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A Full-Diversity Rate-One STBC for Four Tx Antennas with Single-Symbol Decoding

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ABSTRACT

Space-Time block codes (STBCs) based on Orthogonal Designs (ODs) is well known due to its simple (single-symbol) ML decoding. A full-diversity, symbol rate one, square complex OD exists only for 2 antennas which is basically the Alamouti scheme. For larger number of antennas either the symbol-rate or full-diversity or the advantage of single-symbol decoding needs to be sacrificed in general. In this paper we introduce a STBC for four transmit antennas based on a non-orthogonal square design, which we call Coordinate-Interleaved Design (CID), that achieves full-diversity (four), symbol-rate one as well as single-symbol ML decoding. This scheme can be combined with any number M of receive antennas to achieve a diversity of $4M$. We identify the parameters that determine the coding gain of the resulting STBC. Simulation results are presented for BPSK, QPSK and 16 QAM modulations.

Keywords: Fading channels, transmit diversity, orthogonal-designs, space-time block codes, wireless communication, coordinate interleaved design.

1 Introduction

Receive antenna diversity is a widely applied technique to reduce the detrimental effects of multi path fading, but as this makes the receiver complex and costly, is generally used exclusively at the base station. Transmit diversity and, if affordable, a combination of transmit and receive diversity has been suggested as a remedy to the above problem. Several interesting approaches to transmit diversity have been suggested, like the delay diversity scheme [1], switched diversity [2], [3], [4]. A natural extension of these works is Space-Time Codes, which take advantage of both space and time diversity available with time varying fading [5]. A brief review of the model [5] for a space-time coded system follows.

Let the number of transmit antennas be N and the number of receive antennas be M. At each time slot t, a complex signal points, s_{it}, $i = 0, 1, \cdots, N-1$ are transmitted from the N antennas simultaneously. Let $h_{ij} = \alpha_{ij} e^{\mathbf{j}\theta_{ij}}$ denote the path gain from the transmit antenna i to the receive antenna j, where $\mathbf{j} = \sqrt{-1}$. Assuming that the path gains are constant over a frame length $L \geq N$, $t = 0, \cdots, L-1$, the received signal v_{jt} at the antenna j at time t, is given by

$$v_{jt} = \sum_{i=0}^{N-1} h_{ij} s_{it} + n_{jt}, \quad j = 0, \cdots, M-1; \quad t = 0, \cdots, L-1. \tag{1}$$

In matrix notation,

$$\mathbf{V} = \mathbf{SH} + \mathbf{N} \tag{2}$$

where $\mathbf{V} \in \mathbb{C}^{L \times M}$ ($\mathbb{C}$ denotes the complex field) is the received signal matrix, $\mathbf{S} \in \mathbb{C}^{L \times N}$ is the transmission matrix (codeword matrix) and $\mathbf{N} \in \mathbb{C}^{L \times M}$ has entries that are Gaussian distributed with zero mean and unit variance and also are temporally and spatially white. In $\mathbf{V}, \mathbf{S}$ and $\mathbf{N}$ time runs vertically and space runs horizontally. $\mathbf{H} \in \mathbb{C}^{N \times M}$ defines the channel matrix. The channel matrix $\mathbf{H}$ and the transmitted codeword $\mathbf{S}$ are assumed to have unit variance entries. (Through out the paper, for a matrix $\mathbf{A}$, $\mathbf{A}^H$ represents the Hermitian (conjugate transpose), $\mathbf{A}^T$ the transpose and $|\mathbf{A}|$ the determinant of $\mathbf{A}$). If the transmission power constraint is $E\ tr\{\mathbf{SS}^H\} = LN$ then

$$\mathbf{V} = \sqrt{\frac{\rho}{N}} \mathbf{SH} + \mathbf{N} \tag{3}$$

where ρ is the SNR at each receiver. Assuming that perfect channel state information (CSI) is available at the receiver, the decision rule for ML decoding is to minimize the metric

$$\sum_{t=0}^{L-1} \sum_{j=0}^{M-1} \left| v_{jt} - \sum_{i=0}^{N-1} h_{ij} s_{it} \right|^2 \tag{4}$$

over all codeword matrices $\mathbf{S}$. This metric (4) results in exponential decoding complexity, because of the joint decision on all the symbols s_{it} in the matrix $\mathbf{S}$. If the throughput rate of such a scheme is R in bits/sec/Hz, then $(2^R)^L$ metric calculations are required; one for each possible transmission matrix

S. Even for modest antenna configurations and rates this could be very large resulting in search for codes that admit a simple decoding while providing full diversity gain. Space-Time Block Codes (STBCs) obtained from Orthogonal Designs (ODs) [6] achieve these. A Generalized Complex Linear Processing Orthogonal Design (GCLPOD) in k complex indeterminates $x_1, x_2, \cdots, x_k$ of size n and rate $\mathcal{R} = k/p$, $p \geq N$ is a $p \times N$ matrix Θ, such that

- the entries of Θ are complex linear combinations of $0, \pm x_i, \;\; i = 1, \cdots, k$ and their conjugates.
- $\Theta^H \Theta = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix whose entries are a linear combination of $|x_i|^2, i = 1, \cdots, k$ with all strictly positive real coefficients.

When k=n=p and no entry is zero, then Θ is called a Complex Linear Processing Orthogonal Design (CLPOD). Furthermore, when the entries are only from $\{\pm x_1, \pm x_2, \cdots, \pm x_k\}$, their conjugates and multiples of $\mathbf{j}$ then Θ is called a Complex Orthogonal Design (COD). STBCs from ODs are obtained by replacing x_i by s_i and allowing s_i to ytake all values from a signal set $\mathcal{A}$. Note that if the signal set is of size 2^b the throughput rate R in bits per second per Hertz is related to the rate of the design $\mathcal{R}$ as $R = \mathcal{R}b$. A GCLPOD is said to be of minimal-delay if $n = p$. The Alamouti scheme [7], which is of minimal-delay, full-rank and full-rate is basically the STBC arising from the size 2 COD $\begin{bmatrix} x_0 & x_1 \\ -x_1^* & x_0^* \end{bmatrix}$. For STBCs from ODs, the decoding metric (4) can be written as sum of several square terms, one for each variable of the design [6] which reduces the decoding complexity enormously. We will refer to this as "single-symbol decoding". If the decoding metric is writtenable as sum of several terms each involving two variables and a variable appearing in one term do not appear in any other term, we refer to the decoding as "double-symbol decoding".

A STBC scheme based on Quasi-Orthogonal Design (QOD) that trades diversity for double-symbol decoding is presented in [8] for four and eight antennas. For the case of four antennas several authors have studied the performance of STBCs obtained from different versions of ODs [6, 9] and QODs [8, 10, 11, 12] combining with rotated signal sets or allowing some variables of the design to take values from a signal set and the remaining variables from a rotation of it. Table 1 summarizes the performance of these schemes and also shows (the last row) what is achieved in this correspondence. The scheme proposed in this correspondence is called (by the authors) "Coordinate Interleaved Design (CID)" in which we interleave the inphase and quadrature components of the variables of a design so that they face different fading gains in the channel and in the receiver we deinterleave before decoding. This enables us to get STBCs which admit single-symbol decoding with diversity four for four transmit antennas and rate one. Note that delay-minimal STBCs from ODs [6] that achieve rate one and admit ML single-symbol decoding for four transmit antennas do not exist and other known schemes shown in the table do not admit single-symbol decoding. Limitations on the rates of Generalized ODs have been discussed in detail in [14, 15, 13]. The diversity and coding gain of our scheme are linked to a quantity called "co-ordinate product distance (CPD) of the signal set. We then derive the optimal angle of rotation for 4-QAM constellation, that maximizes the CPD. Simulation results are presented for BPSK, QPSK and 16 QAM modulations. Also we show that such a coordinate interleaving-deinterleaving does not help for two transmit antenna systems.

Table 1: Comparison of known orthogonal, quasi-orthogonal and coordinate-interleaved designs for four antennas

	Symbol Rate	Diversity	Decoding
Orthogonal Designs[6, 9]	3/4	4	single-symbol
QOD; Jhafarkhani [8]	1	2	double-symbol
QOD; Su and Xia [10, 12]	1	4	double-symbol
QOD; Sharma and Papadias[11]	1	4	double-symbol
CID; (proposed scheme)	1	4	single-symbol

The content of this correspondence has been organized as follows. In Section 2, we introduce the CID for four transmit antenna systems and highlight its differences with the well known CODs. Also, we show that the resulting STBC scheme from the CID admits single-symbol decoding. In Section 3 we present conditions on the signal set under which full diversity is achieved and also the conditions for the maximum coding gain. Simulation results are given in Section 4 and concluding remarks and several directions for further research constitute Section 5. In the appendix we show that the coordinate interleaving-deinterleaving when applied to 2 antenna system with Alamouti scheme buys nothing.

2 Coordinate-Interleaved Design (CID) and decoding

In this section we introduce a design called Coordinate-Interleaved Design (CID) suitable for four antennas and show that STBCs obtained from this design admit single-symbol decoding.

Definition 2.1 (Coordinate-Interleaved Design (CID)) *Let* $x_i = x_{i,I} + \mathbf{j}x_{i,Q},\ i = 0,1,2,3,$ *denote four complex indeterminates, where* $x_{i,I}$ *and* $x_{i,Q}$ *denote, respectively, the real and imaginary parts of* x_i. *Let*

$$\tilde{x}_i = x_{i,I} + \mathbf{j}x_{(i+2)_4,Q} \quad i = 0,1,2,3$$

where $(a)_4$ *denotes* $a \pmod 4$. *The indeterminate* $\tilde{x}_i$ *will be referred as the coordinate interleaved version of* $x_i,\ i = 0,1,2,3$. *The CID* $\mathcal{S}(x_0,x_1,x_2,x_3)$ *in variables* $x_i,\ i = 0,1,2,3$ *is defined to be the* 4×4 *matrix*

$$\mathcal{S}(x_0,x_1,x_2,x_3) = \begin{bmatrix} \tilde{x}_0 & \tilde{x}_1 & 0 & 0 \\ -\tilde{x}_1^* & \tilde{x}_0^* & 0 & 0 \\ 0 & 0 & \tilde{x}_2 & \tilde{x}_3 \\ 0 & 0 & -\tilde{x}_3^* & \tilde{x}_2^* \end{bmatrix} \tag{5}$$

$$= \begin{bmatrix} x_{0,I} + \mathbf{j}x_{2,Q} & x_{1,I} + \mathbf{j}x_{3,Q} & 0 & 0 \\ -x_{1,I} + \mathbf{j}x_{3,Q} & x_{0,I} - \mathbf{j}x_{2,Q} & 0 & 0 \\ 0 & 0 & x_{2,I} + \mathbf{j}x_{0,Q} & x_{3,I} + \mathbf{j}x_{1,Q} \\ 0 & 0 & -x_{3,I} + \mathbf{j}x_{1,Q} & x_{2,I} - \mathbf{j}x_{0,Q} \end{bmatrix}.$$

As in the case of Orthogonal Designs a STBC is obtained from the CID above by replacing x_i by s_i and allowing each $s_i,\ i = 0,1,2,3,$ to take values from a signal set $\mathcal{A}$. Henceforth a STBC obtained from the CID above will be referred as Coordinate Interleaved STBC (CI-STBC) and for notational simplicity we will use only $\mathcal{S}$ for $\mathcal{S}(x_0,x_1,x_2,x_3)$ dropping the arguments, whenever it is clear from the context.

It is important to note that when the variables $x_i,\ i = 0,1,2,3,$ take values from a complex signal set $\mathcal{A}$ the transmission matrix have entries which are coordinate interleaved versions of the variables and hence the actual signal points transmitted are not from $\mathcal{A}$ but from an *expanded version of* $\mathcal{A}$ which we denote by $\tilde{\mathcal{A}}$. Figure 1(a) shows $\tilde{\mathcal{A}}$ when $\mathcal{A} = \{1, -1, \mathbf{j}, -\mathbf{j}\}$ which is shown in Figure 1(c). Notice that $\tilde{\mathcal{A}}$ has 8 signal points whereas $\mathcal{A}$ has 4. Figure 1(b) shows $\tilde{\mathcal{A}}'$ where $\mathcal{A}'$ is the four point signal set obtained by rotating $\mathcal{A}$ by 13.2825 degrees counter clockwise i.e., $\mathcal{A}' = \{e^{\mathbf{j}\theta}, -e^{\mathbf{j}\theta}, \mathbf{j}e^{\mathbf{j}\theta}, -\mathbf{j}e^{\mathbf{j}\theta}\}$ where $\theta = 13.2825$ degrees as shown in Figure 1(d). Notice that now the expanded signal set has 16 signal points (The value $\theta = 13.2825$ has been chosen so as to maximize the parameter called Coordinate Product Distance of the signal set which is related to diversity and coding gain of the CI-STBC, discussed in detail in Section V).

Another important aspect to notice in the above matrix is that during the time interval the first two antennas transmit the remaining two antennas transmit nothing and vice versa. However, at any time the symbols that are transmitted by any pair of antennas carry a component (in- or quadrature) of all the four indeterminates s_0, s_1, s_2, s_3 . Moreover, as we shall see in the subsection ahead, the decoding will be for the variables s_i's and not for the interleaved versions of these.

Further differences between CI-STBC and STBCs from ODs are:

- $\mathcal{S}$ is not an orthogonal matrix since

$$\mathcal{S}^H \mathcal{S} = \begin{bmatrix} |\tilde{x}_0|^2 + |\tilde{x}_1|^2 & 0 & 0 & 0 \\ 0 & |\tilde{x}_0|^2 + |\tilde{x}_1|^2 & 0 & 0 \\ 0 & 0 & |\tilde{x}_2|^2 + |\tilde{x}_3|^2 & 0 \\ 0 & 0 & 0 & |\tilde{x}_2|^2 + |\tilde{x}_3|^2 \end{bmatrix}. \tag{6}$$

- CID is defined only for 4 variables unlike the CODs. However, its significance is due to the fact that a square COD in four variables does not exist and, as will be seen in the sequel, CID enjoys all the advantages of a general COD.

- At any given time only two antennas (either the first two or the last two) transmit signals and the other two transmit nothing. So, the peak-to-average power ratio for any one antenna is high compared to those STBCs obtained from COD. This can be taken care of by "power uniformization" techniques as discussed in [9] for CODs with entries zeros.

In the following subsection we show that CI-STBCs admit single-symbol ML decoding.

2.1 Single-Symbol Decoding of CI-STBC

Fig. 2 shows the baseband representation of the scheme that uses the CI-STBC with only the jth, $j = 0, 1, ..., M-1$, receive antenna, where M is the number of receive antennas. If the signals $s_0, s_1, s_2, s_3 \in \mathcal{A}$ are to be communicated, their interleaved version as given in Definition 2.1 is transmitted. The signal transmission matrix is $\mathcal{S}(s_0, s_1, s_2, s_3)$ which is obtained by replacing x_i by s_i in (5). The received signals at the different time slots, v_{jt}, $t = 0, 1, 2, 3$ and $j = 0, 1, \cdots, M-1$ for the M receive antennas are given by

$$v_{j0} = h_{0j}\tilde{s}_0 + h_{1j}\tilde{s}_1 + n_{j0}; \;\; v_{j1} = -h_{0j}\tilde{s}_1^* + h_{1j}\tilde{s}_0^* + n_{j1}; \tag{7}$$
$$v_{j2} = h_{2j}\tilde{s}_2 + h_{3j}\tilde{s}_3 + n_{j2}; \;\; v_{j3} = -h_{2j}\tilde{s}_3^* + h_{3j}\tilde{s}_2^* + n_{j3} \tag{8}$$

where n_{ji} , $i = 0, 1, 2, 3$ and $j = 0, \cdots, M-1$ are complex independent Gaussian random variables. Letting

$$\mathbf{V}_j = [v_{j0}, \; v_{j1}^*, \; v_{j2}, \; v_{j3}^*]^T,$$
$$\tilde{\mathbf{S}} = [\tilde{s}_0, \; \tilde{s}_1, \; \tilde{s}_2, \; \tilde{s}_3]^T,$$
$$\mathbf{N}_j = [n_{j0}, \; n_{j1}^*, \; n_{j2}, \; n_{j3}^*]^T \text{ and}$$

$$\mathbf{H}_j = \begin{bmatrix} h_{0j} & h_{1j} & 0 & 0 \\ h_{1j}^* & -h_{0j}^* & 0 & 0 \\ 0 & 0 & h_{2j} & h_{3j} \\ 0 & 0 & h_{3j}^* & -h_{2j}^* \end{bmatrix}$$

for $j = 0, 1, \cdots, M-1$, (7) can be put in matrix notation as

$$\mathbf{V}_j = \mathbf{H}_j\tilde{\mathbf{S}} + \mathbf{N}_j. \tag{9}$$

Let $\mathbf{V}_j' = [v'_{j0}, \; v'_{j1}, \; v'_{j2}, \; v'_{j3}]^T = \mathbf{H}_j^H\mathbf{V}_j$. Then, we have

$$\mathbf{V}'_j = \begin{bmatrix} (|h_{0j}|^2 + |h_{1j}|^2)I_2 & 0 \\ 0 & (|h_{2j}|^2 + |h_{3j}|^2)I_2 \end{bmatrix} \tilde{\mathbf{S}} + \mathbf{H}_j^H\mathbf{N}_j. \tag{10}$$

Rearranging the in-phase and quadrature-phase components of v'_{ji}'s, define, for $i = 0, 1$,

$$\hat{v}_i = \sum_{j=0}^{M-1} v'_{ji,I} + \mathbf{j}v'_{ji+2,Q} = as_{i,I} + \mathbf{j}bs_{i,Q} + u_{0i} \tag{11}$$

$$\hat{v}_{i+2} = \sum_{j=0}^{M-1} v'_{ji+2,I} + \mathbf{j}v'_{ji,Q} = bs_{i+2,I} + \mathbf{j}as_{i+2,Q} + u_{1i} \tag{12}$$

where

$$a = \sum_{j=0}^{M-1}\{\alpha_{0j}^2 + \alpha_{1j}^2\}, \qquad b = \sum_{j=0}^{M-1}\{\alpha_{2j}^2 + \alpha_{3j}^2\} \tag{13}$$

and u_{0i}, u_{1i} are complex Gaussian random variables. Let $\tilde{\mathbf{N}}_j = [\tilde{n}_{j0} \; \tilde{n}_{j1} \; \tilde{n}_{j2} \; \tilde{n}_{j3}]^T = \mathbf{H}_j^H\mathbf{N}_j$. Then, for $i = 0, 1$,

$$u_{0i} = \sum_{j=0}^{M-1} \tilde{n}_{ji,I} + \mathbf{j}\tilde{n}_{ji+2,Q} \qquad \text{and} \qquad u_{1i} = \sum_{j=0}^{M-1} \tilde{n}_{ji+2,I} + \mathbf{j}\tilde{n}_{ji,Q}.$$

Note that u_{00} and u_{01} have the same variance and similarly u_{10} and u_{11}. The variance of the in-phase component of u_{00} is a and that of the quadrature-phase component is b. The in-phase component of u_{10} has the same variance as that of the quadrature-phase component of u_{00} and vice versa. Equations (11) and (13) shows that the decisions on the variables s_i can be done separately for each i with the difference that the inphase and quadrature phases of the variable have faced different noise variances. In what follows we first derive the ML decision rule for such a situation in a general setting and then apply it to our case.

ML decision rule for unequal variance noise components: Consider the received signal r, given by

$$r = c_1 s_I + \mathbf{j} c_2 s_Q + n \tag{14}$$

where c_1, c_2 are *real* constants and $s_I,\ s_Q$ are in-phase and quadrature-phase components of transmitted signal s. The ML decision rule when the in-phase, n_I, and quadrature-phase component, n_Q, of the Gaussian noise, n have different variances $c_1\sigma^2$ and $c_2\sigma^2$ is derived as follows: The pdf of n is given by

$$p_n(n) = \frac{1}{2\pi\sigma^2\sqrt{c_1 c_2}} e^{-\frac{n_I^2}{2c_1\sigma^2}} e^{-\frac{n_Q^2}{2c_2\sigma^2}}. \tag{15}$$

The ML rule is: decide in favor of s_i, if and only if

$$p_n(r/s_i) \geq p_n(r/s_k),\ \forall\ i \neq k. \tag{16}$$

Substituting from (14) and (15) into (16) and simplifying we have

$$\begin{aligned} c_2|r_I - as_{i,I}|^2 + c_1|r_Q - bs_{i,Q}|^2 \leq \\ c_2|r_I - as_{k,I}|^2 + c_1|r_Q - bs_{k,Q}|^2, \forall\ i \neq k. \end{aligned} \tag{17}$$

By substituting $c_1 = a$ and $c_2 = b$, and then $c_1 = b$ and $c_2 = a$, we obtain, for our problem, the ML decisions given by;

For $\hat{v}_j$, $j = 0, 1$, choose signal $s_i \in \mathcal{A}$ iff

$$b|\hat{v}_{j,I} - as_{i,I}|^2 + a|\hat{v}_{j,Q} - bs_{i,Q}|^2 \leq b|\hat{v}_{j,I} - as_{k,I}|^2 + a|\hat{v}_{j,Q} - bs_{k,Q}|^2, \quad \forall\ i \neq k \tag{18}$$

and for $\hat{v}_j$, $j = 2, 3$, choose signal s_i iff

$$a|\hat{v}_{j,I} - bs_{i,I}|^2 + b|\hat{v}_{j,Q} - as_{i,Q}|^2 \leq a|\hat{v}_{j,I} - bs_{k,I}|^2 + b|\hat{v}_{j,Q} - as_{k,Q}|^2, \quad \forall\ i \neq k. \tag{19}$$

From the above two equations it is clear that decoupling of the variables has been achieved as in the case of STBCs from CODs.

Basically, the codes of this paper use the orthogonality of the in-phase and quadrature-phase components of the complex signal points more efficiently than the known codes. Although Ganesan and Stoica [16], [17] consider the use of the in-phase and quadrature-phase components separately, they require that each component (in-phase and quadrature) of the transmitted symbols be transmitted from all the antennas. Essentially, our designs have been obtained by relaxing this constraint by observing that it is sufficient to transmit one component of a signal point from some of the antennas and the other component from the remaining antennas.

3 Diversity and Coding Gain

In this section we drive the conditions under which the scheme presented in the previous section achieves full diversity and maximum coding gain.

It is known [5] that maximum diversity gain is achieved when the codeword difference matrix $B(\mathbf{S},\mathbf{S}') = \mathbf{S} - \mathbf{S}'$ is of full rank for two distinct codeword matrices $\mathbf{S},\mathbf{S}'$. For our case, we have

$$B(\mathbf{S},\mathbf{S}') = \mathbf{S} - \mathbf{S}' = \begin{bmatrix} \tilde{s}_0 - \tilde{s'}_0 & \tilde{s}_1 - \tilde{s'}_1 & 0 & 0 \\ -\tilde{s}_1^* + \tilde{s'}_1^* & \tilde{s}_0^* - \tilde{s'}_0^* & 0 & 0 \\ 0 & 0 & \tilde{s}_2 - \tilde{s'}_2 & \tilde{s}_3 - \tilde{s'}_3 \\ 0 & 0 & -\tilde{s}_3^* + \tilde{s'}_3^* & \tilde{s}_2^* - \tilde{s'}_2^* \end{bmatrix} \qquad (20)$$

and

$$B^H(\mathbf{S},\mathbf{S}')B(\mathbf{S},\mathbf{S}') = \begin{bmatrix} aI_2 & 0 \\ 0 & bI_2 \end{bmatrix} \qquad (21)$$

where $a = |\tilde{s}_0 - \tilde{s'}_0|^2 + |\tilde{s}_1 - \tilde{s'}_1|^2$ and $b = |\tilde{s}_2 - \tilde{s'}_2|^2 + |\tilde{s}_3 - \tilde{s'}_3|^2$. We get full rank iff $ab \neq 0$ (both a and b are strictly positive) for all possible distinct pairs of s_i and s'_i with each taking values taken from the signal set $\mathcal{A}$. It is easily verified by straight forward manipulation of terms that this is true iff

$$|s_{i,I} - s'_{i,I}||s_{i,Q} - s'_{i,Q}| \neq 0 \qquad \forall i = 0,1,2,3. \qquad (22)$$

In other words, no two signal points in $\mathcal{A}$ can have the same ordinate or coordinate, i.e., no two signal points can lie on a horizontal or vertical line in the complex plane. **This is not at all a restrictive condition, since given any signal set $\mathcal{A}$ one can always get the above condition satisfied by rotating it. In fact, there are infinitely many angles of rotations that will satisfy the required condition and only finitely many which will not. Moreover, appropriate rotation leads to more coding gain also.** To translate this condition into a neat form (Theorem 3.1) on the signal set $\mathcal{A}$ from which $s_i,\ i = 0,1,2,3$ take values as well as to get a closed form expression for the coding gain we introduce

Definition 3.1 *The Coordinate Product Distance (CPD) between any two signal points $u = u_I + \mathbf{j}u_Q$ and $v = v_I + \mathbf{j}v_Q$, $u \neq v$, in the signal set $\mathcal{A}$ is defined as*

$$CPD(u,v) = |u_I - v_I||u_Q - v_Q| \tag{23}$$

and the minimum of this value among all possible pairs is defined as the CPD of $\mathcal{A}$.

Then we have

Theorem 3.1 *The STBC obtained from the CID with variables taking values from a signal set $\mathcal{A}$ gives full-rank (equal to 4) iff the CPD of $\mathcal{A}$ is not zero.*

Now, suppose two codeword matrices $\mathbf{S}$ and $\mathbf{S}'$ are such that they differ by only one variable, say s_0 taking different values from the signal set $\mathcal{A}$. Then, for this case, the determinant of the matrix (21) is $|s_{0,I} - {s'}_{0,I}||s_{0,Q} - {s'}_{0,Q}|$, which implies, using the determinant criterion [5]

Theorem 3.2 *The coding gain of a full-rank STBC based on the CID with the variables taking values from a signal set is equal to the CPD of that signal set.*

From the above results it follows that signal constellations with $CPD = 0$ like $M - ary\ QAM, M - ary\ PSK$ will not achieve full diversity. But the situation gets salvaged by simply rotating the signal set to get this condition satisfied as also indicated in [18]. This result is similar to the ones on co-ordinate interleaved schemes like co-ordinate interleaved trellis coded modulation [18] and bit and co-ordinate interleaved coded modulation [19], [20], [21] for single antenna transmit systems.

3.1 Optimal angle of rotation for 4-QAM constellations

In this subsection we derive the optimal angle of rotation for 4-QAM constellation so that CPD and hence the coding gain is maximized.

Theorem 3.3 *Consider the 4-QAM constellation $\mathcal{A}$, with signal points from $(2k-3)d + \mathbf{j}(2l-3)d$ where $k, l \in [1,2]$ where d is an arbitrary scaling factor, rotated by an angle θ. The CPD of $\mathcal{A}$ is maximized at $\theta = \frac{\arctan(2)}{2} = 31.7175°$ and this maximum CPD is $\frac{4d^2}{\sqrt{5}}$.*

Proof: For an arbitrary point (x, y) let (x_R, y_R) denote the point to which (x, y) goes when rotated by an angle $\theta \in [0, 90°]$. We have

$$\begin{bmatrix} x_R \\ y_R \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}}_{R} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{24}$$

Let $(x_1, y_1), (x_2, y_2)$ be two distinct points in $\mathcal{A}$ and $\triangle_x = x_1 - x_2, \triangle_y = y_1 - y_2$. Observe that $\triangle_x, \triangle_y = 0, \pm 2d$ and both $\triangle_x, \triangle_y$ cannot be zero simultaneously. Since, rotation is a linear operation,

$$\begin{bmatrix} \triangle x_R \\ \triangle y_R \end{bmatrix} = R \begin{bmatrix} \triangle x \\ \triangle y \end{bmatrix}, \tag{25}$$

where $\triangle x_R = x_{1R} - x_{2R}, \triangle y_R = y_{1R} - y_{2R}$. It is easily seen that there are only two possible values for the CPDs $= 2d^2 \sin(2\theta)$ and $4d^2 \cos(2\theta)$. As sine is an increasing function and cosine a decreasing function of θ in the first quadrant, the optimal angle of rotation, θ_{opt} for which the minimum of these two CPDs is largest is the one for which these two are equal. It follows that $\theta_{opt} = \frac{\arctan(2)}{2} = 31.7175°$ and $CPD = 2d^2 \sin(2\theta_{opt}) = 4d^2 \cos(2\theta_{opt})$. $\square$

Remark: The 4 QAM constellation in Fig. 1 is rotated by 13.2825° as the original QAM signal set in the figure is a rotated version (45°) of the QAM signal set considered in Theorem 3.3.

4 Simulation Results

In this section we present simulation results for 4-QAM and 16-QAM modulation over a quasi-static fading channel. The fading is assumed to be constant over a fade length of 120 symbol durations.

First, we compare the scheme of this paper, with (i) the STBC (denoted by STBC-CR in Fig.3 and 4) of [22], (ii) rate 1/2, COD and (iii) rate 3/4 COD for four transmit antennas for the **identical throughput** of 2 bits/sec/Hz. For CI-STBC the transmitter chooses symbols from a QPSK signal set rotated by an angle of 13.2825° so as to maximize the CPD. For STBC-CR the symbols are from a QPSK signal set and rate 1/2 COD from 16-QAM signal set. For rate 3/4 COD, the symbols are chosen from 6-PSK for a throughput of 1.94 bits/sec/Hz which is close to 2 bits/sec/Hz. The average transmitted power is equal in all the cases i.e. $\mathrm{E}\{tr(S^H S)\} = 16$, so that average energy per bit using the channel model of (4) is equal. Fig. 3. shows the BER performance for these schemes. Observe that the scheme of this paper outperforms rate 1/2 COD by 3.0 dB, rate 3/4 COD by 1.3

Table 2: The coding gains of CID, STBC-CR, rate 3/4 COD and rate 1/2 COD for 4 tx. antennas and QAM constellations

R (bits/sec/Hz)	Λ_{CID}	$\Lambda_{STBC-CR}$	$\Lambda_{rate\ 3/4\ COD}$	$\Lambda_{rate\ 1/2\ COD}$
2	1.788	2	1.333	0.8
3	0.5963	0.66	0.5333	0.1905
4	0.3587	0.4	-	0.0471

dB and STBC-CR by 1.2 dB at $P_b = 10^{-5}$. A comparison of the coding gain, Λ, of these schemes is given in tabular form in Table 2. For CI-STBC, $\Lambda_{CID} = 1.788$ while for STBC-CR $\Lambda_{STBC-CR} = 2$ at $R = 2$ bits/sec/Hz, but still CI-STBC outperforms STBC-CR because the coding gain is derived on the basis of an upper bound. The results for rest of the schemes are in accordance with their coding gains;

$$10\log_{10}\left(\frac{\Lambda_{CID}}{\Lambda_{rate\ 1/2\ COD}}\right) = 3.5 \text{ and } 10\log_{10}\left(\frac{\Lambda_{CID}}{\Lambda_{rate\ 3/4\ COD}}\right) = 1.3.$$

Observe that rate 3/4 COD and STBC-CR have almost similar performance at 2 bits/sec/Hz, and around 1.6dB coding gain over rate 1/2 COD. A possible apparent inconsistancy of these with the results in [24, 25], which report coding gain of over 2 dB, is due to the fact that symbol error rate (SER) vs. ρ is plotted in [24, 25]. As rate 1/2 COD chooses symbols from 16 QAM and STBC-CR from 4 QAM, SER vs. ρ plot gives an overestimate of the errors for STBC-OD as compared to STBC-CR and therefore bit error rate (BER) vs. E_b/N_0 is a more appropriate plot for comparison at the same through put (2 bits/sec/Hz).

From the table 2, which gives the coding gains of various schemes at spectral efficiencies of 2,3,4 bits/sec/Hz, we see that the coding gain of STBC-CR and CID are nearly equal (differ by a factor of 1.11) and significantly greater than other schemes. But, the main factor in favor of CID as compared to STBC-CR (as also any STBC other than STBC-OD) is that CID allows linear complexity ML decoding while STBC-CR has exponential ML decoding complexity. At a modest rate of 4 bits/sec/Hz, CID requires 64 metric computations while STBC-CR requires $16^4 = 65,536$ metric computations. Even the sphere-decoding algorithm is quite complex requiring order of $32^3 = 16384$ metric computations [26].

For 4-QAM and 16-QAM constellations, Fig. 4 shows the performance for CI-STBC, STBC-CR and Diagonal Algebraic Space Time (DAST) codes of

[23]. As expected CI-STBC shows better performance due to its ML decoding whereas DAST decoding is not ML.

5 Discussion

In this correspondence we have presented a rate 1 STBC for four transmit antennas that achieves full diversity for complex constellations satisfying the condition that no two points of it lie on one horizontal or vertical line. This scheme admits a singly-symbol ML decoder wherein the transmitted symbols are decoupled at the receiver using linear processing. Simulation results were presented to compare the performance of this scheme with known STBCs for four transmit antennas. As half of the antennas are idle during any given symbol period, this scheme has higher peak-to-average ratio (PAR) compared to the ones using Orthogonal Designs. But, since for four transmit antennas there is no Complex Orthogonal Design, because of simple decoder, this scheme is attractive.

From the CID we can also obtain a non-minimal delay design by taking any three rows of $\mathcal{S}$ in (9). For example, if we take the first three rows, we have

$$S = \begin{bmatrix} \tilde{x}_0 & \tilde{x}_1 & 0 \\ -\tilde{x}_1^* & \tilde{x}_0^* & 0 \\ 0 & 0 & \tilde{x}_2 \\ 0 & 0 & -\tilde{x}_3^* \end{bmatrix}, \tag{26}$$

where $\tilde{x}_i, i = 0, \cdots, 3$ are as defined in definition III.2. Other than being non-minimal delay, this design inherits all the properties of CID. In particular, it is rate 1. As a result a full rate STBC for three antennas can be obtained from this design as in Section IV. Also observe that the codes presented here do not achieve full diversity over fast fading channels. This problem can be solved by pre-multiplying with a Hadamard matrix as is done for DAST codes in [23]. This pre-multiplication by a Hadamard matrix will not change the decoding complexity while more evenly distributing the transmitted power across space and time and achieve full diversity over fast fading channels resulting in 'smart and greedy' CID codes. Few possible directions for further research are listed below.

- It is not clear whether the CID presented in this paper can be extended to more than four antennas. It will be interesting to develop a general theory of coordinate-interleaved designs like the one available for Orthogonal Designs.

- Non-minimal delay CID involving rectangular matrices is another interesting direction to pursue.
- It is known that Alamouti Scheme achieves capacity only for the 2 transmit and 1 receive antenna and does not achieve for 2 transmit and 2 receive antennas [27]. Such an analysis of capacity of the schemes based on CID is being carried out.
- Su and Xia [28] have reported GODs for 5 and 6 transmit antennas. It will be interesting to investigate whether coordinate interleaving leads to any advantages in terms of rate or some other aspect.

APPENDIX

Coordinate Interleaving for Two Transmit Antennas

It is known that the Alamouti scheme is optimum in the following senses: (i) This is the only COD [6] (ii) Achieves capacity for two transmit and one receive antennas [27]. In this section, for the sake of completeness, we show that the interleaving technique described above for four transmit antennas do not lead to any advantage when employed for two transmit antennas in comparison to the Alamouti scheme.

Let $s_0 = s_{0,I} + \mathbf{j}s_{0,Q}$ and $s_1 = s_{1,I} + \mathbf{j}s_{1,Q}$ be two signals chosen from the signaling constellation $\mathcal{A}$. We exchange the in-phase and quadrature-phase components of s_0 and s_1 to obtain

$$\begin{aligned} \tilde{s}_0 &= s_{0,I} + \mathbf{j}s_{1,Q} \\ \tilde{s}_1 &= s_{1,I} + \mathbf{j}s_{0,Q}. \end{aligned} \tag{27}$$

The signals $\tilde{s}_0, \tilde{s}_1$ are then transmitted according to the transmission matrix $\mathbf{S}$ given by,

$$\mathbf{S} = \begin{bmatrix} \tilde{s}_0 & 0 \\ 0 & \tilde{s}_1 \end{bmatrix}. \tag{28}$$

Note that the scheme is of minimal-delay and of rate 1. The received signals at the different time slots, v_{jt}, $t = 0, 1$ and $j = 0, 1, \cdots, M-1$, where M is the number of receive antennas, is given by

$$\begin{array}{l} v_{j0} = h_{0j}\tilde{s}_0 + n_{j0} \\ v_{j1} = h_{1j}\tilde{s}_1 + n_{j1} \end{array}, \quad 0 \leq t \leq T \tag{29}$$

where n_{j0} and n_{j1} are complex Gaussian random variables. Multiplication of the above signals by $h_{ij}{}^*$, $i = 0, 1$, gives

$$\begin{aligned} \tilde{v}_{j0} &= v_{j0}h_{0j}^* = {\alpha_{0j}}^2\tilde{s}_0 + {h_{0j}}^* n_{j0} \\ \tilde{v}_{j1} &= v_{j1}h_{1j}^* = {\alpha_{1j}}^2\tilde{s}_1 + h_{1j}^* n_{j1}. \end{aligned} \tag{30}$$

Separating and rearranging the in-phase and quadrature-phase components of the two signals we have

$$\hat{v}_0 = \sum_{j=0}^{M-1} \tilde{v}_{j0,I} + \mathbf{j}\tilde{v}_{j1,Q} = \sum_{j=0}^{M-1} {\alpha_{0j}}^2 s_{0,I} + \mathbf{j}{\alpha_{1j}}^2 s_{0,Q} + n_3$$
$$\hat{v}_1 = \sum_{j=0}^{M-1} \tilde{v}_{j0,I} + \mathbf{j}\tilde{v}_{j1,Q} = \sum_{j=0}^{M-1} {\alpha_{1j}}^2 s_{1,I} + \mathbf{j}{\alpha_{0j}}^2 s_{1,Q} + n_4 \qquad (31)$$

where $n_3,\ n_4$ are complex Gaussian random variables, the in-phase component of n_3 and quadrature-phase component of n_4 have the same variance $\sum_{j=0}^{M-1} \alpha_{0j}^2$ and the in-phase component of n_4 and the quadrature-phase component of n_3 have the same variance $\sum_{j=0}^{M-1} \alpha_{1j}^2$.

The above equation is the counterpart of (11) and (12) for the four transmit antenna case discussed in Section 2. Following the same arguments as in that section and Section 3 it follows that the diversity and coding gain of this scheme also depends on the CPD of the signal set $\mathcal{A}$ as in the case of the four transmit antenna system. Since the maximum possible diversity gain achieved is two, in this as well as the Alamouti scheme, and considering the fact that the scheme given by (28) has larger peak-to-average power ratio compared the Alamouti scheme we conclude that coordinate interleaving for two transmit antennas does not help.

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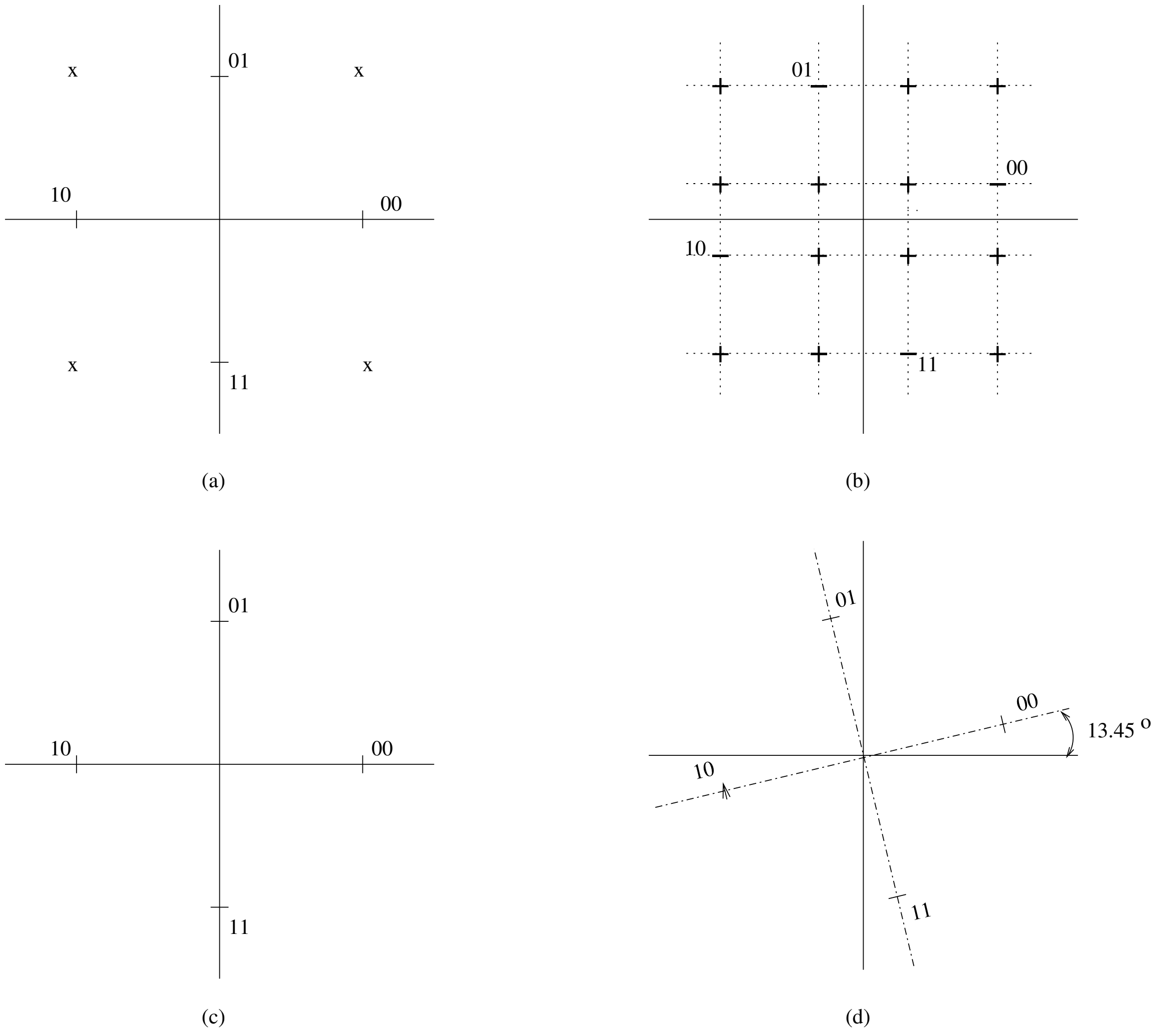

Figure 1: Expanded signal sets $\tilde{\mathcal{A}}$ for $\mathcal{A} = \{1, -1, \mathbf{j}, -\mathbf{j}\}$ and a rotated version of it.

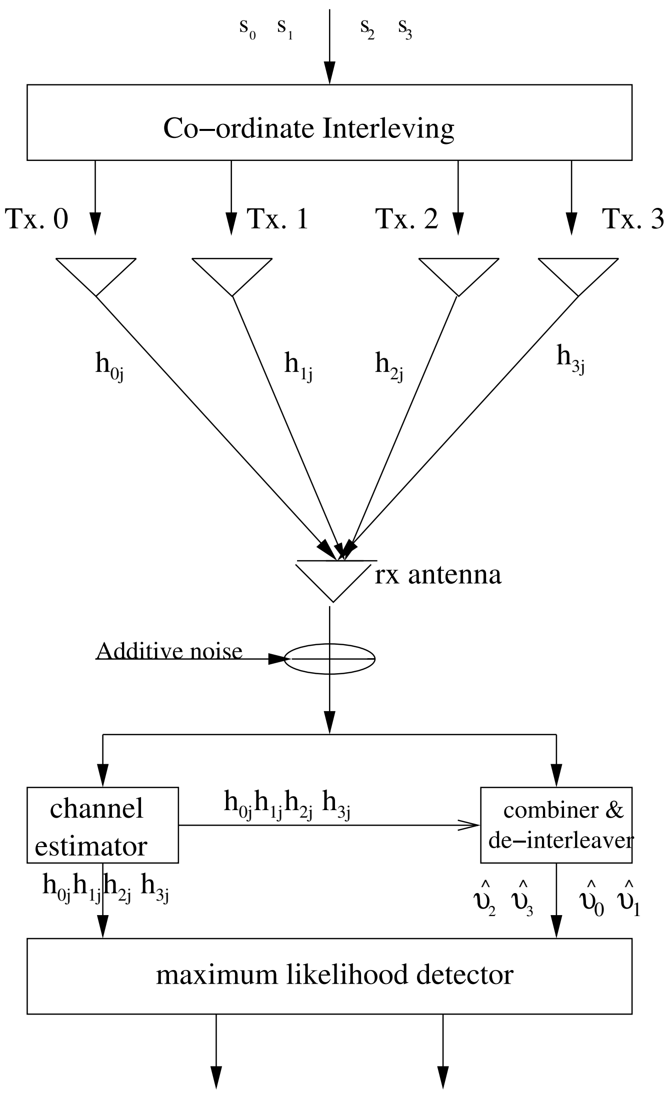

Figure 2: Baseband representation of new scheme for four transmit and the j-th receive antennas.

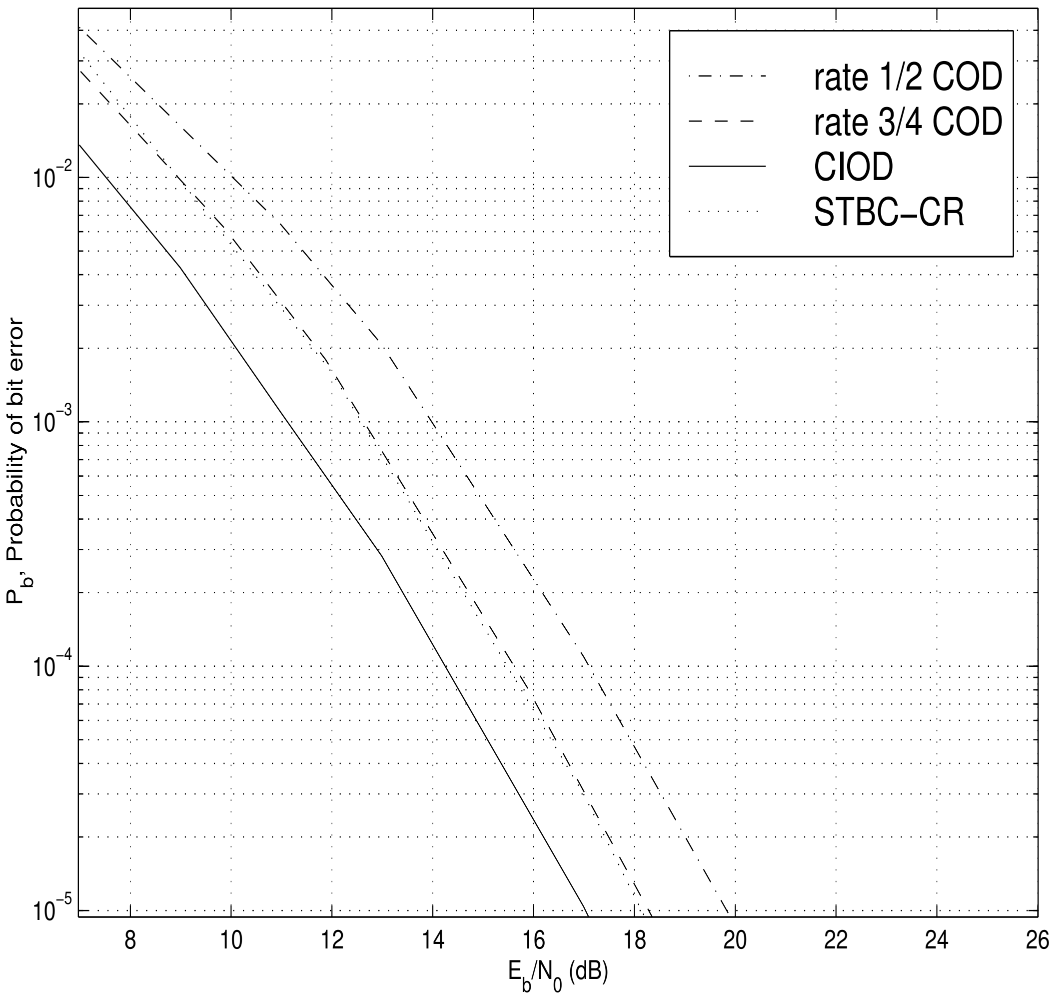

Figure 3: The BER performance of coherent QPSK rotated by an angle of 13.2885° for of the CID scheme for 4 transmit and 1 receive antenna compared with STBC-CR, rate 1/2 COD and rate 3/4 COD at a throughout of 2 bits/sec/Hz in Rayleigh fading for the same number of transmit and receive antennas.

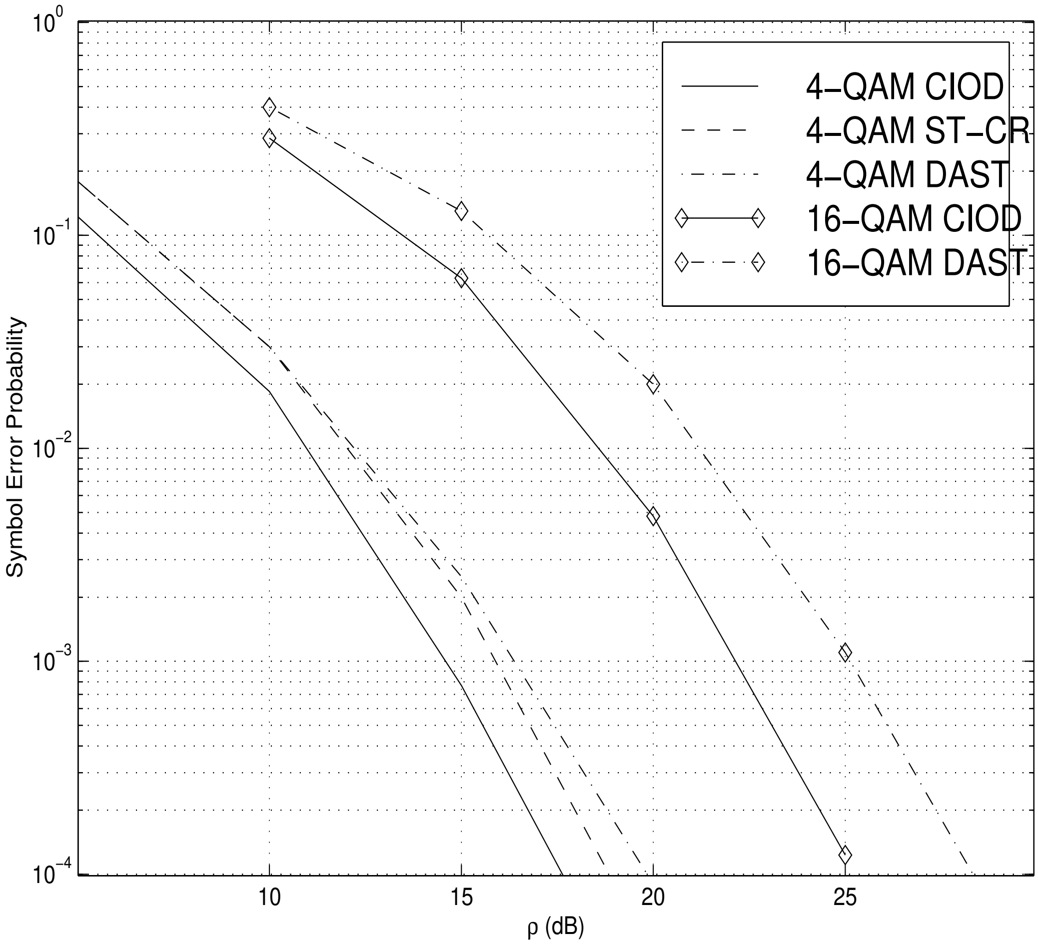

Figure 4: The BER performance of the CI-STBC with 4- and 16-QAM modulation and comparison with ST-CR and DAST schemes