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Matrix characterization of linear codes with arbitrary Hamming weight hierarchy ☆

G. Viswanath, B. Sundar Rajan *

Department of Electrical Communication Engineering, Indian Institute of Science, Bangalore 560 012, India

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Abstract

The support of an $[n, k]$ linear code C over a finite field F_q is the set of all coordinate positions such that at least one codeword has a nonzero entry in each of these coordinate position. The rth generalized Hamming weight $d_r(C)$, $1 \leqslant r \leqslant k$, of C is defined as the minimum of the cardinalities of the supports of all $[n, r]$ subcodes of C. The sequence $(d_1(C), d_2(C), \ldots, d_k(C))$ is called the Hamming weight hierarchy (HWH) of C. The HWH, $d_r(C) = n - k + r$; $r = 1, 2, \ldots, k$, characterizes maximum distance separable (MDS) codes. Therefore the matrix characterization of MDS codes is also the characterization of codes with the HWH $d_r(C) = n - k + r$; $r = 1, 2, \ldots, k$. A linear code C with systematic check matrix $[I|P]$, where I is the $(n-k) \times (n-k)$ identity matrix and P is a $(n-k) \times k$ matrix, is MDS iff every square submatrix of P is nonsingular. In this paper we extend this characterization to linear codes with arbitrary HWH. Using this result, we characterize Near-MDS codes, Near-Near-MDS (N^2-MDS) codes and A^μ-MDS codes. The MDS-rank of C is the smallest integer η such that $d_{\eta+1} = n - k + \eta + 1$ and the defect vector of C with MDS-rank η is defined as the ordered set $\{\mu_1(C), \mu_2(C), \mu_3(C), \ldots, \mu_\eta(C), \mu_{\eta+1}(C)\}$, where $\mu_i(C) = n - k + i - d_i(C)$. We call C a dually defective code if the defect vector of the code and its dual are the same. We

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* Corresponding author.
E-mail addresses: gviswa@protocol.ece.iisc.ernet.in (G. Viswanath), bsrajan@ece.iisc.ernet.in (B. Sundar Rajan).

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also discuss matrix characterization of dually defective codes. Further, the codes meeting the generalized Greismer bound are characterized in terms of their generator matrix. The HWH of dually defective codes meeting the generalized Greismer bound are also reported.

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1. Introduction and preliminaries

Let C be an $[n, k]$ linear code over F_q. Let $\chi(C)$ be the support of C, defined by, $\chi(C) = \{i|x_i \neq 0 \text{ for some } (x_1, x_2, \ldots, x_n) \in C\}$. The rth generalized Hamming weight of C is then defined as $d_r(C) = min\{|\chi(D)| :$ D is an r-dimensional subcode of C$\}$ [1–3]. The sequence $(d_1(C), d_2(C), \ldots, d_k(C))$ is called the Hamming weight hierarchy (HWH) of C. The notion of HWH has been found to be useful in several applications. HWH characterizes the performance of C on Type-II wire-tap channels [4]. HWH also finds application in state complexity of trellis diagrams of codes [5], t-resilient functions [3] and designing codes for switching multiple access channel [6].

A linear $[n, k, d]$ code satisfying the Singleton bound $d \leqslant n - k + 1$ with equality is called a maximum distance separable (MDS) code [7]. All Reed–Solomon (RS) codes are MDS. The length of RS codes are at most the size of the alphabet and hence are short. Moreover, all known MDS codes are such that there is an RS code (with slight modifications) with identical parameters [8]. The problem of obtaining the maximal length MDS codes, in general, is still open. Algebraic geometric (AG) codes [9] are a generalization of RS codes with minimum distances deviating from the Singleton bound by the genus of the curve over which the code is defined, which is small in some cases. The length of the AG code is determined by the number of rational points on the curve. It is shown in [10] that the class of AG codes contains codes that exceed the Varshamov–Gilbert bound [11]. Hence, to find long codes, codes with minimum distance not reaching the Singleton bound, but deviating from it only slightly needs to be studied in general. An explicit approach to this problem was developed by Dodunekov and Landgev [12,13] by considering Near-MDS (NMDS) codes. The class of NMDS codes contains remarkable representatives such as the ternary Golay code, the quaternary code with parameters $[11, 6, 5]$, the quaternary code with parameters $[12, 6, 6]$ and a large class of algebraic geometric codes [14]. The importance of NMDS codes is that there exist NMDS codes which are considerably longer than the longest possible MDS codes for a given size of the code and the alphabet. Also, these codes have good error detecting capabilities [15]. AMDS codes [16] and several classes of codes with distances close to the Singleton bound are studied in [17]. One such class of codes is the Near-Near-MDS codes, which we denote by N^2-MDS codes [18]. Classes of codes with generalized Hamming weights close to the generalized Singleton bound also include A^μ-MDS codes and dually A^μ-MDS codes [17]. The generalized Singleton bound for an $[n, k]$ code C is given by

$$d_r(C) \leqslant n - k + r; \quad 1 \leqslant r \leqslant k \tag{1}$$

and it is a well known fact that the sequence of generalized Hamming weights is strictly increasing [3], i.e.

$$d_1(C) < d_2(C) < \cdots < d_k(C) = n. \tag{2}$$

The HWH of a code is related to that of its dual code $C^\perp$ as follows:

$$\begin{aligned} &\{d_r(C) \mid r = 1, 2, \ldots, k\} \cup \{n + 1 - d_r(C^\perp) \mid r = 1, 2, \ldots, n - k\} \\ &\quad = \{1, 2, \ldots, n\}. \end{aligned} \tag{3}$$

MDS, *AMDS*, *NMDS*, N^2-MDS and A^μ-*MDS codes*: Linear $[n, k, d]$ codes meeting the generalized Singleton bound (Eq. (1)) with equality are MDS codes. AMDS codes are the class of codes with $d_1(C) = n - k$. NMDS codes are those with the following HWH: $d_1(C) = n - k$ and $d_i(C) = n - k + i$ for $i = 2, 3, 4, \ldots, k$. Equivalently a code is NMDS iff $d_1(C) = n - k$ and $d_1(C^\perp) = k$. N^2-MDS codes are codes with the property that $d_1(C) = n - k - 1$, $d_2(C) = n - k + 1$ and $d_i(C) = n - k + i$ for $i = 3, 4, \ldots, k$ [17,18]. A^μ-MDS codes are those with $d_1(C) = n - k + 1 - \mu$ [17].

An equivalent condition for an $[n, k]$ code to be NMDS is that $d_1(C) + d_1(C^\perp) = n$, where $d_1(C^\perp)$ is the minimum Hamming distance of the dual code [12]. This implies that an $[n, k]$ NMDS as well as its dual code are AMDS. Further NMDS codes are characterized in terms of their parity check matrices and generator matrices as follows [12]: A linear $[n, k]$ code is NMDS iff its parity check matrix satisfies the following conditions: (i) any $n - k - 1$ columns of the parity check matrix are linearly independent, (ii) there exists a set of $n - k$ linearly dependent columns in the parity check matrix and (iii) any $n - k + 1$ columns of the parity check matrix are of rank $n - k$. A linear $[n, k]$ code is NMDS iff its generator matrix satisfies the following conditions: (i) any $k - 1$ columns of the generator matrix are linearly independent (ii) there exists a set of k linearly dependent columns in the generator matrix and (iii) any $k + 1$ columns of the generator matrix are of rank k.

Definition 1 (*Defect and MDS-rank*). The defect $\mu_i(C)$ of the ith generalized Hamming weight of a code C is defined as $\mu_i(C) = n - k + i - d_i(C)$ ($\mu_i(C)$ is zero for MDS codes for every i, $1 \leqslant i \leqslant k$) and the MDS-rank of an $[n, k]$ code C is defined as the smallest η such that $d_{\eta+1} = n - k + \eta + 1$.

Definition 2 (*Dually A^μ-MDS codes*). A code C is dually A^μ-MDS if $\mu_1(C) = \mu_1(C^\perp) = \mu$.

It is well known that a linear MDS code can be described in terms of its systematic generator matrix as follows: a linear code with systematic generator matrix $[I|P]$ is MDS iff every square submatrix of P is nonsingular. Since MDS codes are characterized by the HWH $d_r(C) = n - k + r$ for $1 \leqslant r \leqslant k$, the systematic generator matrix characterization of MDS codes can be viewed as the systematic generator matrix characterization of linear codes with HWH $d_r(C) = n - k + r$ for $1 \leqslant r \leqslant k$. In this paper, we generalize this characterization to all linear codes in terms of their HWH. We also characterize NMDS and N^2-MDS codes in terms of their systematic generator matrices. Codes meeting the generalized Greismer bound are also characterized in terms of their systematic generator matrices. The HWH of dually defective codes meeting the generalized Greismer bound is also reported.

The contents of this paper is organized as follows: In Section 2 we discuss the systematic check matrix characterization of an arbitrary linear code with a specified HWH. We also apply this systematic matrix characterization to A^{μ}-MDS codes, NMDS codes and N^2-MDS codes. In Section 3 we define a class of codes which we call dually defective codes and discuss the matrix characterization of these codes. The codes meeting the generalized Greismer bound are also characterized in terms of their generator matrix in Section 3. Also we present the conditions for dually defective codes to meet the generalized Greismer bound.

2. Systematic check matrix characterization in terms of HWH

The following result from [19] gives a check matrix characterization of the HWH for a linear code. We will use this result to prove our main result presented in Theorem 2.

Proposition 1. *An $[n, k]$ code C with MDS-rank η and a check matrix H has the HWH $\{d_i(C)|1 \leqslant i \leqslant k\}$ iff*

1. *For every i, every $(d_i(C) - 1)$ columns of H have rank greater than or equal to $(d_i(C) - i)$.*
2. *There exist $d_i(C)$ columns of H for every i, with rank equal to $(d_i(C) - i)$.*

For $[n, k]$ linear MDS codes or equivalently for linear codes with the HWH $d_i(C) = n - k + i$ for all $i = 1, 2, \dots, k$ the systematic check matrix characterization assumes that the check matrix is in the form $[I|P]$ where I is the $n - k \times n - k$ identity matrix. For MDS codes any k coordinate positions can be taken as information symbols and the remaining co-ordinate positions can be taken as check locations. There always exists such a systematic check matrix. This need not be possible for arbitrary linear codes in general. However, with suitable column permutations on the check matrix one can obtain a check matrix in the systematic form $[I|P]$ for any linear code. In the strict sense the resulting matrix is a check matrix for an equivalent code obtained by the coordinate permutation corresponding to the column permutations that led to the systematic form. In this paper we will always assume that the code under consideration has a check matrix in the systematic form $[I|P]$ with the understanding that we are dealing with the corresponding equivalent code. Then the conditions on P should be taken as conditions on the submatrix of the original code that corresponds to check positions. With this understanding we present our main result in the following theorem.

Theorem 2. *An $[n, k]$ code with parity check matrix $H = [I|P]$ and MDS-rank η has the HWH, $\{d_i(C) = n - k + i - \mu_i(C)\}$ where $\mu_i(C) \geqslant 0$ for $1 \leqslant i \leqslant k$ iff the following conditions are satisfied*:

1. *For $i < g \leqslant \min\{d_i(C) - 1,\ k\}$, every $g + \mu_i(C) + 1 - i \times g$ submatrix of P have rank $\geqslant g - i + 1$.*
2. *There exists a g, $i < g \leqslant \min\{d_i(C), k\}$, such that the rank of some $g + \mu_i(C) - i \times g$ submatrix of P is $g - i$.*

Proof. We establish equivalence between the conditions of Proposition 1 and Theorem 2. In the Part (i) of the proof we prove that the conditions of Proposition 1 imply those of Theorem 2 and in Part (ii) we prove the converse.

Part (i): Let $d_i(C) = n - k + i - \mu_i(C)$. From the condition 1 of Proposition 1, we know that every $n - k + i - 1 - \mu_i(C)$ columns of H have rank greater than or equal to $n - k - \mu_i(C)$. Choose a set of $n - k + i - 1 - \mu_i(C)$ columns of H. If all these columns are from the P submatrix then the $n - k \times n - k + i - 1 - \mu_i(C)$ submatrix of P has rank greater than or equal to $n - k - \mu_i(C)$. This leads to the condition 1 of Theorem 2. Consider the case where g columns are from the P submatrix. Then $n - k + i - 1 - \mu_i(C) - g$ columns are from the I submatrix of H. The rank of these $n - k + i - 1 - \mu_i(C) - g$ columns is $n - k + i - 1 - \mu_i(C) - g$. Hence the g columns from the P submatrix of H have rank greater than or equal to $g - i + 1$. Therefore every $g + \mu_i(C) - i + 1 \times g$ submatrix has rank greater than or equal to $g - i + 1$, where $i \leqslant g \leqslant \min\{d_i(C) - 1, k\}$. The range of g follows from $g \leqslant k$, $g + \mu_i(C) - i + 1 \leqslant n - k$ and $g - i + 1 \geqslant 0$.

Using the condition 2 of Proposition 1 we prove the second condition of our theorem as follows: Choose a set of $n - k + i - \mu_i(C)$ columns of H with rank $n - k - \mu_i(C)$. If all these columns are from P, then we have an $n - k \times n - k + i - \mu_i(C)$ submatrix of P with rank $n - k - \mu_i(C)$. Let g' of the $n - k + i - \mu_i(C)$ columns be from the P submatrix. Then $n - k + i - \mu_i(C) - g'$ columns are from I. These $n - k + i - \mu_i(C) - g'$ columns of the I submatrix have rank equal to $n - k + i - \mu_i(C) - g'$. Therefore in the set of g' columns from the P submatrix we have a $g' - i + \mu_i(C) \times g'$ submatrix of P with rank $g' - i$.

Part (ii): To prove the condition 1 of Proposition 1, pick any $g + \mu_i(C) - i + 1 \times g$ submatrix from P. Take a set of $n - k + i - 1 - \mu_i(C) - g$ columns from the I submatrix such that these columns have zeros in the $g + \mu_i(C) - i + 1$ rows associated with the $g + \mu_i(C) - i + 1 \times g$ submatrix of P. The $g + \mu_i(C) - i + 1 \times g$

submatrix has rank $\geqslant n - k - \mu_i(C)$. Since we have chosen an appropriate set of columns from the I submatrix of H, the $n - k + i - \mu_i(C)$ columns of H has rank $\geqslant n - k - \mu_i(C)$ (which is the sum of the ranks of the columns from the I submatrix and the P submatrix of H).

The condition 2 in Proposition 1 is obtained from the second condition of our theorem as follows: From our second condition, it follows that there exists a $g' + \mu_i(C) - i \times g'$ submatrix of P with rank equal to $g' - i + 1$. Choose $n - k + i - \mu_i(C) - g'$ columns from the I submatrix of H such that these columns have zeros in the $g' + \mu_i(C) - i$ rows associated with the $g' + \mu_i(C) - i \times g'$ submatrix of P. These columns have rank $n - k + i - \mu_i(C) - g'$. Thus we have $n - k + i - \mu_i(C)$ columns of H with rank $n - k - \mu_i(C)$ which is equal to the sum of the ranks of columns from the I submatrix and the P submatrix. This completes the proof. □

The well known systematic check or generator matrix characterization of MDS codes is obtained from Theorem 2 by putting $\mu_1(C) = 0$. From the second condition of the theorem we see that every $g \times g$ submatrix of the P submatrix has rank g. This systematic generator matrix characterization is used for constructing MDS codes in [20].

Now we apply our systematic parity matrix characterization of Theorem 2 to other well known codes which are close to achieving the generalized Singleton bound.

Corollary 3. *An $[n, k]$ code with parity check matrix $[I|P]$ is NMDS iff*

- *For $1 < g \leqslant min\{n - k - 1, k\}$ every $g + 1 \times g$ submatrix of P has rank $\geqslant g$.*
- *For $1 < g \leqslant min\{n - k, k\}$ there exists a $g \times g$ submatrix of P with rank equal to $g - 1$.*
- *For $1 < g \leqslant min\{n - k, k - 1\}$ every $g \times g + 1$ submatrix of P has rank g.*

Corollary 3 has been reported in [22] as an independent result with a different proof.

Lemma 4 [18]. *If $k > q > 3$ and $n < 2q - 1 + k$ then every $[n, k, n - k - 1]$ code C over F_q is an N^2-MDS code.*

Corollary 5. *For $k > q > 3$ and $n > 2q - 1 + k$, a code with the systematic parity check matrix $[I|P]$ is N^2-MDS iff every $g + 2 \times g$ submatrix of P has rank $\geqslant g$.*

Proof. This corollary is obtained by combining following two results: (i) for the given range of n and k any code with $d_1(C) = n - k - 1$ is a N^2-MDS code (this follows from Lemma 4) and (ii) the systematic matrix characterization given in Theorem 2. We substitute $\eta_1 = 2$ in Theorem 2 to get this characterization of N^2-MDS code. □

Corollary 6. *If $n > k + q$ the $[n, k]$ code with systematic parity check matrix $H = [I|P]$ is NMDS iff every $g + 1 \times g$ submatrix of P has rank g.*

Proof. This corollary is obtained by using the fact that for the given range of k and n any $n-k$ code is a NMDS code. Therefore, all we need to show is $d_1(C)=n-k$. We know that $d_1(C)=n-k$ iff every $n-k-1$ columns of H are linearly independent and there exist $n-k$ linearly dependent columns. Therefore it follows that every $g+1 \times g$ submatrix of the P submatrix has rank g. $\square$

The above result is useful in decoding codes for the erasure channel [11].

Proposition 7. *A code C is dually A^{μ}-MDS iff it's MDS-rank is μ.*

Proof. The MDS-rank is μ implies that $d_{\mu+1}(C)=n-k+\mu+1$ and $\mu+1$ is the first i such that $d_i(C)=n-k+i$ for $1 \leqslant i \leqslant k$. Therefore $d_{\mu+1}(C)-d_{\mu}(C) \geqslant 2$. It follows that $d_1(C^{\perp})=k+1-\mu$. Hence the code is dually A^{μ}-MDS. $\square$

Proposition 8. *For an A^{μ}-MDS code C with parity check matrix H the following conditions hold*

1. *Every $n-k-\mu$ columns of H are linearly independent.*
2. *There exists $n-k+1-\mu$ linearly dependent columns of H.*

Proof. We have $d_1(C)=n-k+1-\mu$. Therefore there exist $n-k+1-\mu$ linearly dependent columns in the H matrix. Also every $n-k-\mu$ columns of H are linearly independent. $\square$

The following corollary characterizes dually A^{μ}-MDS codes.

Corollary 9. *An $[n,k]$ code C with systematic generator matrix $[I|P]$ is dually A^{μ}-MDS iff every $g+\mu$, g and $g \times g+\mu$ submatrix of the P submatrix has rank greater than or equal to g.*

Proof. For an A^{μ}-MDS code to be dually defective we know that its MDS-rank has to be μ. The proof follows from Theorem 2 by taking $\mu_1=\mu$. For A^{μ}-MDS codes we specify only $d_1(C)$ and other Hamming weights are arbitrary. Therefore we need to ensure only $d_1(C)$. Hence the result follows from Theorem 2. $\square$

3. Matrix characterization of dually defective codes and codes meeting the generalized Greismer bound

We begin with

Definition 3. The defect vector of an $[n,k]$ code C with MDS-rank η is defined as the ordered set $\{\mu_1(C), \mu_2(C), \ldots, \mu_{\eta}(C), \mu_{\eta+1}(C)\}$, where $\mu_i(C)=n-k+i-d_i(C)$. (Note that $\mu_{\eta+1}(C)$ is equal to zero.) A code is called dually defective if the defect vector is same for the code and its dual. The difference set of the defect vector of an $[n, k]$ code C with MDS-rank η is the ordered set $\{(\mu_1(C) - \mu_2(C)), (\mu_2(C) - \mu_3(C)), \ldots, (\mu_{\eta-1}(C) - \mu_\eta(C)), (\mu_\eta(C) - \mu_{\eta+1}(C))\}$.

Example 1. Consider an N^2-MDS code. The defect vector is $\{2, 1, 0\}$ and the difference set is the ordered set $\{(2 - 1) = 1, (1 - 0) = 1\}$. Therefore between the first three Hamming weights of the code there is a gap. From (3) we see that the HWH of the dual code is $\{d_1(C^\perp) = k - 1, d_2(C^\perp) = k + 1, d_3(C^\perp) = k + 3, \ldots, d_{n-k}(C^\perp) = n\}$. Therefore the defect vector of the dual code $C^\perp$ is $\{2, 1, 0\}$ and hence C is dually defective.

Example 2. Consider a code C with $\mu_1(C) = 8$. For C to be dually defective it's MDS-rank should be 9 and $(\mu_8(C) - \mu_9(C)) > 1$. Let the code have the defect vector as $\{8, 7, 7, 7, 6, 5, 4, 1, 0\}$. The difference set is $\{1, 0, 0, 1, 1, 1, 3, 1\}$. Since $\mu_2(C) = \mu_3(C) = \mu_4(C) = 7$ and $\mu_5(C) = 6$ for the code to be dually defective $(\mu_8(C) - \mu_9(C)) = 1$ and $(\mu_7(C) - \mu_8(C)) = 3$. Similarly since $\mu_5(C) = 6$ and $\mu_6(C) = 5$ we have $(\mu_6(C) - \mu_5(C)) = 1$.

Now we proceed to study the properties of the defect vector.

Lemma 10. *For an $[n, k]$ code C with MDS-rank η, we have $\mu_1(C) \geqslant \mu_2(C) \geqslant \ldots \mu_\eta(C) \geqslant \mu_{\eta+1}(C)$.*

Proof. This result can be proved from the monotonicity of the HWH. We know that $d_{i+1}(C) > d_i(C)$, i.e. $n - k + i + 1 - \mu_{i+1}(C) > n - k + i - \mu_i(C)$. Simplifying the inequality we get $\mu_i(C) + 1 > \mu_{i+1}(C)$. □

Lemma 11. *If C is an $[n, k]$ code with MDS-rank η, then $\sum_{i=1}^{\eta}(\mu_i(C) - \mu_{i+1}(C)) = \mu_1(C)$.*

Proof. Since the MDS-rank of C is η we have $\mu_{\eta+1}(C) = 0$. In the sum all terms except $\mu_1(C)$ and $\mu_{\eta+1}(C)$ cancel out. Therefore the sum is equal to μ_1. □

The class of dually defective codes include MDS codes, $NMDS$ codes, N^2-MDS codes and self dual codes. The following lemma gives the conditions for a dually A^μ-MDS code to be dually defective. The proof of the following lemma follows from Theorem 5.27 in [17].

Lemma 12. *Let C be an $[n, k, d]$ dually A^μ-MDS code over F_q with $s \geqslant 2$. If $n \leqslant s(q + 1) - 1 + k$ and $2 \leqslant s \leqslant q$. Then C is a dually defective code with $d_i(C) = n - k + 1 - s$ and $d_i(C) = n - k + i - 1$ for $2 \leqslant i \leqslant s$.*

The following proposition gives a matrix characterization of dually defective codes.

Theorem 13. *An $[n, k]$ code C with MDS-rank η and systematic generator matrix $[I|P]$ is dually defective iff the following conditions are satisfied*

1. *For $i < g \leqslant min\{d_i(C) - 1, k\}$, every $g + \mu_i(C) + 1 - i \times g$ and $g \times g + \mu_i(C) + 1 - i$ submatrix of P has rank $\geq g - i + 1$.*
2. *There exists a g, $i < g \leqslant min\{d_i(C), k\}$, such that the rank of every $g - i + \mu_i(C) \times g$ and $g \times g - i + \mu_i(C)$ submatrix of P is $g - i$.*
3. *For $1 < g \leqslant min\{n - k, k - \eta\}$ every g, $g + \eta$ and $g + \eta \times g$ submatrix of P has rank g.*

Proof. Let us assume that C is dually defective. Then $d_i(C) = n - k + i - \mu_i(C)$ and $d_i(C^{\perp}) = k + i - \mu_i(C)$. Therefore, from Theorem 2 every submatrix of $-P^{\mathrm{T}}$, where P^{T} denotes the transpose of P, of the type $g + \mu_i(C) - i \times g$ has rank $\geqslant g - i + 1$ since $d_i(C) = n - k + i - \mu_i(C)$ [22]. For dually defective code $d_i(C^{\perp}) = k + i - \mu_i(C)$. Therefore every $g + \mu_i(C) - i \times g$ submatrix of P matrix has rank $\geqslant g - i + 1$. Therefore it follows that every $g + \mu_i(C) + 1 - i \times g$ and $g \times g + \mu_i(C) + 1 - i$ submatrix of P has rank $\geqslant g - i + 1$.

The fact that $d_i(C) = n - k + i - \mu_i(C)$ and $d_i(C^{\perp}) = k + i - \mu_i(C)$ leads to the condition that there exist $g - i + \mu_i(C) \times g$ and $g \times g - i + \mu_i(C)$ submatrices of P such that the rank is $g - i$. The third condition of the theorem follows similarly. Establishing that the code is dually defective assuming the three conditions is straight forward. □

Proposition 14 (The generalized Greismer bound [1]). *If C is an $[n, k, d]$ code over F_q, then $d_r(C) \geqslant \sum_{i=0}^{r-1} \left\lceil \frac{d}{q^i} \right\rceil$ for $1 \leqslant r \leqslant k$.*

Theorem 15. *If an $[n, k, d]$ code C over F_q meets the generalized Greismer bound then the code will have a generator matrix whose structure is as follows*:

$$\begin{array}{ccccc} d_1(C) & d_2(C) - d_1(C) & d_3(C) - d_2(C) & \dots & d_k(C) - d_{k-1}(C) \\ \overbrace{\qquad} & \overbrace{\qquad} & \overbrace{\qquad} & & \overbrace{\qquad} \\ * & 0 & 0 & \dots & 0 \\ \bigstar & * & 0 & \dots & 0 \\ \bigstar & \bigstar & * & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \bigstar & \bigstar & \bigstar & \dots & * \end{array} \tag{4}$$

where $*$ denotes any non-zero element of F_q, $\bigstar$ denotes any element of F_q and $(d_{i+1}(C) - d_i(C))$ denotes the number of columns with the structure as shown below it.

Proof. As the code meets the generalized Greismer bound for all the values of $d_r(C)$ we can construct the matrix in the proposition as follows. Since the minimum distance is d we can choose a generator matrix with the first row having d consecutive non zeros followed by $n-d$ zeros. Next, we know that $d_2(C) = d + \left\lceil \frac{d}{q} \right\rceil$. Therefore, we can choose a second row such that the first d elements can be any element from the field followed by $\left\lceil \frac{d}{q} \right\rceil$ non zero elements of the field. Thus, we have constructed a two dimensional subcode with support $d_2(C)$ meeting the generalized Greismer bound. It is possible to construct rows with a sequence of zeros and non-zeros as we can permute the columns of the generator matrix without affecting the weight distribution of the code. We can repeat the above construction for all $d_i(C)$ where $3 \leqslant i \leqslant k$. Thus, we can construct the generator matrix for a code meeting the Greismer bound as given in the proposition. $\square$

From this matrix characterization we can obtain the systematic matrix characterization of codes meeting the generalized Greismer bound by elementary row operations and permutations of the columns.

Theorem 16. *Consider an* $[n,\ k]$ *code C of MDS rank* η *with defect vector* $\{\mu_1(C), \mu_2(C), \ldots, \mu_\eta(C), \mu_{\eta+1}(C)\}$ *with* $\mu_{\eta+1}(C) = 0$ *and the difference set between successive elements of the defect vector be* $\{(\mu_1(C) - \mu_2(C)), (\mu_2(C) - \mu_3(C)), \ldots, (\mu_i(C) - \mu_{i+1}(C)), \ldots, (\mu_\eta(C) - \mu_{\eta+1}(C))\}$. *If the difference set is not the all zero vector or the all one vector (i.e.,* $(0, 0, \ldots, 0)$ *or* $(1, 1, \ldots, 1)$*) then C is not a dually defective code meeting the generalized Greismer bound.*

Proof. We need to consider the cases (i) $(\mu_1(C) - \mu_2(C)) > 1$ and (ii) $(\mu_1(C) - \mu_2(C) = \cdots = \mu_i - \mu_{i+1} = 0; (\mu_{i+1}(C) - \mu_{i+2}(C)) > 0$ for some i and show that in both these two cases dually defective code meeting the generalized Greismer bound do not exist.

Assume that a dually defective code C meeting the generalized Greismer bound exists with $(\mu_1(C) - \mu_2(C)) > 1$. Let $d_1(C) = n - k + 1 - \mu_1(C)$ and $d_2(C) = n - k + 2 - \mu_2(C)$. Let $\mu_1(C) - \mu_2(C) = \delta > 1$. For the code to be dually defective $d_{\eta+1}(C) = n - k + \eta + 1, d_\eta(C) = n - k + \eta - 1$ and $d_{\eta-i}(C) = n - k + \eta - i - 1$ for $1 \leqslant i \leqslant (\delta - 1)$. Since the code also meets the generalized Greismer bound we have $d_{\eta+1}(C) - d_\eta(C) = \left\lceil \frac{d_1(C)}{q^\eta} \right\rceil$. Further since the code is assumed to be dually defective we have $\left\lceil \frac{d_1(C)}{q^\eta} \right\rceil = 2$. For codes meeting generalized Greismer bound $d_\eta(C) - d_{\eta-1}(C) = \left\lceil \frac{d_1(C)}{q^{\eta-1}} \right\rceil$. This difference must be equal to 1 for the code to be dually defective. But this is not possible since $\left\lceil \frac{d_1(C)}{q^\eta} \right\rceil = 2$. Therefore C can not meet the generalized Greismer bound.

Assume that a dually defective code meeting the generalized Greismer bound exists with the following difference set $(\mu_j - \mu_{j+1}) = 0$ for all $1 \leqslant j \leqslant i$ and $(\mu_i - \mu_{i+1}) \geqslant 1$. For dually defective code we have $(\mu_\eta - \mu_{(\eta+1)} = i$ and $(\mu_{(\eta-1)} - \mu_\eta) \leqslant 1$. Further since the code is assumed to meet the generalized Greismer bound we have $d_{\eta+1}(C) - d_\eta(C) = \left\lceil \frac{d_1(C)}{q^\eta} \right\rceil$. This difference $d_{\eta+1}(C) - d_\eta(C) = (i + 1)$.

Moreover $d_\eta(C) - d_{\eta-1}(C) = \left\lceil \frac{d_1(C)}{q^{\eta-1}} \right\rceil \leqslant 2$. But this is not possible since $\left\lceil \frac{d_1(C)}{q^{\eta}} \right\rceil \leqslant \left\lceil \frac{d_1(C)}{q^{(\eta-1)}} \right\rceil$. This completes the proof. □

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