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Space Time Block Codes from Designs for Fast-Fading Wireless Communication

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ABSTRACT

Space-Time block codes (STBC) obtained from Orthogonal Designs (OD), Quasi-Orthogonal Designs and their variations [3]-[21] are attractive due to their fast ML decoding (single/double-symbol decoding) when used over quasi-static fading channels. However, these STBCs from Designs have not been studied well for use in fast-fading channels. In this report, we study these codes for use in fast-fading channels by giving a matrix representation of the multi-antenna fast-fading channels. We first characterize all linear STBCs that allow single-symbol ML decoding when used in fast-fading channels. Then, among these we identify those with full diversity, i.e., those with diversity L when the STBC is of size $L \times N, (L \geq N)$, where N is the number of transmit antennas and L is the time interval. The maximum rate for such a full-diversity, single-symbol decodable code is shown to be $2/L$ from which it follows that rate 1 is possible only for 2 Tx. antennas. The co-ordinate interleaved design (CID) for 2 Tx (introduced in [13]) is shown to be one such full-rate, full-diversity and single-symbol decodable code. (It turns out that Alamouti code is not single-symbol decodable for fast-fading channels.) This code performs well even when the channel is varying in the sense that sometimes it is quasi-static and other times it is fast-fading. We present simulation results for this code in such a scenario.

1 Introduction

A design criterion for designing codes for multiple transmit and receive antennas suitable for fading channels was first proposed in [1] and the implementation of multi-antenna systems in the context of

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transmit diversity was extensively investigated in [2] with space-time trellis codes (STTC). However, most of the subsequent research was directed toward Space-Time Block Codes (STBC) over quasi-static fading channels due to their amenability for fast decoding: Following Alamouti [3], lot of work has gone into fast decoding of STBC over quasi-static fading channels [4]-[12],[17]-[21]. These are based on Orthogonal Designs (OD) and Quasi-Orthogonal Designs (QOD). The authors introduced co-ordinate interleaved designs (CID) in [13, 14] and a complete characterization of single-symbol ML decodable, full diversity linear STBC has been reported in [16]. All these are applicable for quasi-static fading channels. The use of ODs, QODs and their variations like CIDs for fast-fading channels have not been studied so far to the best of our knowledge. In this report, we study linear STBCs for fast-fading channels with emphasis on fast decoding (single-symbol decoding).

By a linear STBC[1] we mean those covered by the following definition [7]: *Let S be a $L\times N$ matrix whose entries are complex linear combinations of K complex indeterminates $x_k = x_{kI} + \mathbf{j}x_{kQ}, k = 0,\cdots,K-1$ and their complex conjugates. The STBC obtained by letting each indeterminate to take values from a complex constellation $\mathcal{A}$ is called a linear STBC over $\mathcal{A}$. Notice that S is basically a "design" and by the STBC $(S,\mathcal{A})$ we mean the STBC obtained using the design S with the indeterminates taking values from the signal constellation $\mathcal{A}$.* The matrix S can be written as $S = \sum_{k=0}^{K-1} A_{2k}x_{kI} + A_{2k+1}x_{kQ}$ where the $2K$ complex matrices $\{A_{2k}, A_{2k+1}\}, \quad k = 0,1,\cdots,K-1$ are called **the weight matrices** of S.

Let the number of transmit antennas be N and the number of receive antennas be M. At each time slot t, complex signal points, $s_{it}, \ i = 0,1,\cdots,N-1$ are transmitted from the N transmit antennas simultaneously. Let $h_{ijt} = \alpha_{ijt}e^{\mathbf{j}\theta_{ijt}}$ denote the path gain from the transmit antenna i to the receive antenna j at time t, where $\mathbf{j} = \sqrt{-1}$. The received signal v_{jt} at the antenna j at time t, is given by

$$v_{jt} = \sum_{i=0}^{N-1} h_{ijt}s_{it} + n_{jt}, \tag{1}$$

$j = 0,\cdots,M-1; \quad t = 0,\cdots,L-1$. Assuming that perfect channel state information (CSI) is available at the receiver, the decision rule for ML decoding is to minimize the metric

$$\sum_{t=0}^{L-1}\sum_{j=0}^{M-1}\left|v_{jt} - \sum_{i=0}^{N-1} h_{ijt}s_{it}\right|^2 \tag{2}$$

over all codewords. This results in exponential decoding complexity, because of the joint decision on all the symbols s_{it} in the matrix $\mathbf{S}$. If the throughput rate of such a scheme is R in bits/sec/Hz, then 2^{RL} metric calculations are required; one for each possible transmission matrix $\mathbf{S}$. Even for modest antenna configurations and rates this could be very large resulting in search for codes that admit a simple decoding while providing full diversity gain.

[1]Also referred as a linear dispersion (LD) code [10]

2 Single-Symbol Decoding for Quasi-Static Fading

For quasi-static fading channels $h_{ijt} = h_{ij}$ and (1) can be written in matrix notation as,

$$\mathbf{V} = \mathbf{SH} + \mathbf{W} \tag{3}$$

where $\mathbf{V} \in \mathbb{C}^{L\times M}$ ($\mathbb{C}$ denotes the complex field) is the received signal matrix, $\mathbf{S} \in \mathbb{C}^{L\times N}$ is the transmission matrix (codeword matrix), $\mathbf{H} \in \mathbb{C}^{N\times M}$ denotes the channel matrix and $\mathbf{W} \in \mathbb{C}^{L\times M}$ has entries that are Gaussian distributed with zero mean and unit variance and also are temporally and spatially white. In $\mathbf{V}, \mathbf{S}$ and $\mathbf{W}$ time runs vertically and space runs horizontally. The channel matrix $\mathbf{H}$ and the transmitted codeword $\mathbf{S}$ are assumed to have unit variance entries. (Through out the report, for a matrix $\mathbf{A}$, $\mathbf{A}^H$ represents the Hermitian (conjugate transpose), $\mathbf{A}^T$ the transpose, $|\mathbf{A}|$ the determinant of $\mathbf{A}$) and the in-phase/quadrature component of a complex symbol x_k is represented as $x_{k,I}/x_{k,Q}$. The ML metric can then be written as

$$M(\mathbf{S}) = \mathrm{tr}\left((\mathbf{V} - \mathbf{SH})^H(\mathbf{V} - \mathbf{SH})\right). \tag{4}$$

This ML metric (4) results in exponential decoding complexity with the rate of transmission in bits/sec/Hz. If r is the minimum of the rank of $B(\mathbf{S}, \hat{\mathbf{S}}) = \mathbf{S} - \hat{\mathbf{S}}$ among all distinct pairs of codeword matrices $\mathbf{S}$ and $\hat{\mathbf{S}}$, then, the diversity order is rM. If $r = N$ the code is said to be of full-rank.

For a linear STBC with K variables if the ML metric (4) can be written as sum of several terms with each term involving only one variable then the STBC is said to be single-symbol decodable.

In [16] all linear STBCs (not necessarily of full-rank) that admit single-symbol decoding for quasi-static fading channels have been characterized as follows. For any linear STBC, S can be expressed as [7],

$$S = \sum_{k=0}^{K-1} A_{2k}x_{kI} + A_{2k+1}x_{kQ} \tag{5}$$

where $\{A_k\}_{k=0}^{2K-1}$ is a set of complex matrices called *weight matrices*. The following proposition characterizes all S, in terms of the weight matrices, that will allow single-symbol decoding for any complex signal set.

Proposition 2.1. *For a linear STBC in K variables, $S = \sum_{k=0}^{K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$, the ML metric, $M(S)$ defined in (4) decomposes as $M(S) = \sum_{k=0}^{K-1} M_k(x_k) + M_C$ where $M_C = -(K-1)\mathrm{tr}\left(V^HV\right)$, iff*

$$A_k^HA_l + A_l^HA_k = 0 \quad \begin{cases} \forall l \neq k, k+1 \text{ if } k \text{ is even} \\ \forall l \neq k, k-1 \text{ if } k \text{ is odd} \end{cases}. \tag{6}$$

Proof: From (4) we have

$$M(S) = \mathrm{tr}\left(\mathbf{V}^H\mathbf{V}\right) - \mathrm{tr}\left((S\mathbf{H})^H\mathbf{V}\right) - \mathrm{tr}\left(\mathbf{V}^H S\mathbf{H})\right)$$
$$+\mathrm{tr}\left(S^H S\mathbf{H}\mathbf{H}^H\right).$$

Observe that $\mathrm{tr}\left(\mathbf{V}^H\mathbf{V}\right)$ is independent of S. The next two terms in $M(S)$ are functions of S, S^H and hence linear in x_{kI}, x_{kQ}. In the last term,

$$\begin{aligned} S^H S = &\sum_{k=0}^{K-1}(A_{2k}^H A_{2k}x_{kI}^2 + A_{2k+1}^H A_{2k+1}x_{kI}^2)+ \\ &\sum_{k=0}^{K-1}\sum_{l=k+1}^{K-1}(A_{2k}^H A_{2l} + A_{2l}^H A_{2k})x_{kI}x_{lI}+ \\ &\sum_{k=0}^{K-1}\sum_{l=k+1}^{K-1}(A_{2k+1}^H A_{2l+1} + A_{2l+1}^H A_{2k+1})x_{kQ}x_{lQ}+ \\ &\sum_{k=0}^{K-1}\sum_{l=0}^{K-1}(A_{2k}^H A_{2l+1} + A_{2l+1}^H A_{2k})x_{kI}x_{lQ}. \end{aligned} \tag{7}$$

(a) Proof for the “if part”: If (6) is satisfied then (7) reduces to

$$\begin{aligned} S^H S &= \textstyle\sum_{k=0}^{K-1}\left(A_{2k}^H A_{2k}x_{kI}^2 + A_{2k+1}^H A_{2k+1}x_{kQ}^2\right. \\ &\qquad \left.+ \left(A_{2k}^H A_{2k+1} + A_{2k+1}^H A_{2k}\right)x_{kI}x_{kQ}\right) \\ &= \textstyle\sum_{k=0}^{K-1} T^H T, \text{ where } T = A_{2k}x_{kI} + A_{2k+1}x_{kQ} \end{aligned} \tag{8}$$

and using linearity of the trace operator, $M(S)$ can be written as

$$\begin{aligned} M(S) = \mathrm{tr}\left(\mathbf{V}^H\mathbf{V}\right) - &\sum_{k=0}^{K-1}\left\{\mathrm{tr}\left((T\mathbf{H})^H\mathbf{V}\right)\right. \\ &-\mathrm{tr}\left(\mathbf{V}^H T\mathbf{H})\right) + \mathrm{tr}\left(T^H T\mathbf{H}\mathbf{H}^H\right) \end{aligned} \tag{9}$$

$$= \sum_k \underbrace{\left\|\mathbf{V} - (A_{2k}x_{kI} + A_{2k+1}x_{kQ})\mathbf{H}\right\|^2}_{M_k(x_k)} + M_C \tag{10}$$

where $M_C = -(K-1)\mathrm{tr}\left(V^H V\right)$ and $\|.\|$ denotes the Frobenius norm.

(b) Proof for the “only if part”: If (6) is not satisfied for any $A_{k_1}, A_{l_1}, k_1 \neq l_1$ then

$$M(S) = \sum_k \left\|\mathbf{V} - (A_{2k}x_{kI} + A_{2k+1}x_{kQ})\mathbf{H}\right\|^2$$
$$+\mathrm{tr}\left((A_{k_1}^H A_{l_1} + A_{l_1}^H A_{k_1})\mathbf{H}^H\mathbf{H}\right)y + M_C \text{ where}$$
$$y = \begin{cases} x_{k_1/2I}x_{l_1/2I} & \text{if both } k_1, l_1 \text{ are even} \\ x_{(k_1-1)/2I}x_{(l_1-1)/2I} & \text{if both } k_1, l_1 \text{ are odd} \\ x_{(k_1-1)/2I}x_{(l_1)/2I} & \text{if } k_1 \text{ odd, } l_1 \text{ even.} \end{cases}$$

Now, from the above it is clear that $M(S)$ can not be decomposed into terms involving only one variable. $\square$

The condition (6) can also be given as

$$A_k A_l^H + A_l A_k^H = 0 \quad \begin{cases} \forall l \neq k, k+1 \text{ if k is even} \\ \forall l \neq k, k-1 \text{ if k is odd} \end{cases}$$

due to the identity $\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA)$, for square S.

It is important to notice that when $A_{2k'} = cA_{2k'+1}$ for some specific k', where c is a real number, then when (6) is satisfied the code is single symbol decoding. However, in this case, $S = \sum_{k=0,k\neq k'}^{K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1} + A_{2k'}y_{k'}$,where $y_{k'} = (x_{k'I} + cx_{k'Q})$ and S is a STBC in $2K-1$ real variables. Since we consider each complex indeterminate to assume all values of the same complex constellation, these situations need to be excluded from further consideration and we argue that it is sufficient to consider a subclass of linear STBCs, $S = \sum_{k=0}^{K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$, which have the property that the weight matrices of the inphase and quadrature components of any variable are orthogonal, that is $A_{2k}^H A_{2k+1} + A_{2k+1}^H A_{2k} = 0, 0 \leq k \neq l \leq K-1$. With an abuse of notation we call such linear STBCs as *complex linear STBCs* and consider such STBCs only in the rest of this paper. We mention in the passing that all known STBCs are complex linear STBCs. Proposition 2.1 for this case specializes to:

Theorem 2.1. *For a complex linear STBC in K complex variables, $S = \sum_{k=0}^{K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$, the ML metric, $M(S)$ defined in (4) decomposes as $M(S) = \sum_{k=0}^{K-1} M_k(x_k) + M_C$ where $M_C = -(K-1)\mathrm{tr}\left(V^H V\right)$, iff*

$$A_k^H A_l + A_l^H A_k = 0, 0 \leq k \neq l \leq 2K-1. \quad (11)$$

Theorem 2.1 characterizes all SSDDs, i.e., designs which lead to single-symbol decodable STBCs (not necessarily giving full-diversity) in terms of the weight matrices.

Since we exploit this result for fast-fading channels in the sequel, we illustrate the above theorem with an example below:

Example 1: For $N = K = L = 4$ consider the design

$$S = \begin{bmatrix} x_{0I} + \mathbf{j}x_{2Q} & x_{1I} + \mathbf{j}x_{3Q} & 0 & 0 \\ -x_{1I} + \mathbf{j}x_{3Q} & x_{0I} - \mathbf{j}x_{2Q} & 0 & 0 \\ 0 & 0 & x_{2I} + \mathbf{j}x_{0Q} & x_{3I} + \mathbf{j}x_{1Q} \\ 0 & 0 & -x_{3I} + \mathbf{j}x_{1Q} & x_{2I} - \mathbf{j}x_{0Q} \end{bmatrix}. \quad (12)$$

The corresponding weight matrices are

$$\mathbf{A}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{j} & 0 \\ 0 & 0 & 0 & -\mathbf{j} \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{A}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{j} \\ 0 & 0 & \mathbf{j} & 0 \end{bmatrix},$$

$$\mathbf{A}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{A}_5 = \begin{bmatrix} \mathbf{j} & 0 & 0 & 0 \\ 0 & -\mathbf{j} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \mathbf{A}_7 = \begin{bmatrix} 0 & \mathbf{j} & 0 & 0 \\ \mathbf{j} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easily seen that the above weight matrices satisfy (11) and hence the code is single-symbol decodable. This is precisely the CID of [13, 14] for 4 Tx. antennas. (This code is of full-diversity also.)

3 Fast-Fading Channels

We recall that the design criteria for fast-fading channels are [2]:

- The Distance Criterion : In order to achieve the diversity rM in fast-fading channels, for any two distinct codeword matrices $\mathbf{S}$ and $\hat{\mathbf{S}}$, the strings $s_{0t}, s_{1t}, \cdots, s_{(N-1)t}$ and $\hat{s}_{0t}, \hat{s}_{1t}, \cdots, \hat{s}_{(N-1)t}$ must differ at least for r values of $0 \leq t \leq L-1$. (Essentially, the distance criterion implies that if a codeword is viewed as a L length vector with each row of the transmission matrix viewed as a single element of $\mathbb{C}^N$, then the diversity gain is equal to the Hamming distance of this L length codeword over $\mathbb{C}^N$.
- The Product Criterion : Let $\mathcal{V}(\mathbf{S}, \hat{\mathbf{S}})$ be the indices of the non-zero rows of $\mathbf{S} - \hat{\mathbf{S}}$ and let$|\mathbf{s_t} - \mathbf{\hat{s}_t}|^\mathbf{2} = \sum_{\mathbf{i=0}}^{\mathbf{N-1}} |\mathbf{s_{it}} - \mathbf{\hat{s}_{it}}|^\mathbf{2}$, where $\mathbf{s_t}$ is the t-th row of $\mathbf{S}$, $0 \leq t \leq L-1$. Then the coding gain is

$$\min_{\mathbf{s} \neq \hat{\mathbf{s}}} \prod_{t \in \mathcal{V}(\mathbf{s}, \hat{\mathbf{s}})} |\mathbf{s_t} - \mathbf{\hat{s}_t}|^\mathbf{2}.$$

Extended Codeword Matrix and the Equivalent Matrix Channel: The inability to write (1) in the matrix form as in (3) for fast-fading channels seems to be the reason for scarce study of these codes for use in fast-fading channels. In this report we solve this problem by introducing proper matrix representations for the codeword matrix and the channel. In what follows we assume that $M = 1$, for simplicity. For a fast-fading channel (1) can be written as

$$V = SH + W \tag{13}$$

where $V \in \mathbb{C}^{L\times 1}$ ($\mathbb{C}$ denotes the complex field) is the received signal vector, $S \in \mathbb{C}^{L\times NL}$ is the **Extended codeword matrix (ExCM)** (as opposed to codeword matrix $\mathbf{S}$) given by

$$S = \begin{bmatrix} S_0 & 0 & 0 & 0 \\ 0 & S_1 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & S_{L-1}a \end{bmatrix} \tag{14}$$

where $S_t = \begin{bmatrix} s_{0t} & s_{1t} & \cdots & s_{(N-1)t} \end{bmatrix}$, $H \in \mathbb{C}^{NL\times 1}$ denotes the **equivalent channel matrix (EChM)** formed by stacking the channel vectors for different t i.e.

$$H = \begin{bmatrix} H_0 \\ H_1 \\ \vdots \\ H_{L-1} \end{bmatrix} \text{ where } H_t = \begin{bmatrix} h_{0t} \\ h_{1t} \\ \cdots \\ h_{(N-1)t} \end{bmatrix},$$

and $\mathbf{W} \in \mathbb{C}^{L\times 1}$ has entries that are Gaussian distributed with zero mean and unit variance and also are temporally and spatially white. We denote the codeword matrices by boldface letters and the ExCMs by normal letters. For example, the ExCM S for the Alamouti code,

$$\mathbf{S} = \begin{bmatrix} x_0 & x_1 \\ -x_1^* & x_0^* \end{bmatrix},$$

is given by

$$S = \begin{bmatrix} x_0 & x_1 & 0 & 0 \\ 0 & 0 & -x_1^* & x_0^* \end{bmatrix}. \tag{15}$$

Observe that for a linear space-time code, its ExCM S is also linear in the indeterminates $x_k, k = 0, \cdots, K-1$ and can be written as $S = \sum_{k=0}^{K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$, where A_k are referred to as **extended weight matrices** to differentiate from weight matrices corresponding to the codeword matrix $\mathbf{S}$.

With these notions of ExCM, and EChM and the similarity between (3) and (13) we observe that,

1. The **distance criterion** on the difference of two distinct codeword matrices is equivalent to the **rank criterion** for the difference of two ExCM.
2. The **product criterion** on the difference of two distinct codeword matrices is equivalent to the **determinant criterion** for the difference of two ExCM.

3. The trace criterion on the difference of two distinct codeword matrices derived for quasi-static fading in [22] applies to fast-fading channels also-following the observation that $\mathrm{tr}\left(\mathbf{S}^H\mathbf{S}\right) = \mathrm{tr}\left(S^H S\right)$.

4. The ML metric (2) can again be represented as (4) with the code word $\mathbf{S}$ replaced by the ExCM, S i.e.

$$M(S) = \mathrm{tr}\left((V - SH)^H (V - SH)\right). \tag{16}$$

   This amenability to write the ML decoding metric in matrix form for fast-fading channels (16) allows the results on single-symbol decodable designs [16] to be applied to fast-fading channels.

In particular, Theorem 1 on substitution of the codeword matrix $\mathbf{S}$ by the ExCM, S reduces to

Theorem 2: For a linear STBC in K complex variables, whose ExCM is given by, $S = \sum_{k=0}^{K-1} x_{kI} A_{2k} + x_{kQ} A_{2k+1}$, the ML metric, $M(S)$ defined in (16) decomposes as $M(S) = \sum_k M_k(x_k) + M_o$ where $M_o = -(K-1)\mathrm{tr}\left(V^H V\right)$, iff

$$A_k^H A_l + A_l^H A_k = 0, 0 \leq k \neq l \leq 2K - 1. \tag{17}$$

Theorem 2 characterizes all linear designs which admit single-symbol decoding over fast-fading channels in terms of the extended weight matrices.

Example 2: The Alamouti code is not single-symbol decodable for fast-fading channels. The extended weight matrices are

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A_1 = \begin{bmatrix} \mathbf{j} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{j} \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & \mathbf{j} & 0 & 0 \\ 0 & 0 & \mathbf{j} & 0 \end{bmatrix}.$$

It is easily checked that the pair A_0, A_2 does not satisfy equation (17).

4 Full-diversity Single-Symbol decodable codes

We proceed to identify all full-diversity codes among single-symbol decodable codes. Recall that for single-symbol decodability in quasi-static fading the weight matrices have to satisfy (11) while for fast-fading the extended weight matrices, have to satisfy (17).

In contrast to quasi-static fading (17) is not easily satisfied for fast-fading due to the structure of the equivalent weight matrices imposed by the structure of S given in (14). The weight matrices A_k are block diagonal of the form (14)

$$A_k = \begin{bmatrix} A_k^{(0)} & 0 & 0 & 0 \\ 0 & A_k^{(1)} & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & A_k^{(L-1)} \end{bmatrix}. \tag{18}$$

where $A_k^{(t)} \in \mathbb{C}^{1\times N}$. In other words even for square codeword matrix the equivalent transmission matrix is rectangular. For example consider the Alamouti code,

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

etc., (17) is not satisfied as a result we have

$$S^H S = \begin{bmatrix} |x_0|^2 & x_0^* x_1 & 0 & 0 \\ x_1^* x_0 & |x_1|^2 & 0 & 0 \\ 0 & 0 & |x_1|^2 & -x_1 x_0^* \\ 0 & 0 & -x_1^* x_0 & |x_0|^2 \end{bmatrix}, \tag{19}$$

and hence single-symbol decoding is not possible for the Alamouti code over fast-fading channels.

The structure of equivalent weight matrices that satisfy (17) is given in Proposition 1.

Proposition 1: All the matrices A_l that satisfy (17), with a specified non-zero matrix A_k in (18) are of the form

$$\begin{bmatrix} a_0 A_k^{(0)} & 0 & 0 & 0 \\ 0 & a_1 A_k^{(1)} & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & a_{L-1} A_k^{(L-1)} \end{bmatrix}. \tag{20}$$

where $a_i = 0, \mathbf{j}\ \forall i$.

Proof: The the matrix A_k can satisfy the condition of Proposition 1 iff $A_k^{(t)H} A_l^{(t)} = -A_l^{(t)H} A_k^{(t)}$, $\forall t$. For a given t, $A_k^{(t)H} A_l^{(t)}$ is skew-Hermitian and rank one, it follows that $A_k^{(t)H} A_l^{(t)} = UDU^H$ where U is unitary and D is diagonal with one imaginary entry only. Therefore $A_k^{(t)} = \pm \mathbf{j} c A_l^{(t)}$ where c is a real constant-in fact only the values $c = 0, 1$ are of interest as other values can be normalized to 1, completing the proof. □

We give a necessary condition in terms of the extended weight matrices A_k for the code to achieve diversity $r \leq L$,

Lemma 1: If a linear STBC in K variables, whose ExCM is given by, $S = \sum_{k=0}^{K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$, achieves diversity r then the matrices A_{2k}, A_{2k+1} together have at least r different non-zero rows for every k, $0 \leq k \leq K-1$.

This follows from the Hamming distance interpretation of the distance criterion. The conditions of Lemma 1 is only a necessary condition since either $(x_{kI} - \hat{x}_{kI})$ or $(x_{kQ} - \hat{x}_{kI})$ may be zero for $x_k \neq \hat{x}_k$. The sufficient condition is given by $(x_{kI} - \hat{x}_{kI})(x_{kQ} - \hat{x}_{kQ}) \neq 0$ or both A_{2k} and A_{2k+1} individually have at least r non-zero rows for every k, $0 \leq k \leq K-1$ [16].

Using the above lemma with $r = L$, we have

Theorem 3: For fast-fading channel, the maximum rates possible for a full-diversity single-symbol decodable STBC using N transmit antennas is $2/L$.

Proof: We have two cases *Case 1: A_k has L non-zero rows $\forall k$.* The number of matrices that satisfy Proposition 1 are 2, and the maximal rate is $R = 1/L$. The corresponding STBC is given by its equivalent transmission matrix $S = x_0 A_0$, where A_0 is of the form given in (18). *Case 2: A_k has less than L non-zero rows for some k.* As Lemma 1 requires A_{2k}, A_{2k+1} to have L non-zero rows, we can assume that A_{2k} has r_1 non-zero rows and A_{2k+1} has non-overlapping $L - r_1$ non-zero rows. The number of such matrices that satisfy Proposition 1 are 4, and hence the maximal rate is $R = 2/L$. $\square$

From Theorem 3 it follows that the maximal rate full-diversity single-symbol decodable code is given by its ExCM

$$S = x_{0I}A_0 + x_{0Q}A_1 + x_{1I}A_2 + x_{1Q}A_3, \tag{21}$$

where $A_{2k}, A_{2k+1}, k = 0, 1$ are of the form

$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \mathbf{j}A & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{j}B \end{bmatrix} \tag{22}$$

where A, B are of the form given in (18) with $L = r_1$ and $L = L - r_1$ respectively. Observe that other STBC's can be obtained from the above, by change of variables, multiplication by unitary matrices etc. Of interest is the code for $L = 2$ due to its full rate. Setting $A = [1\ 0], B = [0\ 1]$ we have the ExCM,

$$S = \begin{bmatrix} x_{0I} + \mathbf{j}x_{1Q} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{1I} + \mathbf{j}x_{0Q} \end{bmatrix} \tag{23}$$

and the corresponding codeword matrix is

$$\mathbf{S} = \begin{bmatrix} x_{0I} + \mathbf{j}x_{1Q} & 0 \\ 0 & x_{1I} + \mathbf{j}x_{0Q} \end{bmatrix}. \tag{24}$$

Observe that this is the CID of size 2 reported in [13, 14]. Also observe that other full rate STBC's that achieve full diversity can be achieved from S by performing linear operations (not necessarily unitary) on S and/or permutation of the real symbols(for each complex symbol there are two real symbols). Consequently the most general full-diversity single-symbol decodable code for $N = 2$ is given by the codeword matrix

$$\mathbf{S} = \left[\begin{array}{cc} x_{0I} + \mathbf{j}x_{1Q} & b(x_{0I} + \mathbf{j}x_{1Q}) \\ c(x_{1I} + \mathbf{j}x_{0Q}) & x_{1I} + \mathbf{j}x_{0Q} \end{array} \right], \mathbf{b}, \mathbf{c} \in \mathbb{C}. \tag{25}$$

An immediate consequence is

Theorem 4: A rate 1 full-diversity single-symbol decodable design for fast-fading channel exists iff $L = N = 2$.

Follows from the results of [13, 14].

Theorem 5: The CID of size 2 is the only STBC that achieves full diversity over both quasi-static and fast-fading channels and provides single-symbol decoding.

Other STBC that achieves full diversity over both quasi-static fading channels and provides single-symbol decoding are unitarily equivalent to the CID.

Remark 1: Contrast Theorem 3 with that of square matrix OD [7] and CID [14] over quasi-static fading channels that provide rates of $\frac{\lceil \log_2 N \rceil + 1}{2^{\lceil \log_2 N \rceil}}$ and $\frac{\lceil \log_2 N/2 \rceil + 1}{2^{\lceil \log_2 N \rceil - 1}}$ respectively. The maximal rate is independent of the number of transmit antennas in fast-fading channels.

5 Robustness of CID to channel variations

The CID for $N = 2$ is suitable for both quasi-static fading and fast-fading channels, in the sense it gives full diversity and amenable for single-symbol decoding in both the cases, whereas Alamouti code is not. This makes the CID suitable for varying-fading scenario, i.e., the channel sometimes quasi-static and other times fast-fading. Now we provide simulation results for the case when the channel is either fast-fading or quasi-static with different probabilities to show that the CID is robust to channel variations. The corresponding curves for Alamouti are also presented for comparison. The quasi-static symbol duration is assumed to be 12 symbols. The probability that the channel is quasi-static is denoted by p and perfect CSI is assumed to be available at the receiver when the channel is quasi-static and fast-fading. At the decoder, single-symbol decoding is performed for both Alamouti and the CID schemes-observe that this results in loss of diversity in the case of Alamouti scheme. Observe from Fig. 1 and Fig.2 that the BER of CID is invariant for all p while the BER curves for Alamouti scheme show considerable loss of performance.

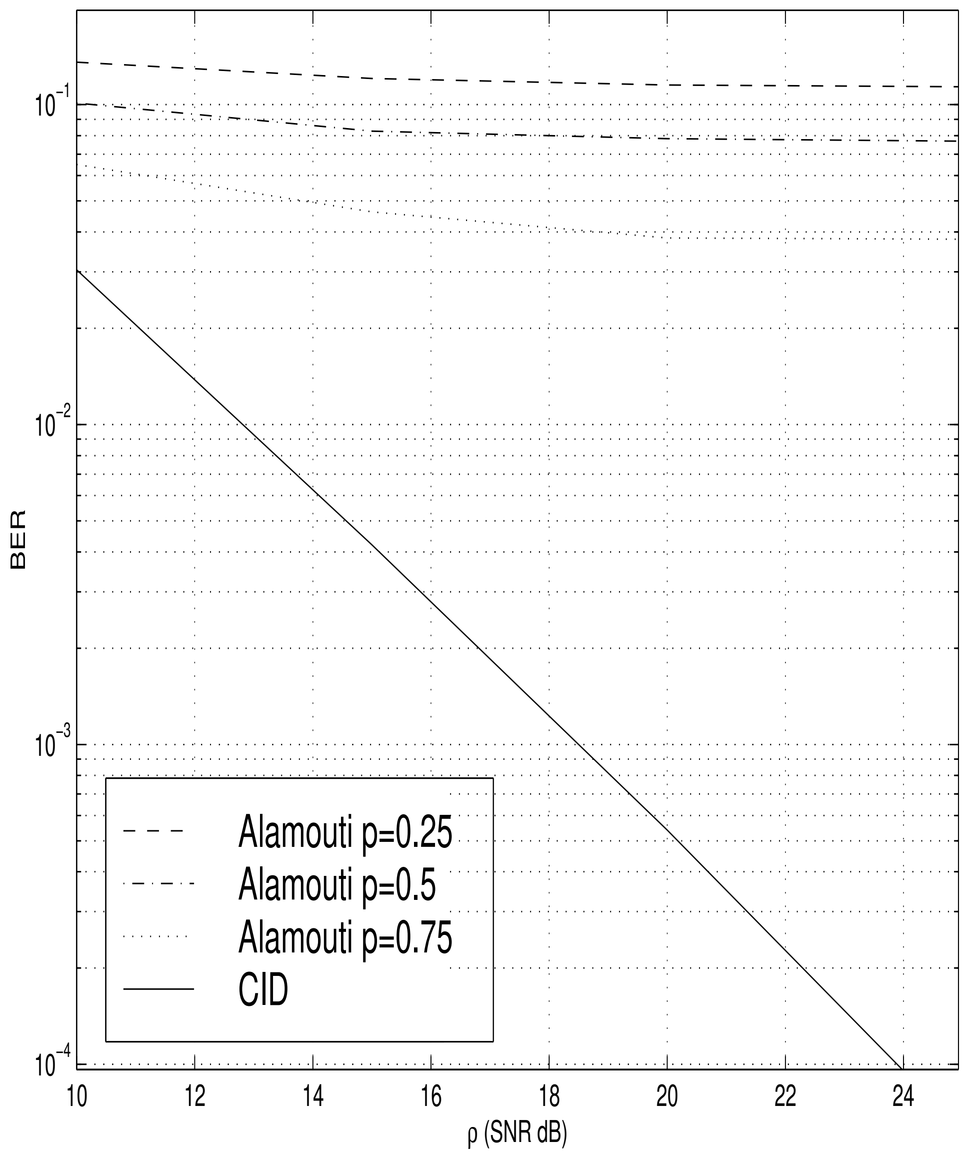

Figure 1: BER curves for Alamouti and CID schemes for QPSK signal set over varying fading channels where the probability that the channel is quasi-static is p and single-symbol decoding.

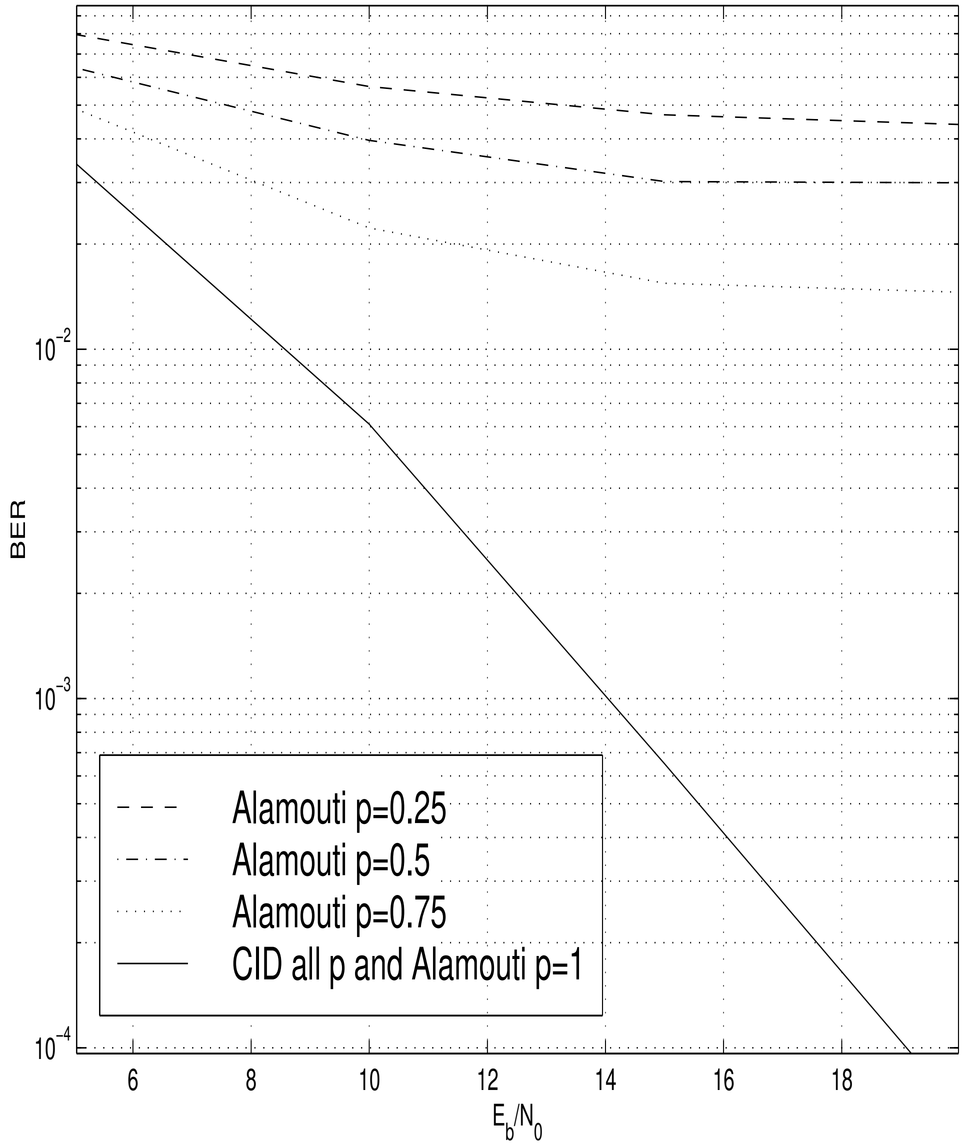

Figure 2: BER curves for Alamouti and CID schemes for BPSK transmission over varying fading channels where the probability that the channel is quasi-static is p and single-symbol decoding.

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