	$(0, 7, 6, 1, 5, 2, 3, 4)$
98	$(x_1 + x_2 + x_3 + x_4 + x_5, x_2 + x_3 + x_4 + x_5, x_1 + x_4 + x_5)$	$(0, 7, 3, 4, 5, 2, 6, 1)$
99	$(x_1 + x_2 + x_3 + x_4 + x_5, x_2 + x_3 + x_4 + x_5, x_1 + x_4 + x_5)$	$(0, 1, 6, 2, 5, 4, 3, 7)$
100	$(x_1 + x_2 + x_3 + x_4 + x_5, x_2 + x_3 + x_4 + x_5, x_1 + x_4 + x_5)$	$(0, 4, 3, 2, 5, 1, 6, 7)$
101	$(x_1 + x_4 + x_5, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 7, 2, 5, 4, 3, 6, 1)$
102	$(x_1 + x_4 + x_5, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 7, 6, 1, 4, 3, 2, 5)$
103	$(x_1 + x_4 + x_5, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 5, 2, 3, 4, 1, 6, 7)$
104	$(x_1 + x_4 + x_5, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 1, 6, 3, 4, 5, 2, 7)$
105	$(x_1 + x_2 + x_3 + x_4 + x_5, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 7, 2, 5, 4, 3, 6, 1)$
106	$(x_1 + x_2 + x_3 + x_4 + x_5, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 7, 6, 1, 4, 3, 2, 5)$
107	$(x_1 + x_2 + x_3 + x_4 + x_5, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 5, 2, 3, 4, 1, 6, 7)$
108	$(x_1 + x_2 + x_3 + x_4 + x_5, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 1, 6, 3, 4, 5, 2, 7)$
109	$(x_1 + x_4 + x_5, x_1 + x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 7, 2, 5, 6, 1, 4, 3)$
110	$(x_1 + x_4 + x_5, x_1 + x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 7, 4, 3, 6, 1, 2, 5)$

111	$(x_1 + x_4 + x_5, x_1 + x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 5, 2, 1, 6, 3, 4, 7)$
112	$(x_1 + x_4 + x_5, x_1 + x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 3, 4, 1, 6, 5, 2, 7)$
113	$(x_1 + x_2, x_2 + x_3, x_4 + x_5)$	$(0, 1, 2, 3, 4, 5, 6, 7)$
114	$(x_1 + x_2, x_2 + x_3, x_4 + x_5)$	$(0, 1, 6, 7, 4, 5, 2, 3)$
115	$(x_1 + x_2, x_2 + x_3, x_4 + x_5)$	$(0, 3, 2, 5, 4, 7, 6, 1)$
116	$(x_1 + x_2, x_2 + x_3, x_4 + x_5)$	$(0, 7, 6, 5, 4, 3, 2, 1)$
117	$(x_1 + x_2, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 1, 3, 2, 4, 5, 7, 6)$
118	$(x_1 + x_2, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 1, 7, 6, 4, 5, 3, 2)$
119	$(x_1 + x_2, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 2, 3, 5, 4, 6, 7, 1)$
120	$(x_1 + x_2, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 6, 7, 5, 4, 2, 3, 1)$
121	$(x_1 + x_2, x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 1, 2, 3, 5, 4, 7, 6)$
122	$(x_1 + x_2, x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 1, 7, 6, 5, 4, 2, 3)$
123	$(x_1 + x_2, x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 3, 2, 4, 5, 6, 7, 1)$
124	$(x_1 + x_2, x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 6, 7, 4, 5, 3, 2, 1)$
125	$(x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 1, 3, 2, 5, 4, 6, 7)$
126	$(x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 1, 6, 7, 5, 4, 3, 2)$
127	$(x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 2, 3, 4, 5, 7, 6, 1)$
128	$(x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 7, 6, 4, 5, 2, 3, 1)$
129	$(x_1 + x_2, x_1 + x_2 + x_3, x_4 + x_5)$	$(0, 1, 2, 3, 6, 7, 4, 5)$
130	$(x_1 + x_2, x_1 + x_2 + x_3, x_4 + x_5)$	$(0, 1, 4, 5, 6, 7, 2, 3)$
131	$(x_1 + x_2, x_1 + x_2 + x_3, x_4 + x_5)$	$(0, 3, 2, 7, 6, 5, 4, 1)$
132	$(x_1 + x_2, x_1 + x_2 + x_3, x_4 + x_5)$	$(0, 5, 4, 7, 6, 3, 2, 1)$
133	$(x_1 + x_2, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 3, 2, 1, 4, 7, 6, 5)$
134	$(x_1 + x_2, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 3, 6, 5, 4, 7, 2, 1)$
135	$(x_1 + x_2, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 1, 2, 7, 4, 5, 6, 3)$
136	$(x_1 + x_2, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 5, 6, 7, 4, 1, 2, 3)$
137	$(x_1 + x_2, x_1 + x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 3, 2, 1, 6, 5, 4, 7)$
138	$(x_1 + x_2, x_1 + x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 3, 4, 7, 6, 5, 2, 1)$
139	$(x_1 + x_2, x_1 + x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 1, 2, 5, 6, 7, 4, 3)$
140	$(x_1 + x_2, x_1 + x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 7, 4, 5, 6, 1, 2, 3)$
141	$(x_1 + x_2, x_1 + x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 1, 3, 2, 6, 7, 5, 4)$

142	$(x_1+x_2, x_1+x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,1,5,4,6,7,3,2)$
143	$(x_1+x_2, x_1+x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,2,3,7,6,4,5,1)$
144	$(x_1+x_2, x_1+x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,4,5,7,6,2,3,1)$
145	$(x_1+x_2, x_1+x_2+x_3, x_1+x_4+x_5)$	$(0,1,2,3,7,6,5,4)$
146	$(x_1+x_2, x_1+x_2+x_3, x_1+x_4+x_5)$	$(0,1,5,4,7,6,2,3)$
147	$(x_1+x_2, x_1+x_2+x_3, x_1+x_4+x_5)$	$(0,3,2,6,7,4,5,1)$
148	$(x_1+x_2, x_1+x_2+x_3, x_1+x_4+x_5)$	$(0,4,5,6,7,3,2,1)$
149	$(x_1+x_2, x_1+x_2+x_3, x_1+x_2+x_3+x_4+x_5)$	$(0,1,3,2,7,6,4,5)$
150	$(x_1+x_2, x_1+x_2+x_3, x_1+x_2+x_3+x_4+x_5)$	$(0,1,4,5,7,6,3,2)$
151	$(x_1+x_2, x_1+x_2+x_3, x_1+x_2+x_3+x_4+x_5)$	$(0,2,3,6,7,5,4,1)$
152	$(x_1+x_2, x_1+x_2+x_3, x_1+x_2+x_3+x_4+x_5)$	$(0,5,4,6,7,2,3,1)$
153	$(x_1+x_2, x_2+x_3+x_4+x_5, x_1+x_4+x_5)$	$(0,3,2,1,5,6,7,4)$
154	$(x_1+x_2, x_2+x_3+x_4+x_5, x_1+x_4+x_5)$	$(0,3,7,4,5,6,2,1)$
155	$(x_1+x_2, x_2+x_3+x_4+x_5, x_1+x_4+x_5)$	$(0,1,2,6,5,4,7,3)$
156	$(x_1+x_2, x_2+x_3+x_4+x_5, x_1+x_4+x_5)$	$(0,4,7,6,5,1,2,3)$
157	$(x_1+x_2, x_1+x_4+x_5, x_1+x_2+x_3+x_4+x_5)$	$(0,3,1,2,7,4,6,5)$
158	$(x_1+x_2, x_1+x_4+x_5, x_1+x_2+x_3+x_4+x_5)$	$(0,3,6,5,7,4,1,2)$
159	$(x_1+x_2, x_1+x_4+x_5, x_1+x_2+x_3+x_4+x_5)$	$(0,2,1,4,7,5,6,3)$
160	$(x_1+x_2, x_1+x_4+x_5, x_1+x_2+x_3+x_4+x_5)$	$(0,5,6,4,7,2,1,3)$
161	$(x_1+x_3, x_2+x_3, x_4+x_5)$	$(0,1,2,3,4,5,6,7)$
162	$(x_1+x_3, x_2+x_3, x_4+x_5)$	$(0,1,6,7,4,5,2,3)$
163	$(x_1+x_3, x_2+x_3, x_4+x_5)$	$(0,3,2,5,4,7,6,1)$
164	$(x_1+x_3, x_2+x_3, x_4+x_5)$	$(0,7,6,5,4,3,2,1)$
165	$(x_1+x_2+x_4+x_5, x_2+x_3, x_4+x_5)$	$(0,5,2,7,4,1,6,3)$
166	$(x_1+x_2+x_4+x_5, x_2+x_3, x_4+x_5)$	$(0,5,6,3,4,1,2,7)$
167	$(x_1+x_2+x_4+x_5, x_2+x_3, x_4+x_5)$	$(0,7,2,1,4,3,6,5)$
168	$(x_1+x_2+x_4+x_5, x_2+x_3, x_4+x_5)$	$(0,3,6,1,4,7,2,5)$
169	$(x_1+x_3+x_4+x_5, x_2+x_3, x_4+x_5)$	$(0,5,2,7,4,1,6,3)$
170	$(x_1+x_3+x_4+x_5, x_2+x_3, x_4+x_5)$	$(0,5,6,3,4,1,2,7)$
171	$(x_1+x_3+x_4+x_5, x_2+x_3, x_4+x_5)$	$(0,7,2,1,4,3,6,5)$
172	$(x_1+x_3+x_4+x_5, x_2+x_3, x_4+x_5)$	$(0,3,6,1,4,7,2,5)$

173	$(x_1 + x_3, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 1, 3, 2, 4, 5, 7, 6)$
174	$(x_1 + x_3, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 1, 7, 6, 4, 5, 3, 2)$
175	$(x_1 + x_3, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 2, 3, 5, 4, 6, 7, 1)$
176	$(x_1 + x_3, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 6, 7, 5, 4, 2, 3, 1)$
177	$(x_1 + x_3, x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 1, 3, 2, 5, 4, 6, 7)$
178	$(x_1 + x_3, x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 1, 6, 7, 5, 4, 3, 2)$
179	$(x_1 + x_3, x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 2, 3, 4, 5, 7, 6, 1)$
180	$(x_1 + x_3, x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 7, 6, 4, 5, 2, 3, 1)$
181	$(x_1 + x_3, x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 1, 2, 3, 5, 4, 7, 6)$
182	$(x_1 + x_3, x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 1, 7, 6, 5, 4, 2, 3)$
183	$(x_1 + x_3, x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 3, 2, 4, 5, 6, 7, 1)$
184	$(x_1 + x_3, x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 6, 7, 4, 5, 3, 2, 1)$
185	$(x_1 + x_2 + x_4 + x_5, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 5, 7, 2, 4, 1, 3, 6)$
186	$(x_1 + x_2 + x_4 + x_5, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 5, 3, 6, 4, 1, 7, 2)$
187	$(x_1 + x_2 + x_4 + x_5, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 2, 7, 1, 4, 6, 3, 5)$
188	$(x_1 + x_2 + x_4 + x_5, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 6, 3, 1, 4, 2, 7, 5)$
189	$(x_1 + x_3 + x_4 + x_5, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 5, 2, 7, 4, 1, 6, 3)$
190	$(x_1 + x_3 + x_4 + x_5, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 5, 6, 3, 4, 1, 2, 7)$
191	$(x_1 + x_3 + x_4 + x_5, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 7, 2, 1, 4, 3, 6, 5)$
192	$(x_1 + x_3 + x_4 + x_5, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 3, 6, 1, 4, 7, 2, 5)$
193	$(x_1 + x_3, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 3, 2, 1, 4, 7, 6, 5)$
194	$(x_1 + x_3, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 3, 6, 5, 4, 7, 2, 1)$
195	$(x_1 + x_3, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 1, 2, 7, 4, 5, 6, 3)$
196	$(x_1 + x_3, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 5, 6, 7, 4, 1, 2, 3)$
197	$(x_1 + x_3, x_1 + x_4 + x_5, x_4 + x_5)$	$(0, 3, 2, 1, 6, 5, 4, 7)$
198	$(x_1 + x_3, x_1 + x_4 + x_5, x_4 + x_5)$	$(0, 3, 4, 7, 6, 5, 2, 1)$
199	$(x_1 + x_3, x_1 + x_4 + x_5, x_4 + x_5)$	$(0, 1, 2, 5, 6, 7, 4, 3)$
200	$(x_1 + x_3, x_1 + x_4 + x_5, x_4 + x_5)$	$(0, 7, 4, 5, 6, 1, 2, 3)$
201	$(x_1 + x_3, x_1 + x_2 + x_3 + x_4 + x_5, x_2 + x_3 + x_4 + x_5)$	$(0, 3, 2, 1, 5, 6, 7, 4)$
202	$(x_1 + x_3, x_1 + x_2 + x_3 + x_4 + x_5, x_2 + x_3 + x_4 + x_5)$	$(0, 3, 7, 4, 5, 6, 2, 1)$
203	$(x_1 + x_3, x_1 + x_2 + x_3 + x_4 + x_5, x_2 + x_3 + x_4 + x_5)$	$(0, 1, 2, 6, 5, 4, 7, 3)$

204	$(x_1+x_3, x_1+x_2+x_3+x_4+x_5, x_2+x_3+x_4+x_5)$	$(0,4,7,6,5,1,2,3)$
205	$(x_1+x_3, x_1+x_2+x_3+x_4+x_5, x_1+x_4+x_5)$	$(0,3,1,2,7,4,6,5)$
206	$(x_1+x_3, x_1+x_2+x_3+x_4+x_5, x_1+x_4+x_5)$	$(0,3,6,5,7,4,1,2)$
207	$(x_1+x_3, x_1+x_2+x_3+x_4+x_5, x_1+x_4+x_5)$	$(0,2,1,4,7,5,6,3)$
208	$(x_1+x_3, x_1+x_2+x_3+x_4+x_5, x_1+x_4+x_5)$	$(0,5,6,4,7,2,1,3)$
209	$(x_1+x_3+x_4+x_5, x_2+x_3+x_4+x_5, x_1+x_4+x_5)$	$(0,7,6,1,5,2,3,4)$
210	$(x_1+x_3+x_4+x_5, x_2+x_3+x_4+x_5, x_1+x_4+x_5)$	$(0,7,3,4,5,2,6,1)$
211	$(x_1+x_3+x_4+x_5, x_2+x_3+x_4+x_5, x_1+x_4+x_5)$	$(0,1,6,2,5,4,3,7)$
212	$(x_1+x_3+x_4+x_5, x_2+x_3+x_4+x_5, x_1+x_4+x_5)$	$(0,4,3,2,5,1,6,7)$
213	$(x_1+x_2+x_4+x_5, x_2+x_3+x_4+x_5, x_4+x_5)$	$(0,7,2,5,4,3,6,1)$
214	$(x_1+x_2+x_4+x_5, x_2+x_3+x_4+x_5, x_4+x_5)$	$(0,7,6,1,4,3,2,5)$
215	$(x_1+x_2+x_4+x_5, x_2+x_3+x_4+x_5, x_4+x_5)$	$(0,5,2,3,4,1,6,7)$
216	$(x_1+x_2+x_4+x_5, x_2+x_3+x_4+x_5, x_4+x_5)$	$(0,1,6,3,4,5,2,7)$
217	$(x_1+x_3+x_4+x_5, x_2+x_3+x_4+x_5, x_4+x_5)$	$(0,7,2,5,4,3,6,1)$
218	$(x_1+x_3+x_4+x_5, x_2+x_3+x_4+x_5, x_4+x_5)$	$(0,7,6,1,4,3,2,5)$
219	$(x_1+x_3+x_4+x_5, x_2+x_3+x_4+x_5, x_4+x_5)$	$(0,5,2,3,4,1,6,7)$
220	$(x_1+x_3+x_4+x_5, x_2+x_3+x_4+x_5, x_4+x_5)$	$(0,1,6,3,4,5,2,7)$
221	$(x_1+x_2+x_4+x_5, x_1+x_2+x_3+x_4+x_5, x_4+x_5)$	$(0,7,2,5,6,1,4,3)$
222	$(x_1+x_2+x_4+x_5, x_1+x_2+x_3+x_4+x_5, x_4+x_5)$	$(0,7,4,3,6,1,2,5)$
223	$(x_1+x_2+x_4+x_5, x_1+x_2+x_3+x_4+x_5, x_4+x_5)$	$(0,5,2,1,6,3,4,7)$
224	$(x_1+x_2+x_4+x_5, x_1+x_2+x_3+x_4+x_5, x_4+x_5)$	$(0,3,4,1,6,5,2,7)$
225	$(x_1+x_3, x_2+x_3, x_4+x_5)$	$(0,1,2,3,4,5,6,7)$
226	$(x_1+x_3, x_2+x_3, x_4+x_5)$	$(0,1,6,7,4,5,2,3)$
227	$(x_1+x_3, x_2+x_3, x_4+x_5)$	$(0,3,2,5,4,7,6,1)$
228	$(x_1+x_3, x_2+x_3, x_4+x_5)$	$(0,7,6,5,4,3,2,1)$
229	$(x_1+x_3, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,1,3,2,4,5,7,6)$
230	$(x_1+x_3, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,1,7,6,4,5,3,2)$
231	$(x_1+x_3, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,2,3,5,4,6,7,1)$
232	$(x_1+x_3, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,6,7,5,4,2,3,1)$
233	$(x_1+x_3, x_2+x_3, x_1+x_4+x_5)$	$(0,1,2,3,5,4,7,6)$
234	$(x_1+x_3, x_2+x_3, x_1+x_4+x_5)$	$(0,1,7,6,5,4,2,3)$

235	$(x_1 + x_3, x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 3, 2, 4, 5, 6, 7, 1)$
236	$(x_1 + x_3, x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 6, 7, 4, 5, 3, 2, 1)$
237	$(x_1 + x_3, x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 1, 3, 2, 5, 4, 6, 7)$
238	$(x_1 + x_3, x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 1, 6, 7, 5, 4, 3, 2)$
239	$(x_1 + x_3, x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 2, 3, 4, 5, 7, 6, 1)$
240	$(x_1 + x_3, x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 7, 6, 4, 5, 2, 3, 1)$
241	$(x_1 + x_3, x_1 + x_2 + x_3, x_4 + x_5)$	$(0, 1, 2, 3, 6, 7, 4, 5)$
242	$(x_1 + x_3, x_1 + x_2 + x_3, x_4 + x_5)$	$(0, 1, 4, 5, 6, 7, 2, 3)$
243	$(x_1 + x_3, x_1 + x_2 + x_3, x_4 + x_5)$	$(0, 3, 2, 7, 6, 5, 4, 1)$
244	$(x_1 + x_3, x_1 + x_2 + x_3, x_4 + x_5)$	$(0, 5, 4, 7, 6, 3, 2, 1)$
245	$(x_1 + x_3, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 3, 2, 1, 4, 7, 6, 5)$
246	$(x_1 + x_3, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 3, 6, 5, 4, 7, 2, 1)$
247	$(x_1 + x_3, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 1, 2, 7, 4, 5, 6, 3)$
248	$(x_1 + x_3, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 5, 6, 7, 4, 1, 2, 3)$
249	$(x_1 + x_3, x_1 + x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 3, 2, 1, 6, 5, 4, 7)$
250	$(x_1 + x_3, x_1 + x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 3, 4, 7, 6, 5, 2, 1)$
251	$(x_1 + x_3, x_1 + x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 1, 2, 5, 6, 7, 4, 3)$
252	$(x_1 + x_3, x_1 + x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 7, 4, 5, 6, 1, 2, 3)$
253	$(x_1 + x_3, x_1 + x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 1, 3, 2, 6, 7, 5, 4)$
254	$(x_1 + x_3, x_1 + x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 1, 5, 4, 6, 7, 3, 2)$
255	$(x_1 + x_3, x_1 + x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 2, 3, 7, 6, 4, 5, 1)$
256	$(x_1 + x_3, x_1 + x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 4, 5, 7, 6, 2, 3, 1)$
257	$(x_1 + x_3, x_1 + x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 1, 2, 3, 7, 6, 5, 4)$
258	$(x_1 + x_3, x_1 + x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 1, 5, 4, 7, 6, 2, 3)$
259	$(x_1 + x_3, x_1 + x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 3, 2, 6, 7, 4, 5, 1)$
260	$(x_1 + x_3, x_1 + x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 4, 5, 6, 7, 3, 2, 1)$
261	$(x_1 + x_3, x_1 + x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 1, 3, 2, 7, 6, 4, 5)$
262	$(x_1 + x_3, x_1 + x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 1, 4, 5, 7, 6, 3, 2)$
263	$(x_1 + x_3, x_1 + x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 2, 3, 6, 7, 5, 4, 1)$
264	$(x_1 + x_3, x_1 + x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 5, 4, 6, 7, 2, 3, 1)$
265	$(x_1 + x_3, x_2 + x_3 + x_4 + x_5, x_1 + x_4 + x_5)$	$(0, 3, 2, 1, 5, 6, 7, 4)$

266	$(x_1 + x_3, x_2 + x_3 + x_4 + x_5, x_1 + x_4 + x_5)$	$(0, 3, 7, 4, 5, 6, 2, 1)$
267	$(x_1 + x_3, x_2 + x_3 + x_4 + x_5, x_1 + x_4 + x_5)$	$(0, 1, 2, 6, 5, 4, 7, 3)$
268	$(x_1 + x_3, x_2 + x_3 + x_4 + x_5, x_1 + x_4 + x_5)$	$(0, 4, 7, 6, 5, 1, 2, 3)$
269	$(x_1 + x_3, x_1 + x_4 + x_5, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 3, 1, 2, 7, 4, 6, 5)$
270	$(x_1 + x_3, x_1 + x_4 + x_5, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 3, 6, 5, 7, 4, 1, 2)$
271	$(x_1 + x_3, x_1 + x_4 + x_5, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 2, 1, 4, 7, 5, 6, 3)$
272	$(x_1 + x_3, x_1 + x_4 + x_5, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 5, 6, 4, 7, 2, 1, 3)$
273	$(x_1 + x_2, x_2 + x_3, x_4 + x_5)$	$(0, 1, 2, 3, 4, 5, 6, 7)$
274	$(x_1 + x_2, x_2 + x_3, x_4 + x_5)$	$(0, 1, 6, 7, 4, 5, 2, 3)$
275	$(x_1 + x_2, x_2 + x_3, x_4 + x_5)$	$(0, 3, 2, 5, 4, 7, 6, 1)$
276	$(x_1 + x_2, x_2 + x_3, x_4 + x_5)$	$(0, 7, 6, 5, 4, 3, 2, 1)$
277	$(x_1 + x_3 + x_4 + x_5, x_2 + x_3, x_4 + x_5)$	$(0, 5, 2, 7, 4, 1, 6, 3)$
278	$(x_1 + x_3 + x_4 + x_5, x_2 + x_3, x_4 + x_5)$	$(0, 5, 6, 3, 4, 1, 2, 7)$
279	$(x_1 + x_3 + x_4 + x_5, x_2 + x_3, x_4 + x_5)$	$(0, 7, 2, 1, 4, 3, 6, 5)$
280	$(x_1 + x_3 + x_4 + x_5, x_2 + x_3, x_4 + x_5)$	$(0, 3, 6, 1, 4, 7, 2, 5)$
281	$(x_1 + x_2 + x_4 + x_5, x_2 + x_3, x_4 + x_5)$	$(0, 5, 2, 7, 4, 1, 6, 3)$
282	$(x_1 + x_2 + x_4 + x_5, x_2 + x_3, x_4 + x_5)$	$(0, 5, 6, 3, 4, 1, 2, 7)$
283	$(x_1 + x_2 + x_4 + x_5, x_2 + x_3, x_4 + x_5)$	$(0, 7, 2, 1, 4, 3, 6, 5)$
284	$(x_1 + x_2 + x_4 + x_5, x_2 + x_3, x_4 + x_5)$	$(0, 3, 6, 1, 4, 7, 2, 5)$
285	$(x_1 + x_2, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 1, 3, 2, 4, 5, 7, 6)$
286	$(x_1 + x_2, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 1, 7, 6, 4, 5, 3, 2)$
287	$(x_1 + x_2, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 2, 3, 5, 4, 6, 7, 1)$
288	$(x_1 + x_2, x_2 + x_3, x_2 + x_3 + x_4 + x_5)$	$(0, 6, 7, 5, 4, 2, 3, 1)$
289	$(x_1 + x_2, x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 1, 3, 2, 5, 4, 6, 7)$
290	$(x_1 + x_2, x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 1, 6, 7, 5, 4, 3, 2)$
291	$(x_1 + x_2, x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 2, 3, 4, 5, 7, 6, 1)$
292	$(x_1 + x_2, x_2 + x_3, x_1 + x_4 + x_5)$	$(0, 7, 6, 4, 5, 2, 3, 1)$
293	$(x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 1, 2, 3, 5, 4, 7, 6)$
294	$(x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 1, 7, 6, 5, 4, 2, 3)$
295	$(x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 3, 2, 4, 5, 6, 7, 1)$
296	$(x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4 + x_5)$	$(0, 6, 7, 4, 5, 3, 2, 1)$

297	$(x_1+x_3+x_4+x_5, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,5,7,2,4,1,3,6)$
298	$(x_1+x_3+x_4+x_5, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,5,3,6,4,1,7,2)$
299	$(x_1+x_3+x_4+x_5, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,2,7,1,4,6,3,5)$
300	$(x_1+x_3+x_4+x_5, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,6,3,1,4,2,7,5)$
301	$(x_1+x_2+x_4+x_5, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,5,2,7,4,1,6,3)$
302	$(x_1+x_2+x_4+x_5, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,5,6,3,4,1,2,7)$
303	$(x_1+x_2+x_4+x_5, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,7,2,1,4,3,6,5)$
304	$(x_1+x_2+x_4+x_5, x_2+x_3, x_2+x_3+x_4+x_5)$	$(0,3,6,1,4,7,2,5)$
305	$(x_1+x_2, x_2+x_3+x_4+x_5, x_4+x_5)$	$(0,3,2,1,4,7,6,5)$
306	$(x_1+x_2, x_2+x_3+x_4+x_5, x_4+x_5)$	$(0,3,6,5,4,7,2,1)$
307	$(x_1+x_2, x_2+x_3+x_4+x_5, x_4+x_5)$	$(0,1,2,7,4,5,6,3)$
308	$(x_1+x_2, x_2+x_3+x_4+x_5, x_4+x_5)$	$(0,5,6,7,4,1,2,3)$
309	$(x_1+x_2, x_1+x_4+x_5, x_4+x_5)$	$(0,3,2,1,6,5,4,7)$
310	$(x_1+x_2, x_1+x_4+x_5, x_4+x_5)$	$(0,3,4,7,6,5,2,1)$
311	$(x_1+x_2, x_1+x_4+x_5, x_4+x_5)$	$(0,1,2,5,6,7,4,3)$
312	$(x_1+x_2, x_1+x_4+x_5, x_4+x_5)$	$(0,7,4,5,6,1,2,3)$
313	$(x_1+x_2, x_1+x_2+x_3+x_4+x_5, x_2+x_3+x_4+x_5)$	$(0,3,2,1,5,6,7,4)$
314	$(x_1+x_2, x_1+x_2+x_3+x_4+x_5, x_2+x_3+x_4+x_5)$	$(0,3,7,4,5,6,2,1)$
315	$(x_1+x_2, x_1+x_2+x_3+x_4+x_5, x_2+x_3+x_4+x_5)$	$(0,1,2,6,5,4,7,3)$
316	$(x_1+x_2, x_1+x_2+x_3+x_4+x_5, x_2+x_3+x_4+x_5)$	$(0,4,7,6,5,1,2,3)$
317	$(x_1+x_2, x_1+x_2+x_3+x_4+x_5, x_1+x_4+x_5)$	$(0,3,1,2,7,4,6,5)$
318	$(x_1+x_2, x_1+x_2+x_3+x_4+x_5, x_1+x_4+x_5)$	$(0,3,6,5,7,4,1,2)$
319	$(x_1+x_2, x_1+x_2+x_3+x_4+x_5, x_1+x_4+x_5)$	$(0,2,1,4,7,5,6,3)$
320	$(x_1+x_2, x_1+x_2+x_3+x_4+x_5, x_1+x_4+x_5)$	$(0,5,6,4,7,2,1,3)$
321	$(x_1+x_2+x_4+x_5, x_2+x_3+x_4+x_5, x_1+x_4+x_5)$	$(0,7,6,1,5,2,3,4)$
322	$(x_1+x_2+x_4+x_5, x_2+x_3+x_4+x_5, x_1+x_4+x_5)$	$(0,7,3,4,5,2,6,1)$
323	$(x_1+x_2+x_4+x_5, x_2+x_3+x_4+x_5, x_1+x_4+x_5)$	$(0,1,6,2,5,4,3,7)$
324	$(x_1+x_2+x_4+x_5, x_2+x_3+x_4+x_5, x_1+x_4+x_5)$	$(0,4,3,2,5,1,6,7)$
325	$(x_1+x_3+x_4+x_5, x_2+x_3+x_4+x_5, x_4+x_5)$	$(0,7,2,5,4,3,6,1)$
326	$(x_1+x_3+x_4+x_5, x_2+x_3+x_4+x_5, x_4+x_5)$	$(0,7,6,1,4,3,2,5)$
327	$(x_1+x_3+x_4+x_5, x_2+x_3+x_4+x_5, x_4+x_5)$	$(0,5,2,3,4,1,6,7)$

328	$(x_1 + x_3 + x_4 + x_5, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 1, 6, 3, 4, 5, 2, 7)$
329	$(x_1 + x_2 + x_4 + x_5, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 7, 2, 5, 4, 3, 6, 1)$
330	$(x_1 + x_2 + x_4 + x_5, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 7, 6, 1, 4, 3, 2, 5)$
331	$(x_1 + x_2 + x_4 + x_5, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 5, 2, 3, 4, 1, 6, 7)$
332	$(x_1 + x_2 + x_4 + x_5, x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 1, 6, 3, 4, 5, 2, 7)$
333	$(x_1 + x_3 + x_4 + x_5, x_1 + x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 7, 2, 5, 6, 1, 4, 3)$
334	$(x_1 + x_3 + x_4 + x_5, x_1 + x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 7, 4, 3, 6, 1, 2, 5)$
335	$(x_1 + x_3 + x_4 + x_5, x_1 + x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 5, 2, 1, 6, 3, 4, 7)$
336	$(x_1 + x_3 + x_4 + x_5, x_1 + x_2 + x_3 + x_4 + x_5, x_4 + x_5)$	$(0, 3, 4, 1, 6, 5, 2, 7)$

Table A.1: Optimal $(\mathcal{C}, \mathcal{M})$ pairs for Example 2.1.

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