

High-rate, Multi-Symbol-Decodable STBCs from Clifford Algebras

Sanjay Karmakar

Beceem Communications Pvt. Ltd.,
Bangalore.
skarmakar@beceem.com

B. Sundar Rajan

Department of ECE
Indian Institute of Science, Bangalore
bsrajan@ece.iisc.ernet.in

Abstract

It is well known that Space-Time Block Codes (STBCs) obtained from Orthogonal Designs (ODs) are single-symbol-decodable (SSD) and from Quasi-Orthogonal Designs (QODs) are double-symbol decodable. However, there are SSD codes that are not obtainable from ODs and DSD codes that are not obtainable from QODs. In this paper a method of constructing g -symbol decodable (g -SD) STBCs using representations of Clifford algebras are presented which when specialized to $g = 1, 2$ gives SSD and DSD codes respectively. For the number of transmit antennas 2^a the rate (in complex symbols per channel use) of the g -SD codes presented in this paper is $\frac{a+1-g}{2^{a-g}}$. The maximum rate of the DSD STBCs from QODs reported in the literature is $\frac{a}{2^{a-1}}$ which is smaller than the rate $\frac{a-1}{2^{a-2}}$ of the DSD codes of this paper, for 2^a transmit antennas. In particular, the reported DSD codes for 8 and 16 transmit antennas offer rates 1 and 3/4 respectively whereas the known STBCs from QODs offer only 3/4 and 1/2 respectively. The construction of this paper is applicable for any number of transmit antennas.

1. Introduction and Preliminaries

We consider a multiple antenna transmission system with N_t number of transmit antennas and N_r number of receive antennas. At each time slot t , the complex signals, s_{ti} , $i = 0, 1, \dots, N_t - 1$ are transmitted from the N_t transmit antennas simultaneously. Let $h_{ij} = \alpha_{ij} e^{j\theta_{ij}}$ denote the path gain from the transmit antenna i to the receive antenna j , where $\mathbf{j} = \sqrt{-1}$. Assuming that the path gain are constant over a frame length N_t (we consider only square designs), the received signal y_{tj} at the receive antenna j at time t is given by,

$$y_{tj} = \sum_{i=0}^{N_t-1} s_{ti} h_{ij} + n_{tj}, \quad (1)$$

for $j = 0, \dots, N_r - 1$, $t = 0, \dots, N_t - 1$, which in matrix notation is,

$$\mathbf{Y} = \mathbf{S}\mathbf{H} + \mathbf{N} \quad (2)$$

where $\mathbf{Y} \in \mathbb{C}^{N_t \times N_r}$ is the received signal matrix, $\mathbf{S} \in \mathbb{C}^{N_t \times N_t}$ is the transmission matrix (also referred as codeword matrix), $\mathbf{N} \in \mathbb{C}^{N_t \times N_r}$ is the additive noise matrix

and $\mathbf{H} \in \mathbb{C}^{N_t \times N_r}$ is the channel matrix, where \mathbb{C} denotes the complex field. The entries of \mathbf{H} are complex Gaussian with zero mean and unit variance and the entries of \mathbf{N} are complex Gaussian with zero mean and variance σ^2 . Both are assumed to be temporally and spatially white. We further assume that transmission power constraint is given by $E[\text{tr}\{\mathbf{S}\mathbf{S}^H\}] = N_t^2$.

An $n \times n$ linear dispersion STBC [1] with gK complex variables x_1, x_2, \dots, x_{gK} , where g and K are positive integers, can be written as

$$\mathbf{S} = \sum_{i=1}^K S_i \quad (3)$$

where,

$$S_i = \sum_{j=1}^g x_{(g(i-1)+j),I} A_{(g(i-1)+j),I} + x_{(g(i-1)+j),Q} A_{(g(i-1)+j),Q} \quad (4)$$

and $x_l = x_{l,I} + jx_{l,Q} \in \mathcal{A}_l \subset \mathbb{C}$ for $1 \leq l \leq gK$ where \mathcal{A}_l is the signal constellation from which the variable x_l takes values. The set of gK number of complex $n \times n$ matrices A_j , $1 \leq j \leq gK$ are called the weight matrices of the code and this set defines the code \mathbf{S} . With $|\mathcal{A}_i|$ denoting the number of points in the constellation, the rate of this code in bits per channel use is $R = \frac{1}{n} \sum_{i=1}^{gK} \log_2(|\mathcal{A}_i|)$. Now assuming that perfect channel state information (CSI) is available at the receiver, the maximum likelihood (ML) decision rule minimizes the metric,

$$\mathbf{M}(\mathbf{S}) \triangleq \min_{\mathbf{S}} \text{tr}((\mathbf{Y} - \mathbf{S}\mathbf{H})^H (\mathbf{Y} - \mathbf{S}\mathbf{H})) = \|\mathbf{Y} - \mathbf{S}\mathbf{H}\|^2. \quad (5)$$

It is clear that there are $\prod_{i=1}^{gK} |\mathcal{A}_i|$ different codewords and, in general, the ML decoding requires $\prod_{i=1}^{gK} |\mathcal{A}_i|$ computations, one for each codeword. But if the set of weight matrices are chosen such that the decoding metric (5) could be decomposed into,

$$\mathbf{M}(\mathbf{S}) = \sum_{i=1}^K f_i(x_{(i-1)g+1}, x_{(i-1)g+2}, \dots, x_{(i-1)g+g})$$

a sum of K positive terms, each involving exactly g complex variables only, then the decoding requires

$\sum_{i=1}^K \{\prod_{j=1}^g |\mathcal{A}_{j+(i-1)g}|\}$ computations and the code is called a *g-symbol decodable code (g-SD code)*. The case $g = 1$ corresponds to Single-Symbol Decodable (SSD) codes that includes the well known codes from Orthogonal Designs (ODs) as a proper subclass, and have been extensively studied [2]-[13], few most recent ones being [12]-[13]. The codes corresponding to $g = 2$, are called *Double-Symbol-Decodable (DSD) codes*. The Quasi-Orthogonal Designs studied in [14]-[17] and [6] are proper subclass of DSD codes. If the weight matrices A_j , $1 \leq j \leq 2K$ of a g -SD code are all unitary then it is said to be a Unitary Weight DSD (UW- g -SD) code. The DSD codes of [14]-[17] are UW-DSD codes and that of [6] is not. Throughout this paper, we consider only UW codes.

The contributions of this paper are

- We obtain a set of sufficient conditions for a general linear dispersion STBC to be g -SD in terms of their weight matrices. Also, another set of sufficient conditions for the code to be g -SD is given which enables us to construct UW- g -SD codes from representations of Clifford algebras.
- For the number of transmit antennas 2^a the maximum rate (in complex symbols per channel use) of all the DSD codes reported in the literature with unitary weight matrices is $\frac{a}{2^{a-1}}$. Whereas we present UW-DSD codes with rate $\frac{a-1}{2^{a-2}}$ for 2^a transmit antennas. In particular, our code for 8 and 16 transmit antennas offer rates 1 and 3/4 respectively, whereas the known QODs offer only 3/4 and 1/2 respectively. The rate of our g -SD codes is $\frac{a+1-g}{2^{a-g}}$.

2. Sufficient conditions for Double-Symbol-Decodability

We begin with presenting a sufficient condition on the set of weight matrices of the code to be g -SD.

Lemma 1 *The linear dispersion STBC given by (3) is a g -SD code if,*

$$S_l^H S_j + S_j^H S_l = 0, \quad \forall 1 \leq l \neq j \leq K. \quad (6)$$

Proof: We see from (3) that,

$$\mathbf{S}^H \mathbf{S} = \left(\sum_{i=1}^K S_i^H \right) \left(\sum_{i=1}^K S_i \right). \quad (7)$$

If the conditions of (6) are satisfied, then it is easy to verify that,

$$\mathbf{S}^H \mathbf{S} = \sum_{i=1}^K S_i^H S_i. \quad (8)$$

Using (8) in (5) we get,

$$\begin{aligned} \mathbf{M}(\mathbf{S}) &= \text{Tr} \left[\sum_{i=1}^K (\mathbf{Y} - S_i \mathbf{H})^H (\mathbf{Y} - S_i \mathbf{H}) - (K-1) \mathbf{Y}^H \mathbf{Y} \right] \\ &= \sum_{i=1}^K \text{Tr} \left[(\mathbf{Y} - S_i \mathbf{H})^H (\mathbf{Y} - S_i \mathbf{H}) \right] + \mathbf{M}_c \end{aligned} \quad (9)$$

where $\mathbf{M}_c = \text{Tr} \left[-(K-1) \mathbf{Y}^H \mathbf{Y} \right]$. Note that in (9) the \mathbf{M}_c term is same for all the codewords in the code book. Hence it is sufficient to minimize the first term only. But the first term,

$$\widetilde{\mathbf{M}}(\mathbf{S}) = \sum_{i=1}^K \text{Tr} \left[(\mathbf{Y} - S_i \mathbf{H})^H (\mathbf{Y} - S_i \mathbf{H}) \right]$$

is a sum of K square terms, each involving only g complex variables. Hence the problem of ML decoding reduces to the problem of minimizing,

$$\text{Tr} \left[(\mathbf{Y} - S_i \mathbf{H})^H (\mathbf{Y} - S_i \mathbf{H}) \right] \quad \forall 1 \leq i \leq K. \quad (10)$$

Hence the code is g -SD if the conditions of (6) is satisfied. This completes the proof.

Next we derive a condition on weight matrices of the code so that (6) is satisfied. Towards this end, we denote,

$$\beta_i = \left\{ A_{g(i-1)+j,I}, A_{g(i-1)+j,Q} \right\}_{j=1}^g, \quad 1 \leq i \leq K. \quad (11)$$

A straight forward verification shows that

Lemma 2 *Conditions of (6) is satisfied if the weight matrices of the code (3) satisfy the following condition,*

$$A^H B + B^H A = 0, \quad \forall A \in \beta_i, B \in \beta_j, \text{ for } i \neq j. \quad (12)$$

We first introduce the notion of *normalizing a linear STBC* which not only simplifies the analysis of the codes but also provides deep insight into various aspects of different classes of codes discussed in this paper. Towards this end, let

$$S_U = \sum_{k=1}^{gK} x_{k,I} A'_{k,I} + x_{k,Q} A'_{k,Q} \quad (13)$$

be a UW- g -SD code. We normalize the weight matrices of the code as

$$\begin{aligned} A_{kI} &= A_{1I}^H A'_{kI} \\ A_{kQ} &= A_{1I}^H A'_{kQ}. \end{aligned} \quad \forall 1 \leq k \leq gK \quad (14)$$

to get the *normalized version* of (13) to be

$$S_N = \sum_{k=1}^{gK} x_{k,I} A_{k,I} + x_{k,Q} A_{k,Q} \quad (15)$$

where $A_{1,I} = I_n$, the $n \times n$ identity matrix. We call the code S_N to be the normalized code of S_U .

Lemma 3 The code S_U is g -SD iff S_N is g -SD. In other words normalization does not affect the DSD property.

Proof: For $1 \leq i_1 \neq i_2 \leq K$, the equation of Lemma 2 is satisfied by the weight matrices of S_U iff they are satisfied by the weight matrices of S_N which is easily verified.

The following theorem identifies a set of sufficient conditions for a UW code to be UW- g -SD.

Theorem 1 A $n \times n$ UW code described by (13) and its normalized version given by (15) are UW- g -SD codes if the weight matrices of the normalized code satisfy the following conditions:

$$A_{g(i-1)+1,I}^H = -A_{g(i-1)+1,I}, \quad 2 \leq i \leq K \quad (16)$$

$$A_{g(i-1)+1,I} \text{ and } A_{g(j-1)+1,I} \text{ anticommute} \\ \forall 2 \leq i \neq j \leq K \quad (17)$$

$$A_{1,Q}, A_{2,I}, A_{2,Q}, \dots, A_{g,I}, A_{g,Q} \text{ are} \\ \text{Hermitian, commute among themselves and} \\ \text{with all the matrices } A_{g(i-1)+1,I}, \quad 1 \leq i \leq K. \quad (18)$$

For all $1 \leq i \leq K$, and $2 \leq j \leq g$;

$$A_{g(i-1)+1,Q} = \pm A_{1,Q} A_{g(i-1)+1,I}, \quad (19)$$

$$A_{g(i-1)+1,I} = \pm A_{j,I} A_{g(i-1)+1,I}, \quad (20)$$

$$A_{g(i-1)+1,Q} = \pm A_{j,Q} A_{g(i-1)+1,I}, \quad (21)$$

Proof: Using the conditions (19), (20) and (21) of the theorem the sets (11) for $1 \leq i_1 \neq i_2 \leq K$ are

$$\beta_{i_1} = \left\{ \pm A_{j,I} A_{g(i_1)+1,I}, \pm A_{j,Q} A_{g(i_1)+1,Q} \right\}_{j=1}^g, \quad (22)$$

$$\beta_{i_2} = \left\{ \pm A_{j,I} A_{g(i_2)+1,I}, \pm A_{j,Q} A_{g(i_2)+1,Q} \right\}_{j=1}^g. \quad (23)$$

Let $x A_{g(i_1)+1,I} \in \beta_{i_1}$, $y A_{g(i_2)+1,I} \in \beta_{i_2}$ where $x, y \in \{I, \pm A_{1,Q}, \pm A_{2,I}, \pm A_{2,Q}, \dots, \pm A_{(g-1),I}, \pm A_{(g-1),Q}\}$. Then

$$\begin{aligned} & x A_{g(i_1)+1,I}^H y A_{g(i_2)+1,I} + y A_{g(i_2)+1,I}^H x A_{g(i_1)+1,I} \\ &= A_{g i_1+1,I}^H x^H y A_{g i_2+1,I} + A_{g i_2+1,I}^H y^H x A_{g i_1+1,I} \\ &= A_{g i_1+1,I}^H x y A_{g i_2+1,I} + A_{g i_2+1,I}^H y x A_{g i_1+1,I} \\ &= A_{g i_1+1,I}^H x y A_{g i_2+1,I} + A_{g i_2+1,I}^H x y A_{g i_1+1,I} \\ &= x A_{g i_1+1,I}^H A_{g i_2+1,I} y + x A_{g i_2+1,I}^H A_{g i_1+1,I} y \\ &= x \left[A_{g i_1+1,I}^H A_{g i_2+1,I} + A_{g i_2+1,I}^H A_{g i_1+1,I} \right] y \\ &= x \left[0 \right] y = 0. \end{aligned}$$

This completes the proof.

The requirements of Theorem 1 for UW- g SD can be easily stated using the Table 2 shown at the top of the next page as follows:

- The matrices of the first row should form a Hurwitz-Radon family of matrices
- The matrices of the first column should be

- Hermitian
- mutually commuting and
- commute with all the matrices of the first row

Definition 1 A UW-DSD code satisfying the conditions of (16) is defined to be a Clifford Unitary Weight DSD (CUW-DSD) codes.

The name in the above definition is due to the fact that such codes are constructable using matrix representations of real Clifford algebras as shown in the following section.

3. Construction of CUW- g -SD codes

In this section we present the construction of a new class of $2^a \times 2^a$ g -SD codes, the Clifford UW- g -SD (CUW- g -SD) codes, for 2^a transmit antennas, from unitary matrix representations of real Clifford algebras. For an excellent introduction to and basic properties of these representations see [4].

Definition 2 The Clifford algebra, denoted by CA_L , is the algebra over the real field \mathbb{R} generated by L objects γ_k , $k = 1, 2, \dots, L$ which are anti-commuting, ($\gamma_k \gamma_j = -\gamma_j \gamma_k$, $\forall k \neq j$), and squaring to -1 , ($\gamma_k^2 = -1 \quad \forall k = 1, 2, \dots, L$).

A matrix representation of an algebra is completely specified by a representation of its generators. For a Clifford algebra, we are thus interested in unitary matrix representation of the generators γ_k 's. Let

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & j \\ j & 0 \end{bmatrix} \text{ and } \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (24)$$

$$\text{and } A^{\otimes m} = \underbrace{A \otimes A \otimes A \cdots \otimes A}_{m \text{ times}}$$

From [4] we know that the representation of the generators of CA_{2a+1} is given by

$$\begin{aligned} R(\gamma_1) &= \pm j \sigma_3^{\otimes a} \\ R(\gamma_2) &= I_2^{\otimes a-1} \otimes \sigma_1 \\ R(\gamma_3) &= I_2^{\otimes a-1} \otimes \sigma_2 \\ &\vdots \\ &\vdots \\ &\vdots \\ R(\gamma_{2k}) &= I_2^{\otimes a-k} \otimes \sigma_1 \otimes \sigma_3^{\otimes k-1} \\ R(\gamma_{2k+1}) &= I_2^{\otimes a-k} \otimes \sigma_2 \otimes \sigma_3^{\otimes k-1} \\ &\vdots \\ &\vdots \\ &\vdots \\ R(\gamma_{2a}) &= \sigma_1 \otimes \sigma_3^{\otimes a-1} \\ R(\gamma_{2a+1}) &= \sigma_2 \otimes \sigma_3^{\otimes a-1}. \end{aligned} \quad (25)$$

We add to this list the $2^a \times 2^a$ identity matrix, denoted by I_{2^a} , and designate it as $R(\gamma_0) = I_{2^a}$.

$I = A_{1I}$	$A_{g+1,I}$	$A_{2g+1,I}$	\cdot	\cdot	\cdot	$A_{ig+1,I}$	\cdot	\cdot	\cdot	$A_{(K-1)g+1,I}$
A_{1Q}	$A_{1Q}A_{g+1,I}$	$A_{1Q}A_{2g+1,I}$	\cdot	\cdot	\cdot	$A_{1Q}A_{ig+1,I}$	\cdot	\cdot	\cdot	$A_{1Q}A_{(K-1)g+1,I}$
A_{2I}	$A_{2I}A_{g+1,I}$	$A_{2I}A_{2g+1,I}$	\cdot	\cdot	\cdot	$A_{2I}A_{ig+1,I}$	\cdot	\cdot	\cdot	$A_{2I}A_{(K-1)g+1,I}$
A_{2Q}	$A_{2Q}A_{g+1,I}$	$A_{2Q}A_{2g+1,I}$	\cdot	\cdot	\cdot	$A_{2Q}A_{ig+1,I}$	\cdot	\cdot	\cdot	$A_{2Q}A_{(K-1)g+1,I}$
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot
A_{gI}	$A_{gI}A_{g+1,I}$	$A_{gI}A_{2g+1,I}$	\cdot	\cdot	\cdot	$A_{gI}A_{ig+1,I}$	\cdot	\cdot	\cdot	$A_{gI}A_{(K-1)g+1,I}$
A_{gQ}	$A_{gQ}A_{g+1,I}$	$A_{gQ}A_{2g+1,I}$	\cdot	\cdot	\cdot	$A_{gQ}A_{ig+1,I}$	\cdot	\cdot	\cdot	$A_{gQ}A_{(K-1)g+1,I}$

3.1. Construction of CUW-g-SD codes

To construct CUW-g-SD codes from the last $2g$ matrices $\{R(\gamma_{2(a+1-g)}), R(\gamma_{2(a+1-g)+1}), \dots, R(\gamma_{2a}), R(\gamma_{2a+1})\}$ we construct the following $2g - 1$ new matrices,

$$\begin{aligned}
\alpha_1 &= \pm j R(\gamma_{2a-2}) R(\gamma_{2a-1}) \\
\alpha_2 &= \pm j R(\gamma_{2a}) R(\gamma_{2a+1}) \\
&\vdots \\
\alpha_g &= \pm j R(\gamma_0) R(\gamma_1) \\
\alpha_{g+1} &= \pm \alpha_1 \alpha_2 \\
\alpha_{g+2} &= \pm \alpha_3 \alpha_4 \\
&\vdots \\
\alpha_{g+\frac{g}{2}} &= \pm \alpha_{g-1} \alpha_g \\
\alpha_{g+\frac{g}{2}+1} &= \pm \alpha_{g+1} \alpha_{g+2} \\
\alpha_{g+\frac{g}{2}+2} &= \pm \alpha_{g+3} \alpha_{g+4} \\
&\vdots \\
\alpha_{g+\frac{g}{2}+\frac{g}{4}} &= \pm \alpha_{g+\frac{g}{2}-1} \alpha_{g+\frac{g}{2}} \\
&\vdots \\
\alpha_{2g-1} &= \prod_{i=1}^g \alpha_g.
\end{aligned} \tag{26}$$

Note that the set of matrices $\{\alpha_i\}_{i=1}^{2g-1}$ have the following properties: (i) They are mutually commuting, (ii) Hermitian and (iii) each commutes with $R(\gamma_1), R(\gamma_2), \dots, R(\gamma_{2a-2g+1})$. In the above construction of α_i s there is nothing special about the last $2g$ matrices. Any $2g$ from the set $\{R(\gamma_1), R(\gamma_2), \dots, R(\gamma_{2a+1})\}$ could have been selected. Next, for $1 \leq i \leq 2a - 2g + 2$, we construct the weight matrices of the code combining the matrices defined above in the following way,

$$\left. \begin{aligned}
A_{g(i-1)+1,I} &= R(\gamma_{i-1}) \\
A_{g(i-1)+2,I} &= \pm \alpha_1 R(\gamma_{i-1}) \\
&\vdots \\
A_{g(i-1)+g,I} &= \pm \alpha_{g-1} R(\gamma_{i-1}) \\
A_{g(i-1)+1,Q} &= \pm \alpha_g R(\gamma_{i-1}) \\
A_{g(i-1)+2,Q} &= \pm \alpha_{g+1} R(\gamma_{i-1}) \\
&\vdots \\
A_{g(i-1)+g,Q} &= \pm \alpha_{2g-1} R(\gamma_{i-1})
\end{aligned} \right\} \tag{27}$$

Theorem 2 *The $2^a \times 2^a$ code with the weight matrices given by (27) is a CUW-g-SD code.*

Proof It is easily checked that the weight matrices satisfy the conditions of Theorem 1 and hence the code is CUW-g-SD.

Corollary 1 *The rate of the $2^a \times 2^a$ CUW-g-SD codes of Theorem 2 is $\frac{a+1-g}{2^a-g}$ complex symbols per channel use.*

Example 1 *Let the representation matrices of the $C_{A_{2a+1}}$, where $a = 3$ be,*

$$\begin{aligned}
R(\gamma_0) &= I_2 \otimes I_2 \times I_2, R(\gamma_1) = I_2 \otimes I_2 \times \sigma_1, \\
R(\gamma_2) &= I_2 \otimes I_2 \times \sigma_2, R(\gamma_3) = I_2 \otimes \sigma_1 \times \sigma_3, \\
R(\gamma_4) &= I_2 \otimes \sigma_2 \times \sigma_3, R(\gamma_5) = \sigma_1 \otimes \sigma_3 \times \sigma_3 \\
R(\gamma_6) &= \sigma_2 \otimes \sigma_3 \times \sigma_3, R(\gamma_7) = j\sigma_3 \otimes \sigma_3 \times \sigma_3
\end{aligned}$$

Now according to the prescription of the construction procedure,

$$\begin{aligned}
\alpha_1 &= j R(\gamma_4) R(\gamma_5) = \sigma_1 \otimes \sigma_1 \times I_2 \\
\alpha_2 &= j R(\gamma_6) R(\gamma_7) = j\sigma_1 \otimes I_2 \times I_2 \\
\alpha_3 &= R(\gamma_4) R(\gamma_5) R(\gamma_6) R(\gamma_7) = jI_2 \otimes \sigma_1 \times I_2
\end{aligned}$$

and the weight matrices given by (28) at the top of the next page. Now if we construct the code, we get the code given by (29) at the top of the next page.

Note that for 8 transmit antennas our CUW-DSD code achieves rate 1 whereas all known QODs with unitary weight matrices for 8 transmit antennas achieve rate only $\frac{3}{4}$. However, the 8 transmit antenna DSD code with non-unitary weight matrices of [6] achieve rate 1, but has larger Peak-to-Average Power Ratio due to the presence of zero entries in the code, compared to our CUW-DSD codes.

4. Acknowledgements

The authors would like to thank G.Susinder Rajan for useful discussions on the subject of this paper. This work was partly supported by the DRDO-IISc Program on Advanced Research in Mathematical Engineering and by the Council of Scientific & Industrial Research (CSIR), India, through Research Grant (22(0365)/04/EMR-II) to B.S. Rajan.

$$\begin{aligned}
A_{1I} &= R(\gamma_0) = I_2 \otimes I_2 \times I_2 & A_{3I} &= R(\gamma_1) = I_2 \otimes I_2 \times \sigma_1 & A_{5I} &= R(\gamma_2) = I_2 \otimes I_2 \times \sigma_2 & A_{7I} &= R(\gamma_3) = I_2 \otimes \sigma_1 \times \sigma_3 \\
A_{1Q} &= \alpha_1 A_{1I} = \sigma_1 \otimes \sigma_1 \times I_2 & A_{3Q} &= \alpha_1 A_{3I} = \sigma_1 \otimes \sigma_1 \times \sigma_1 & A_{5Q} &= \alpha_1 A_{5I} = \sigma_1 \otimes \sigma_1 \times \sigma_2 & A_{7Q} &= \alpha_1 A_{7I} = -\sigma_1 \otimes I_2 \times \sigma_3 \\
A_{2I} &= \alpha_2 R(\gamma_0) = j\sigma_1 \otimes I_2 \times I_2 & A_{4I} &= \alpha_2 R(\gamma_1) = j\sigma_1 \otimes I_2 \times \sigma_1 & A_{6I} &= \alpha_2 R(\gamma_2) = j\sigma_1 \otimes I_2 \times \sigma_2 & A_{8I} &= \alpha_2 R(\gamma_3) = j\sigma_1 \otimes \sigma_1 \times \sigma_3 \\
A_{2Q} &= \alpha_3 R(\gamma_0) = jI_2 \otimes \sigma_1 \times I_2 & A_{4Q} &= \alpha_3 R(\gamma_1) = jI_2 \otimes \sigma_1 \times \sigma_1 & A_{6Q} &= \alpha_3 R(\gamma_2) = jI_2 \otimes \sigma_1 \times \sigma_2 & A_{8Q} &= \alpha_3 R(\gamma_3) = -jI_2 \otimes I_2 \times \sigma_3
\end{aligned} \tag{28}$$

$$\begin{bmatrix}
x_{1I} - jx_{8Q} & x_{3I} + jx_{5I} & x_{7I} + jx_{2Q} & -x_{6Q} + jx_{4Q} & -x_{7Q} + jx_{2I} & -x_{6I} + jx_{4I} & x_{1Q} + jx_{8I} & x_{3Q} + jx_{5Q} \\
-x_{3I} + jx_{5I} & x_{1I} + jx_{8Q} & -x_{6Q} - jx_{4Q} & -x_{7I} + jx_{2Q} & -x_{6I} - jx_{4I} & x_{7Q} + jx_{2I} & -x_{3Q} + jx_{5Q} & x_{1Q} - jx_{8I} \\
-x_{7I} - jx_{2Q} & x_{6Q} - jx_{4Q} & x_{1I} - jx_{8Q} & x_{3I} + jx_{5I} & -x_{1Q} - jx_{8I} & -x_{3Q} - jx_{5Q} & -x_{7Q} + jx_{2I} & -x_{6I} + jx_{4I} \\
x_{6Q} + jx_{4Q} & x_{7I} - jx_{2Q} & -x_{3I} + jx_{5I} & x_{1I} + jx_{8Q} & x_{3Q} - jx_{5Q} & -x_{1Q} + jx_{8I} & -x_{6I} - jx_{4I} & x_{7Q} + jx_{2I} \\
x_{7Q} - jx_{2I} & x_{6I} - jx_{4I} & -x_{1Q} - jx_{8I} & -x_{3Q} - jx_{5Q} & x_{1I} - jx_{8Q} & x_{3I} + jx_{5I} & x_{7I} + jx_{2Q} & -x_{6Q} + jx_{4Q} \\
x_{6I} + jx_{4I} & -x_{7Q} - jx_{2I} & x_{3Q} - jx_{5Q} & -x_{1Q} + jx_{8I} & -x_{3I} + jx_{5I} & x_{1I} + jx_{8Q} & -x_{6Q} - jx_{4Q} & -x_{7I} + jx_{2Q} \\
x_{1Q} + jx_{8I} & x_{3Q} + jx_{5Q} & x_{7Q} - jx_{2I} & x_{6I} - jx_{4I} & -x_{7I} - jx_{2Q} & x_{6Q} - jx_{4Q} & x_{1I} - jx_{8Q} & x_{3I} + jx_{5I} \\
-x_{3Q} + jx_{5Q} & x_{1Q} - jx_{8I} & x_{6I} + jx_{4I} & -x_{7Q} - jx_{2I} & x_{6Q} + jx_{4Q} & -x_{7I} - jx_{2Q} & -x_{3I} + jx_{5I} & x_{1I} + jx_{8Q}
\end{bmatrix} \tag{29}$$

5. References

- [1] B. Hassibi and B. Hochwald, "High-rate codes that are linear in space and time," *IEEE Trans. Inform. Theory*, vol.48, no.7, pp.1804-1824, July 2002.
- [2] S.M. Alamouti, "A simple transmitter diversity scheme for wireless communications," *IEEE J. Select Areas Comm.*, vol 16 pp. 1451-1458, Oct. 1998.
- [3] V. Tarokh, H. Jafarkhani and A. R. Calderbank, "Space-Time block codes from orthogonal designs," *IEEE Trans. Inform. Theory*, vol.45, pp.1456-1467, July 1999.
- [4] O. Tirkkonen and A. Hottinen, "Square-matrix embeddable space-time block codes for complex signal constellations," *IEEE Trans. Inform. Theory*, vol.48, no.2, Feb. 2002, pp.384-395.
- [5] G. Ganesan and P. Stoica, "Space-time diversity using orthogonal and amicable orthogonal designs," in *Proc. IEEE Int. Conf. Acoustics, Speech and Signal Processing (ICASSP 2000)*, Istanbul, Turkey, 2000, pp. 2561-2564.
- [6] Md. Zafar Ali Khan, B. Sundar Rajan and M.H.Lee, "On Single-Symbol and Double-Symbol decodable STBCs," *Proc. ISIT 2003*, Yokohama, Japan, June 29-July 4, 2003, p.127.
- [7] Md. Zafar Ali Khan and B. Sundar Rajan, "Single-Symbol Maximum-Likelihood Decodable Linear STBCs," *IEEE Transactions on Information Theory*, vol. 52, no.5, May 2006, pp.2062-2091.
- [8] H.Wang, D.Wang and X-G.Xia, "On optimal QOSTBC with minimal decoding complexity," Submitted to *IEEE transactions on Information Theory*.
- [9] H.Wang, D.Wang and X-G.Xia, "On Optimal Quasi-orthogonal space-time block codes with minimum decoding complexity," *Proc. ISIT 2005*, Adelaide, Nov. 2005, pp.1168-1172.
- [10] G.Yuen, Y.L.Guan and T.T.Tjhung, "Construction of quasi-orthogonal STBC with minimum decoding complexity," *Proc. ISIT 2004*, Chicago, June/July 2004, p.308.
- [11] G.Yuen, Y.L.Guan and T.T.Tjhung, "Optimizing quasi-orthogonal STBC with group-constrained linear transformation," *Proc. GLOBECOM 2004*, Dallas, Texas, Dec. 2004, pp.550-554.
- [12] Sanjay Karmakar and B. Sundar Rajan, "Minimum-decoding-complexity maximum-rate space-time block codes from Clifford algebras," *Proc. ISIT 2006*, Seattle, USA, July 09-14, 2006, pp.788-792.
- [13] Sanjay Karmakar and B. Sundar Rajan, "Non-unitary weight space-time block codes with minimum decoding complexity," *Proc. ISIT 2006*, Seattle, USA, July 09-14, 2006, pp.793-797.
- [14] Jafarkhani, H., "A quasi-orthogonal space-time block code," *IEEE Transactions on Communications*, Volume 49, Issue 1, Jan. 2001, pp.1-4.
- [15] Weifeng Su and X.G.Xia, "Signal constellations for QOSTBC with full diversity," *IEEE Trans. Inform. Theory*, Vol-50, Oct, 2004, pp.2331-2347.
- [16] O.Tirkkonen, A.Boariu and A.Hottinen, "Minimal and non-orthogonality rate 1 space time block code for 3+ Tx antennas," *IEEE 6th Int. Symp. on Spread Spectrum Tech. and Appl. (ISSSTA 2000)*, pp.429-432.
- [17] N.Sharma and C.B.Papadias, "Improved Quasi-orthogonal codes," in *Proc. IEEE Wireless Communications and Networking Conference (WCNC 2002)*, March 17-21, vol.1, pp.169-171.
- [11] C.Yuen, Y.L. Guan and T.T. Tjhung, "Full rate full diversity STBC with constellation rotation," *Proc. VTC 2003*, Spring vol. 1, pp. 296-300, Seogwipo, Korea, April 2003.