Single-symbol and Double-symbol Decodable STBCs for MIMO Fading Channels

A Thesis

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Abstract

For a communication system operating in a fading environment, receive antenna diversity is a widely applied technique to reduce the detrimental effects of multi path fading. But as this makes the receiver complex and costly, is generally used exclusively at the base station. Recently a combination of transmit and receive diversity has been suggested as a more effective solution to the above problem resulting in a MIMO (Multiple Input Multiple Output) fading channel which has been shown to have the potential of supporting large data rates as compared to SISO (Single Input Single Output) channels. Signal design for such a MIMO channel is known as Space-Time Codes (STC), which take advantage of both space and time diversity available with time varying fading by coding both in space and time.

Space-Time Block Codes (STBC) are promising in this regard as there exist fast decoding algorithms like sphere decoding, etc. Further there exist classes of codes among STBCs that have very simple decoding properties like STBCs from orthogonal designs (ODs) and quasi-orthogonal designs (QODs). This simplicity of decoding for ODs and QODs is due to the fact that the Maximum Likelihood (ML) decoding metric can be written as sum of squares of terms each depending on only one variable in the case of OD and two variables in the case of QODs; as a result each variable can be decoded separately for ODs and pairs of variables for QODs. However the rates of ODs and QODs are restrictive; resulting in search of other codes. It is in this context that STBCs are investigated in this thesis where we classify and characterize all STBCs that have decoding properties similar to ODs and QODs; calling them "single-symbol decodable (SD)" and "double-symbol decodable (DSD)" respectively, for quasi-static and fast-fading channels when channel state information (CSI) is known at the receiver.

The characterization of single-symbol decodable STBCs shows that full-rank single-symbol decodable designs can exist outside the well-known Generalized Linear Processing

Complex Orthogonal Designs. In particular, we present single-symbol decodable, full-rank, rate 1, space-time block codes (STBCs) for 2 and 4 antennas that are not-obtainable from GLCODs.

The characterization of single-symbol decodable STBCs proceeds in two steps: first, we characterize all linear STBCs, that allow single-symbol ML decoding (not necessarily full-diversity) over quasi-static fading channels-calling them single-symbol decodable designs (SDD). The class SDD includes GLCOD as a proper subclass. Among the SDD those that offer full-diversity, called Full-rank SDD (FSDD), are characterized. For square FSDD, complete classification and construction of maximal rate designs are presented. As a consequence, we show that full-rank, rate-one, square SDD exist only for 2 and 4 transmit antennas and GLCODs are not maximal rate FSDD except for $N = 2$.

For non-square FSDD we present a class of non-GLCOD, STBCs called Generalized Co-ordinate Interleaved Orthogonal Designs (GCIODs). Construction of various high rate (>1/2) designs within this class of codes is presented and the coding gain and maximum mutual information (MMI) of these codes is evaluated and compared with known STBCs. We then show that the class of GLCODs and GCIODs arise naturally when all maximal SNR STBCs are characterized.

A characterization of all double-symbol decodable STBCs is then presented, in particular we present a double-symbol decodable, full-rank, rate 1, STBCs for 8 antennas. Next, we propose designs for fast-fading channels and derive maximal rates of single, double symbol decodable designs for fast-fading channels. Of particular interest is the uniqueness of CIOD for 2 Tx. which has single-symbol decodability for both quasi-static and fast-fading channels.

It turns out that co-ordinate interleaving of complex constellations is an effective tool to improve the diversity performance of MIMO fading channels without accruing any complexity penalty at the receiver.

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Abbreviations and Notations

MIMO	multiple-input, multiple-output
MISO	multiple-input, single-output
SIMO	single-input, multiple-output
SISO	single-input, single-output
TX,Tx	transmit antennas
RX,Rx	receive antennas
BER	bit error rate
SER	symbol error rate
ML	maximum likelihood
SNR	signal to noise ratio
PAR	peak to average ratio
CSI	channel state information
MMI	maximum mutual information
STC	space-time codes
STTC	space-time trellis codes
STBC	space-time block codes
QAM	quadrature amplitude modulation
PSK	phase shift keying
OD	orthogonal design
QOD	quasi-orthogonal design
GQOD	generalized QOD
LCOD	linear complex orthogonal design
GLCOD	generalized linear complex orthogonal design

Abbreviations (continuation)

LROD	linear real orthogonal design
GLROD	generalized linear real orthogonal design
RFSDD	restricted full-rank single-symbol decodable design
QCRD	quasi-complex restricted design
UFSDD	unrestricted full-rank single-symbol decodable design
CIOD	co-ordinate interleaved orthogonal design
GCIOD	generalized co-ordinate interleaved orthogonal design
ACIOD	Asymmetric co-ordinate interleaved orthogonal design
CPD	co-ordinate product distance
GCPD	generalized CPD
SD	single-symbol decodable
SDD	single-symbol decodable design
DSD	double-symbol decodable
DSDD	double-symbol decodable design
FSD	full-rank SD
FSDD	full-rank SDD
FDSDD	full-rank DSDD
NLC	Non-reducible lattice constellation
MZD	minimum zeta-distance
GMZD	generalized MZD
ECoM	extended codeword matrix
EChM	extended channel matrix
iid	independent, identically distributed
NLC	non-reducible lattice constellation

Notations

$\triangleq$	variable definition
$\mathbb{C}$	the field of complex numbers
$\mathbb{R}$	the field of real numbers
$\mathbf{j}$	$\sqrt{-1}$
$\mathbb{C}^{L\times N}$	the set of all $L \times N$ complex matrices
x_I,Re(x)	real part of a complex number x
x_Q,Im(x)	imaginary part of a complex number x
x^*	conjugate of a complex number x
$\|x\|$	absolute value of a real number x
$CPD(x,y)$	the CPD between two complex numbers x and $y =\|x_I - y_I\|\|x_Q - y_Q\|$
$A^{\mathcal{T}}$	transpose of a matrix A
$A^{\mathcal{H}}$	hermitian of a matrix A
$\det(A), \|A\|$	determinant of a matrix A
$\mathrm{tr}\,(A)$	trace of a matrix A
$\|A\|_+$	product of non-zero eigen values of a matrix A
$\|\|A\|\|$	Frobonius norm of a matrix A
A^{-1}	Inverse of a matrix A
I_N	$N \times N$ identity matrix
$B(S,S')$	difference of the matrices S, S', i.e., $S - S'$
$\Gamma(x)$	Gamma function
$\|S\|$	cardinality of a set S
$(M)_N$	M mod N
$P(E)$	probability of an event E
$H\,(N) - 1$	number of matrices in a Hurwitz family of order N
$\mathrm{E}\,[X]$	expected value of a random variable X

Chapter 1

Introduction

Wireless mobile communications is characterized by a very lossy and dispersive transmission medium, the *fading* channel, that suffers from extreme random fades. In designing communication systems for the fading channel, the emphasis is on minimizing the effect of channel fluctuations. *Diversity* as a technique for minimizing the detrimental effects of channel fluctuations, has been known to the wireless community since decades. A fundamental requirement of the diversity method is that a number of independent transmission paths be available to carry the message. Independent paths can be obtained by coding in time-like repeatedly sending the same signal; resulting in *time diversity*. Time-diversity results in transmission delay and bandwidth loss. Significantly, time-diversity depends on the mobility of the antennas to provide independent paths and as such is useless when both the receiver and transmitter are stationary [1, chap. 5]. Alternately coding/repetition can be in frequency, wherein the same message is transmitted over different frequencies. This results in *frequency diversity*. But again, frequency diversity results in loss of bandwidth. Polarization and angle diversity have also been proposed but in both the cases the number of diversity paths are limited to two and three respectively.

Antenna or space diversity was and is favored for mobile radio for a variety of reasons; in particular, it does not require additional bandwidth. Figure 1.1 presents a graphical presentation of space diversity, with both the transmitter and receiver being equipped with multiple antennas.

Till a decade ago, antenna diversity was synonymous with receive diversity, wherein multiple antennas were used for reception. The problem with receive diversity for mobile communications is that the receive antennas had to be sufficiently apart so that the signals received at each antenna undergoes independent fade. While this is easily implemented

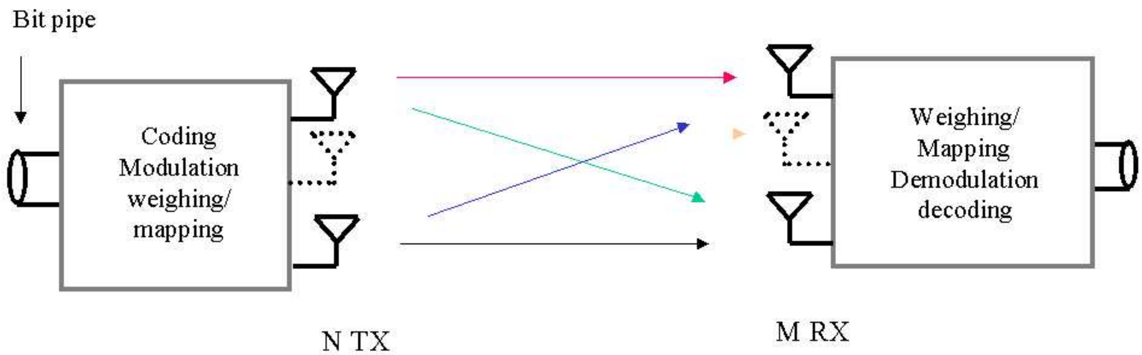


Figure 1.1: Diagram of a MIMO wireless transmission system.

at the base-station, since the mobile unit needs to be small in size it becomes costly to implement this in the mobile unit[2]. In order to reap the benefits of antenna diversity in the down-link (base-station to mobile) transmit diversity and/or a combination of transmit and receive diversity, termed otherwise as a multiple-input multiple-output (MIMO) link, was recently considered, wherein both the receiver and transmitter are equipped with multiple antennas (TX and RX respectively). A key concept in space diversity is the *diversity order* which is defined as the number of decorrelated spatial branches available at the receiver.

In a very short period of about five years MIMO has emerged as one of the most significant technical breakthroughs of modern communication and is poised to penetrate commercial wireless products and networks. In fact the Alamouti scheme is currently a part of W-CDMA and CDMA-2000 standards [2]. The reasons due to which MIMO has and is generating so much interest is many-fold. A properly designed MIMO system not only leads to improved quality of service (Bit Error Rate or BER) but also improved data rate (bits/sec) and therefore revenues of the operator (as is clear from the absolute gains in terms of capacity, reviewed in Section 1.1) . This prospect of performance improvement at no cost of extra spectrum is largely responsible for success of MIMO as a new topic of research. This thesis deals with one of the many open-problems in the area of signal design for MIMO channels; that of achieving simple decodability while retaining the diversity benefits, that have come to fore in these five years of active research.

1.1 The MIMO System

Consider the MIMO system shown in Fig.1.1. A compressed digital source in the form of a binary data stream is fed to a transmitting block encompassing error control coding and mapping to a complex signal constellation (M-QAM, M-PSK). The coding produces several separate symbol streams (not necessarily independent) each of which is mapped onto one of the multiple TX antennas. Mapping may involve linear spatial weighing of antennas (as in the case of channel feedback) or space-time coding [3]. After upward frequency conversion, filtering and amplifying the signals are launched into the wireless domain. At the receiver the signals are captured by possibly multiple antennas and demodulation and demapping operations are performed to recover the message. The selection of coding and antenna mapping can vary a great deal depending upon the application and is primarily decided by factors such as receiver and transmitter complexity, prior channel knowledge among other factors. Here we are interested in exploring the absolute gains offered by a single-user MIMO system and its special cases the SISO, SIMO and MISO systems, in terms of Capacity when perfect CSI (Channel State Information) is available at the receiver.

Consider the traditional SISO system. In a flat-fading channel, the capacity is achieved with continuous Gaussian input and it is [2, 3],

$$C = E[\log_2\left(1+\rho|h|^2\right)] \text{ bits/sec/Hz} \tag{1.1}$$

where the expectation is over the channel realizations h and $\rho = \frac{P}{\sigma^2}$ is the average SNR expressed in terms of the total transmit power P and the average noise power σ^2.

For a matrix channel of N transmit and M receive antennas with continuous Gaussian input and Rayleigh fading the instantaneous capacity version of 1.1 generalizes to [3, 5, 6]

$$C(\mathbf{H}) = \log_2\left[\det\left(1+\frac{\rho}{N}\mathbf{H}\mathbf{H}^{\mathcal{H}}\right)\right] \text{ bits/sec/Hz} \tag{1.2}$$

where now ρ is the SNR for one receive antenna and $\mathbf{H}$ is the $N \times M$ channel matrix. In a SIMO system with M receive antennas the capacity is given by [3]

$$C(\mathbf{H}) = \log_2\left(1+\rho\sum_{i=0}^{M-1}|h_i|^2\right) \text{ bits/sec/Hz} \tag{1.3}$$

where h_i is the channel gain for i-th RX and for a MISO system with N Tx, we have [5, 6]

$$C(\mathbf{H}) = \log_2 \left(1 + \frac{\rho}{N}\sum_{i=0}^{N-1} |h_i|^2\right) \text{ bits/sec/Hz.} \tag{1.4}$$

The ergodic mean capacity is obtained by taking expectation on the expression 1.2 with respect to $\mathbf{H}$. Observe that the factor $\frac{\rho}{N}$ ensures a fixed total transmit power. The important point to observe [5, 6] is that in (1.2) C increases linearly with $\min(N, M)$. while there is a logarithmic increase in C with respect to M in (1.3) and with N in (1.4). The increase in capacity w.r.t. SISO is evident in all the three cases. Interestingly $C(\mathbf{H})$ is a random variable and averaging this over $\mathbf{H}$ leads us to the ergodic mean capacity. From the practical point of view, for a quasi-static channel, what is more useful is the outage capacity, C_p i.e. the capacity that is supported with a probability p defined by Outage capacity C_p: Channel capacity is higher than C_p for p% of the time.

$$Pr\left\{C(\mathbf{H}) > C_p\right\} = \int_{\mathbf{H}:C(\mathbf{H})=C_p}^{\infty} C(\mathbf{H}) f_{\mathbf{H}}(\mathbf{H})\, d\mathbf{H} = p$$

.

Fig. 1.2 gives the outages for various MIMO systems. Observe that the data rate is more than double for a 2×2 MIMO system at all SNRs as compared to a SISO system.

1.2 Space-time Codes

The following subsection describes the channel model, coding, decoding and the performance criteria for MIMO systems employing Space-Time Codes (STC). The primary difference between coded modulation (used for SISO, SIMO) and space-time codes is that in coded modulation the coding is in time only while in space-time codes the coding is in both space and time and hence the name. Space-time Codes (STC) can be thought of as a signal design problem at the transmitter to realize the capacity benefits summarized in the previous section, though, several developments towards STC were presented in [7, 8, 9, 10, 11] which combine transmit and receive, much prior to the results on capacity. Formally, a thorough treatment of STCs were first presented in [12] in the form of trellis codes (space-time trellis codes (STTC)) along with appropriate design and performance criteria, a brief review of which is given in subsection 1.2.1 and 1.2.2 gives a review of STBCs from orthogonal designs.

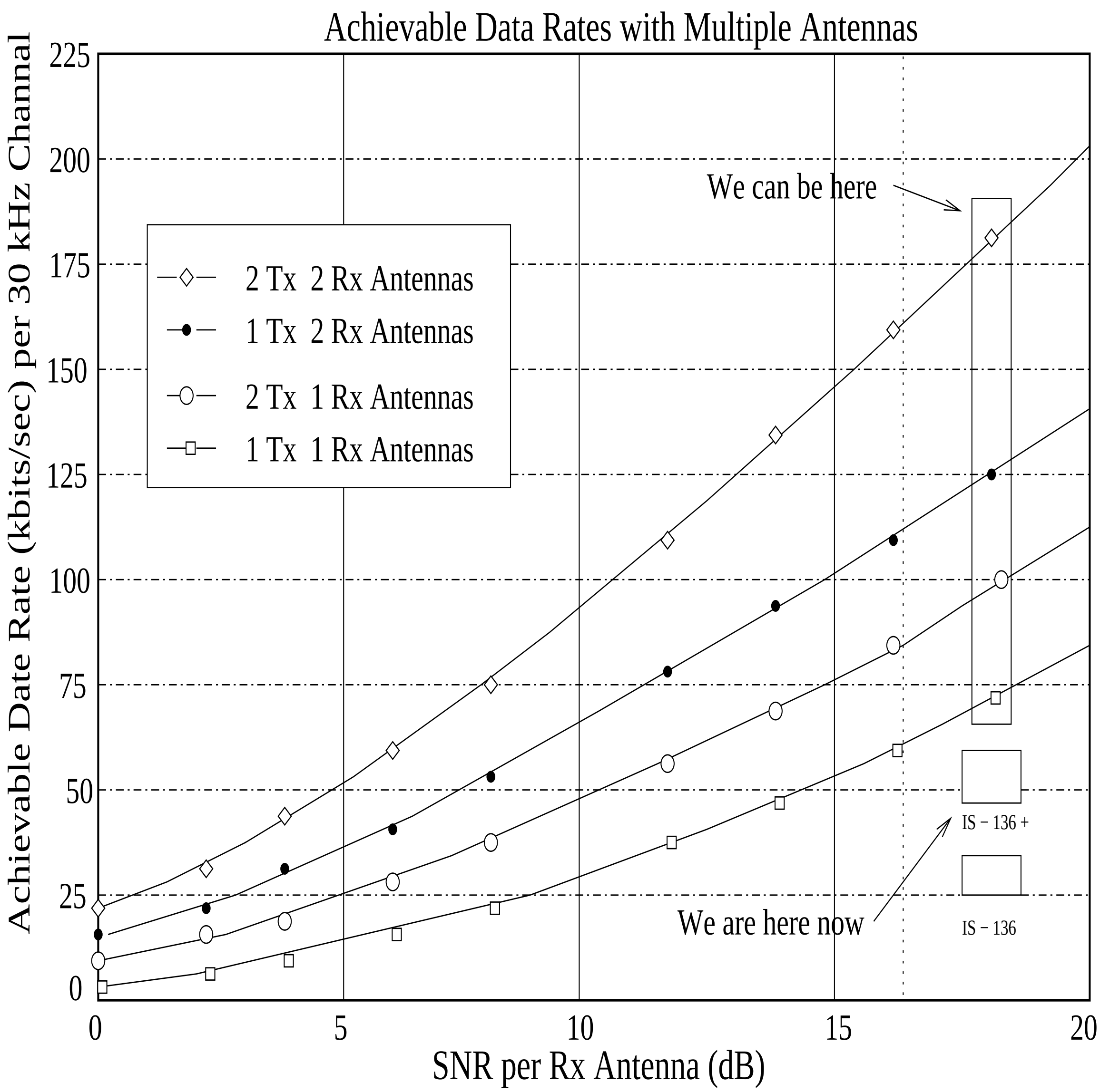


Figure 1.2: The outage Capacity, $C_{0.99}$, for various combinations of $N = 2, 1$ and $M = 2, 1$ MIMO systems in Rayleigh fading

1.2.1 The Channel Model and Performance Criteria for STC

Let the number of transmit antennas be N and the number of receive antennas be M. At each time slot t, the complex signals, s_{it}, $i = 0, 1, \cdots, N-1$ are transmitted from the N antennas simultaneously. Let $h_{ij} = \alpha_{ij} e^{\mathbf{j}\theta_{ij}}$ denote the path gain from the transmit antenna i to the receive antenna j, where $\mathbf{j} = \sqrt{-1}$. Assuming that the path gains are constant over a frame length $L \geq N$, $t = 0, \cdots, L-1$, the received signal v_{jt} at the antenna j at time t, is given by

$$v_{jt} = \sum_{i=0}^{N-1} s_{it} h_{ij} + n_{jt}, \quad j = 0, \cdots, M-1; \qquad t = 0, \cdots, L-1, \tag{1.5}$$

which in matrix notation is,

$$\mathbf{V} = \mathbf{SH} + \mathbf{N} \tag{1.6}$$

where, with $\mathbb{C}$ denoting the complex field,

$\mathbf{V} \in \mathbb{C}^{L\times M}$	is the received signal matrix,
$\mathbf{S} \in \mathbb{C}^{L\times N}$	is the transmission matrix (also referred as codeword matrix),
$\mathbf{N} \in \mathbb{C}^{L\times M}$	is the additive noise matrix, and
$\mathbf{H} \in \mathbb{C}^{N\times M}$	is the channel matrix.

The entries of $\mathbf{N}$ and $\mathbf{H}$ are complex Gaussian distributed with zero mean and unit variance and also are temporally and spatially white. Note that in $\mathbf{V}, \mathbf{S}$ and $\mathbf{N}$ time runs vertically and space runs horizontally and h_{ij} is the entry in the ith row and the jth column of $\mathbf{H}$. Recall that for a matrix A, $A^{\mathcal{H}}$ represents the Hermitian (conjugate transpose) of A, $A^{\mathcal{T}}$ represents the transpose of A and $|A|$ denotes the determinant of A. If the transmission power constraint is given by $E[tr\{\mathbf{SS}^{\mathcal{H}}\}] = LN$ then

$$\mathbf{V} = \sqrt{\frac{\rho}{N}}\mathbf{SH} + \mathbf{N} \tag{1.7}$$

where ρ is the SNR at each receiver. Assuming that perfect channel state information (CSI) is available at the receiver, the decision rule for ML decoding is to minimize the metric

$$\sum_{t=0}^{L-1} \sum_{j=0}^{M-1} \left| v_{jt} - \sum_{i=0}^{N-1} h_{ij} s_{it} \right|^2 \tag{1.8}$$

over all codeword matrices $\mathbf{S}$. In vector form we have

$$M(\mathbf{S}) \triangleq \min_{\mathbf{S}} \operatorname{tr}\left((\mathbf{V}-\mathbf{SH})^{\mathcal{H}}(\mathbf{V}-\mathbf{SH})\right) = \|\mathbf{V}-\mathbf{SH}\|^2. \tag{1.9}$$

This ML metric (1.8)/(1.9) results in exponential decoding complexity, because of the joint decision on all the complex symbols s_{it} in the matrix $\mathbf{S}$. Let $C_{\mathbf{S}}$ denote the set of all possible codeword matrices. Then the throughput rate R of such a scheme in bits/sec/Hz is $\frac{1}{L}\log_2(|C_{\mathbf{S}}|)$ and 2^{RL} metric calculations are required; one for each possible transmission matrix $\mathbf{S}$. Even for modest antenna configurations and rates this could be very large. For a STTC zero padding at the end of each frame is done and Vector Viterbi algorithm [12] is used to perform the minimization.

Towards describing the diversity and coding gains, let $P(\mathbf{S} \rightarrow \hat{\mathbf{S}})$ be the pairwise error probability of $\mathbf{S}$ being transmitted and wrongly decoded as $\hat{\mathbf{S}}$ for the receiver based on (1.8). The Chernoff bound on this error probability takes the form [12, eq. (9)]

$$P(\mathbf{S} \rightarrow \hat{\mathbf{S}}) \leq \frac{1}{\left\{|I + \frac{\rho}{4N}(\mathbf{S}-\hat{\mathbf{S}})(\mathbf{S}-\hat{\mathbf{S}})^{\mathcal{H}}|\right\}^M} \tag{1.10}$$

where I is the identity matrix. For large SNR's (large values of ρ), (1.10) can be written as

$$P(\mathbf{S} \rightarrow \hat{\mathbf{S}}) \leq \left(\frac{1}{\Lambda(\mathbf{S},\hat{\mathbf{S}})\frac{\rho}{4N}}\right)^{rM} \tag{1.11}$$

where r is the rank of the difference matrix, $B(\mathbf{S},\hat{\mathbf{S}}) = \mathbf{S}-\hat{\mathbf{S}}$, and

$$\Lambda(\mathbf{S},\hat{\mathbf{S}}) = |(\mathbf{S}-\hat{\mathbf{S}})(\mathbf{S}-\hat{\mathbf{S}})^{\mathcal{H}}|_+^{1/r} \tag{1.12}$$

where $|A|_+$ represents the product of the non-zero eigen values of the matrix A. The minimum of the ranks of all possible pairs of codeword matrices is referred to as the *diversity gain* and the minimum value of $\Lambda(\mathbf{S},\hat{\mathbf{S}})$ over all possible pairs of codeword matrices is referred to as the *coding gain.* The above pair-wise probability of error analysis for a quasi-static channel leads to

***Design Criteria for STC over quasi-static fading channels*[12]**

- *Rank Criterion:* In order to achieve full-diversity of NM, the matrix $\mathbf{B}(\mathbf{S},\hat{\mathbf{S}})$ has to be full-rank for any two codewords $\mathbf{S}$, $\hat{\mathbf{S}}$. If $\mathbf{B}(\mathbf{S},\hat{\mathbf{S}})$ has rank r, then a diversity gain of rM is achieved.

- *Determinant Criterion:* After maximizing the diversity gain the next criteria is maximizing the minimum of $\Lambda(\mathbf{S}, \hat{\mathbf{S}})$ over all pairs of distinct codewords.

1.2.2 Space-time block codes (STBC)

Definition 1.2.1 (STBC). An $N \times L$ (generally, $L \geq N$) space time block code (STBC) $\mathcal{C}$ consists of a finite number $|\mathcal{C}|$ of $N \times L$ matrices with entries from the complex field. If the entries of the codeword matrices take values from a complex signal set S or complex linear combinations of elements of S then the code is said to be over S. For quasi-static, flat fading channels a primary performance index of $\mathcal{C}$ is the minimum of the rank of the difference of any two codeword matrices, called the rank of the code. $\mathcal{C}$ is of full-rank if its rank is minimum of L and N and is of minimal-delay if $L = N$. The rate of the code in symbols per channel use is given by $\frac{1}{l} \log_{|S|}(|\mathcal{C}|)$.

The study of space-time block codes (STBC) started with the paper by Alamouti [16] and their subsequent generalization in [13] and development in [17, 18, 19, 15, 20, 21, 22, 23] using the theory of Orthogonal Designs (OD) [24, 25, 26, 27]. Many other STBCs like STBCs from Quasi-Orthogonal Designs (QOD) [28, 29, 30, 31, 32, 33, 34], STBCs using unitary precoders [35, 36, 37, 38] and other STBCs like [39, 40, 41] etc. were developed.

We first review the definition and important results of ODs and QODs before continuing our discussion.

Definition 1.2.2 (Generalized Linear Complex Orthogonal Design (GLCOD)[13]). A GLCOD[1] in k complex indeterminates $x_1, x_2, \cdots, x_k$ of size N and rate $\mathcal{R} = k/p$, $p \geq N$ is a $p \times N$ matrix Θ, such that

- the entries of Θ are complex linear combinations of $0, \pm x_i, \quad i = 1, \cdots, k$ and their conjugates.
- $\Theta^{\mathcal{H}}\Theta = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix whose entries are a linear combination of $|x_i|^2, i = 1, \cdots, k$ with all strictly positive real coefficients.

[1] In [13] it is called a Generalized Linear Processing Complex Orthogonal Design where the word "Processing" has nothing to be with the linear processing operations in the receiver and means basically that the entries are linear combinations of the variables of the design. Since we feel that it is better to drop this word to avoid possible confusion we call it GLCOD.

If k=n=p then Θ is called a Linear Complex Orthogonal Design (LCOD). Furthermore, when the entries are only from $\{\pm x_1, \pm x_2, \cdots, \pm x_k\}$, their conjugates and multiples of $\mathbf{j}$ then Θ is called a Complex Orthogonal Design (COD). STBCs from ODs are obtained by replacing x_i by s_i and allowing s_i to take all values from a signal set $\mathcal{A}$. A GLCOD is said to be of minimal-delay if $N = p$.

Actually, according to [13] it is required that $\mathbf{D} = \sum_{i=1}^{k} |x_i|^2 I$, which is a special case of the requirement that D is a diagonal matrix with the conditions in the above definition. In other words, we have presented a generalized version of the definition of GLCOD of [13]. Also we say that a GLCOD satisfies **Equal-Weights** condition if $\mathbf{D} = \sum_{i=1}^{k} |x_i|^2 I$.

The Alamouti scheme [16], which is of minimal-delay, full-rank and full-rate is basically the STBC arising from the size 2 COD.

The main results regarding GLCODs are summarized below:

- when $\mathcal{A}$ is a real constellation, then rate 1 square, real, Orthogonal Designs exist ONLY for N=2,4 and 8 [13].

- when $\mathcal{A}$ is a complex constellation, Square/Non-square GLCODs exist ONLY for N=2 [13, 20, 22].

- when $\mathcal{A}$ is a complex constellation, the maximal rates of Square GLCODs of size $N = 2^a b, b$ odd, satisfying Equal-Weights condition is [15]

$$\mathcal{R} = \frac{a+1}{N}.$$

  Observe that the maximal rate, when the GLCODs do not satisfy the Equal-Weights condition, is not known [22],

- the STBCs from GLCODs satisfying Equal-Weights condition are optimal SNR codes [17, 18, 19, 23].

In the case of QODs, several constructions are given in [28, 29, 31, 32, 30, 33, 34] but the fundamental questions of rate and existence of QODs has not been dealt with, to the best of our knowledge.

All the above STBCs from designs belong to the class of linear STBCs which we define now: Following the spirit of [15], by a linear STBC[2] we mean those covered by the following definition:

Definition 1.2.3 (Linear STBC). A linear design, S, is a $L \times N$ matrix whose entries are complex linear combinations of K complex indeterminates $x_k = x_{kI} + \mathbf{j}x_{kQ}$, $k = 0, \cdots, K-1$ and their complex conjugates. The STBC obtained by letting each indeterminate to take all possible values from a complex constellation $\mathcal{A}$ is called a linear STBC over $\mathcal{A}$. Notice that S is basically a "design"and by the STBC $(S, \mathcal{A})$ we mean the STBC obtained using the design S with the indeterminates taking values from the signal constellation $\mathcal{A}$. The rate of the code/design[3] is given by K/L symbols/channel use. Every linear design S can be expressed as

$$S = \sum_{k=0}^{K-1} x_{kI} A_{2k} + x_{kQ} A_{2k+1} \tag{1.13}$$

where $\{A_k\}_{k=0}^{2K-1}$ is a set of complex matrices called weight matrices of S.

Throughout the thesis, we consider linear STBCs only and also we use the word "linear design" and "linear STBC" interchangeably with the understanding that the signal set, $\mathcal{A}$, in $(S, \mathcal{A})$ is understood from context or will be specified when necessary. Linear STBCs can be decoded using simple linear processing at the receiver with algorithms like sphere-decoding [42, 43] which have polynomial complexity in, N, the number of transmit antennas. But STBCs from ODs and QODs stand out because of very simple (linear complexity in N) decoding. This is because the ML metric of (1.9) can be written as a sum of several square terms, each depending on at-most one variable for OD and at-most two variables for QOD. However, the rates of both ODs and QODs are restrictive; resulting in search of other codes that allow simple decoding similar to ODs and QODs. We call such codes "single-symbol decodable" and "double-symbol decodable" respectively. Formally

Definition 1.2.4 (Single-symbol Decodable (SD) and Double-symbol Decodable (DSD) STBC). A single-symbol decodable STBC of rate K/L in K complex indeterminates $x_k = x_{kI} + \mathbf{j}x_{kQ}$, $k = 0, \cdots, K-1$ is a linear STBC such that the ML decoding metric given by (1.9) can be written as a square of several terms each depending

[2] Also referred to as a Linear Dispersion (LD) code [39]

[3] Note that if the signal set is of size 2^b the throughput rate R in bits per second per Hertz is related to the rate of the design $\mathcal{R}$ as $R = \mathcal{R}b$.

on at most one indeterminate. If (1.9) can be written as a square of several terms each depending on at most two indeterminates, then the STBC is said to be double-symbol decodable.

Examples of single-symbol decodable STBCs are STBCs from Orthogonal Designs (ODs) and examples of double-symbol decodable STBCs are the QODs. To elucidate further, consider the Alamouti code

$$S(x_0, x_1) = \begin{bmatrix} x_0 & x_1 \\ -x_1^* & -x_0^* \end{bmatrix}$$

where x_0, x_1 are signal points from some complex constellation $\mathcal{A}$. That the Alamouti code is a SD STBC can be verified as follows,

$$\begin{aligned} M(S(x_0, x_1)) &= \min_{x_0, x_1 \in \mathcal{A}} \operatorname{tr}\left((\mathbf{V} - S\mathbf{H})^{\mathcal{H}}(\mathbf{V} - S\mathbf{H})\right) = \|\mathbf{V} - S\mathbf{H}\|^2 && (1.14) \\ &= \min_{x_1, x_2 \in \mathcal{A}} \left\{\|\mathbf{V} - S(x_0, 0)\mathbf{H}\|^2 + \|\mathbf{V} - S(0, x_1)\mathbf{H}\|^2\right\} - \operatorname{tr}\left(\mathbf{V}\mathbf{V}^{\mathcal{H}}\right) && (1.15) \\ &= \min_{x_0 \in \mathcal{A}} \left\{\|\mathbf{V} - S(x_0, 0)\mathbf{H}\|^2\right\} + \min_{x_1 \in \mathcal{A}} \left\{\|\mathbf{V} - S(0, x_1)\mathbf{H}\|^2\right\} && (1.16) \end{aligned}$$

and hence each variable x_0, x_1 can be detected separately (see Proposition 2.3.1 for more details).

1.3 Motivation, Overview and Scope of the Thesis

Despite enormous interest and research in STBCs from ODs and QODs, to the best of our knowledge the following fundamental questions related to STBCs from designs either remain unaddressed or are not answered fully:

1. Are STBCs from ODs the **only** single-symbol decodable linear STBCs?
2. Are single-symbol decodable codes necessarily full-rank also?
3. What are the maximal rates of single-symbol decodable linear STBCs, with or without full-rankness?
4. If they exist outside the domain of ODs, are all of them optimal SNR codes?
5. Are STBCs from QODs the **only** double-symbol decodable linear STBCs? What are the maximal rates of square QODs?

6. Are STBCs with single/double-symbol decodablity for quasi-static channel useful, or in some sense related to STBCs suitable, for fast-fading channels?

The above questions are also precisely the set of issues addressed and answered/settled almost completely in this thesis. This is achieved by way of characterizing the entire class of SD and DSD STBCs, without out taking into account their full-rankness and then bringing the full-rankness into consideration. This thesis also deals with the various aspects of these designs including the diversity and coding gain with various signal sets.

The contributions of this thesis are presented in the form of following five chapters

1. **Chapter 2:** Single-symbol Decodable, Full-diversity Linear STBCs[4]
2. **Chapter 3:** Generalized Co-ordinate Interleaved Designs[5]
3. **Chapter 4:** Characterization of Optimal SNR STBCs
4. **Chapter 5:** Double-symbol Decodable Designs[6]
5. **Chapter 6:** STBCs from Designs for Fast-Fading Channels[7]

an outline of each of which follows:

Chapter2: Single-symbol Decodable, Full-diversity Linear STBCs

Space-Time block codes (STBC) from Orthogonal Designs (OD) has been attracting wider attention due to their amenability for fast (single-symbol decoding) ML decoding, and full-rate with full-rank over quasi-static fading channels [16]-[23]. Unfortunately, for complex constellations full-rate, full-rank, square ODs exist only for 2 transmit antennas and for larger number of antennas one needs to pay in terms of reduced symbol-rate or rank or not having single-symbol decodability.

It is natural to ask, if there exist codes other than STBCs form ODs that allow single-symbol decoding? In this chapter, this question is answered in the affirmative by showing that full-rank, full-rate and single-symbol decodable designs can exist outside the well-known Generalized Linear Complex OD (GLCODs) [13]. We first characterize all linear STBCs, that allow single-symbol ML decoding (not necessarily full-diversity)

[4]Part of the results of this chapter are reported in the the publications [50, 52, 53, 55]
[5]Part of the results of this chapter are reported in the the publications [50, 51, 52]
[6]Part of the results of this chapter are reported in the the publications [53, 57]
[7]Part of the results of this chapter are reported in the the publications [54, 56]

over quasi-static fading channels-calling them single-symbol decodable designs (SDD). The class SDD includes GLCOD as a proper subclass. Among the SDD, those that offer full-diversity are called Full-rank SDD (FSDD).

We show that the class of FSDD comprises of (i) an extension of GLCOD which we have called Unrestricted Full-rank SDD (UFSDD) and (ii) a class of non-UFSDDs called Restricted Full-rank SDD (RFSDD)[8].

We then concentrate on square designs and show that the class of square FSDD comprises of (i) square GLCODs (a proper subset of UFSDDs) and (ii) square Co-ordinate Interleaved Orthogonal Designs (CIODs) a proper subset of RFSDDs.

The problem of deriving the maximal rate for square RFSDDs is then addressed and a constructional proof of the same is provided. It follows that, except for $N = 2$, square GLCODs are not maximal rate FSDD. These maximal rate square RFSDDs have a special property that they can be thought of as a combination of co-ordinate interleaving [44, 45, 46, 47, 48, 49, 55] and maximal rate square orthogonal designs. Consequently we call square RFSDDs as square co-ordinate interleaved orthogonal designs (CIODs). Finally, we show that full-rank, rate-1, square FSDD exist only for 2 and 4 transmit antennas: these are respectively the Alamouti scheme and the 2×2 and 4×4 schemes based on the Co-ordinate Interleaved Orthogonal Designs (CIODs).

Chapter 3: Generalized Co-ordinate Interleaved Designs

In this chapter, we generalize the construction of square RFSDDs given in Chapter 2, and give a formal definition for Co-ordinate Interleaved Orthogonal Designs (CIOD) and its generalization, Generalized Co-ordinate Interleaved Orthogonal Designs (GCIOD). This generalization is basically a construction of RFSDD; both square and non-square. We conjecture that this construction can realize maximal rate non-square RFSDDs. We then show that rate 1 GCIODs exist for 2, 3 and 4 transmit antennas and for all other antenna configurations the rate is strictly less than 1. Rate 6/7 designs for 5 and 6 transmit antennas, rate 7/9 designs for 7 transmit antennas, rate 3/4 designs for 8 transmit antennas, rate 7/11 designs for 9 and 10 transmit antennas and rate 3/5 designs for 11 and 12 transmit antennas are also presented. A construction of rate 2/3 GCIOD for $N \geq 8$ is then presented. The signal set expansion due to co-ordinate interleaving is then

[8]the word Restricted in RFSDD reflects the fact that the designs in this class can achieve full-diversity iff the signal set used satisfies a restriction and the word Unrestricted in UFSDD indicates that without any restriction on the complex constellation the codes in this class give full-rank.

highlighted and the coding gain of GCIOD is shown to be equal to generalized co-ordinate product distance (GCPD). A special case of GCPD, the co-ordinate product distance (CPD) is derived for lattice constellations. We then show that, for lattice constellations, GCIODs have higher coding gain as compared to GLCODs. Simulation results are also included for completeness. Finally, the maximum mutual information (MMI) of GCIODs is compared with that of GLCODs to show that, except for $N = 2$, CIODs have higher MMI. In short, this chapter shows that, except for $N = 2$ (the Alamouti code), CIODs are better than GLCODs in terms of rate, coding gain and MMI.

Chapter 4: Characterization of Optimal SNR STBCs

In a recent work [17, 18, 19], space-time block codes (STBC) from Orthogonal designs (OD) were shown to maximize the signal to noise ratio (SNR) and also it was shown that for a linear STBC the maximum SNR is achieved when the weight matrices are unitary. In this chapter we show that STBCs from ODs are not the only codes that maximize SNR; we characterize all linear STBCs that maximize SNR thereby showing that maximum SNR can be achieved with non-unitary weight matrices also, subject to a constraint on the transmitted symbols (which is that the in-phase and quadrature components are of equal energy). This constraint is satisfied by some known signal sets like BPSK rotated by an angle of $45°$, QPSK and X-constellations (defined in Chapter 4). It is then shown that the Generalized Co-ordinate Interleaved orthogonal Designs (GCIOD) presented in the previous chapter achieve maximum SNR and corresponds to a generalized maximal ratio combiner under this constraint. This result is a generalization to a previous result on maximum SNR presented in [17] (in the sense that the necessary conditions for maximal SNR derived in [17] for real symbols are not necessary for the complex symbol case). Also we show that though GCIOD maximizes SNR for QPSK, the same GCIOD with the rotated QPSK performs better in terms of probability of error; although with this rotated QPSK, GCIOD is not maximal SNR. This result is due to the fact that the maximum SNR approach does not necessarily maximize the coding gain also. Note that GCIODs are single-symbol decodable and therefore it is pertinent to talk of maximizing SNR.

Chapter 5: Double-symbol Decodable Designs

In this chapter we consider Double-symbol Decodable Designs. Previous work in this direction is primarily concerned with Space-Time block codes (STBC) from Quasi-Orthogonal designs (QOD) due to their amenability to double-symbol decoding and full-rank along with better performance than STBCs obtained from Orthogonal Designs over quasi-static fading channels for both low and high SNRs [28]-[34]. But the QODs in literature are *instances* of Double-symbol Decodable Designs (DSDD) and in this chapter *all* Double-symbol Decodable Designs are characterized (as Single-symbol Decodable Designs were characterized in Chapter 1).

In this chapter we first characterize all linear STBCs, that allow double-symbol ML decoding (not necessarily full-diversity) over quasi-static fading channels-calling them double symbol decodable designs (DSDD). The class DSDD includes QOD as a proper subclass. Among the DSDDs those that offer full-diversity are called Full-rank DSDDs (FDSDD). We show that the class of FDSDD consists comprises of (i) Generalized QOD (GQOD) and (ii) a class of non-GQODs called Quasi Complex Restricted Designs (QCRD). Among QCRDs we identify those that offer full-rank and call them FQCRD. These full-rank QCRDs along with GQODs constitute the class of FDSDD.

We then upper bound the rates of square GQODs and show that rate 1 GQODs exist for 2 and 4 transmit antennas. Construction of maximal rate square GQODs are then presented to show that the square QODs of [34] are optimal in terms of rate and coding gain. A relation is established between GQODs and GCIODs which leads to the construction of various high rate non-square QODs not obtainable by the constructions of [28]-[34]. The coding gain of GQODs is analyzed which leads to generalization of results of [34]. Also, we upper bound the rates of square QCRDs and show that rate 1 QCRDs exist for 2, 4 and 8 transmit antennas. Construction of maximal rate square FQCRDs are then presented. Simulation results for 8 FQCRD is then presented to show that although the rate is 1, the performance is poorer than that of rate 3/4 QOD.

Chapter 6: Space-Time Block Codes from Designs for Fast-Fading Channels

In the previous chapters we were primarily concerned with STBCs over quasi-static fading channel. STBCs from Designs have not been studied well for use in fast-fading channels previously and literature on this is very scant. In this chapter, we study these codes for use in fast-fading channels by giving a matrix representation of the multi-antenna fast-fading channels. We first characterize all linear STBCs that allow single-symbol ML decoding when used in fast-fading channels. Then, among these we identify those with full-diversity, i.e., those with diversity L when the STBC is of size $L \times N, (L \geq N)$, where N is the number of transmit antennas and L is the time interval. The maximum rate for such a full-diversity, single-symbol decodable code is shown to be $2/L$ from which it follows that rate 1 is possible only for 2 Tx. antennas. The co-ordinate interleaved orthogonal design (CIOD) for 2 Tx (introduced in Chapter 1) is shown to be one such full-rate, full-diversity and single-symbol decodable code. (It turns out that Alamouti code is not single-symbol decodable for fast-fading channels.) This code performs well even when the channel is varying in the sense that sometimes it is quasi-static and other times it is fast-fading. We then present simulation results for this code in such a scenario. Finally, double-symbol decodable codes for fast-fading channels are characterized using which it is shown that maximal rate codes over fast-fading channels are not double-symbol decodable over quasi-static fading channels[9].

[9] In the course of this research, several other topics were pursued but were not presented in this thesis as they are not directly linked to STBC. These works concern differential STBCs from orthogonal designs, bit and co-ordinate interleaved coded modulation (BCICM), bit interleaved coded modulation (BICM) and TCM mapped to asymmetric PSK signal sets and part of these can be found in [58, 59, 60, 61]. In fact the intuition behind this thesis was the application of co-ordinate interleaving to STBCs. The CIODs of Chapter 3 were first obtained by combining co-ordinate interleaving with STBCs from ODs. Since the CIODs were single-symbol decodable, need for characterization of single-symbol decodable codes was immediately felt.

Chapter 2

Single-symbol Decodable STBCs

In this chapter we characterize the class of single-symbol decodable linear STBCs (Throughout the thesis we consider only linear STBCs and hence unless explicitly specified all STBCs in this thesis will mean linear STBCs).

2.1 Introduction

We begin by recollecting few basic notions like linear STBC, weight matrices of a linear STBC and single-symbol decodable (SD) STBCs already given in Chapter 1:

Definition 2.1.1 (Linear STBC). A linear design, S, is a $L \times N$ matrix whose entries are complex linear combinations of K complex indeterminates $x_k = x_{kI} + \mathbf{j}x_{kQ}$, $k = 0, \cdots, K-1$ and their complex conjugates. The STBC obtained by letting each indeterminate to take all possible values from a complex constellation $\mathcal{A}$ is called a linear STBC over $\mathcal{A}$. Notice that S is basically a "design" and by the STBC $(S, \mathcal{A})$ we mean the STBC obtained using the design S with the indeterminates taking values from the signal constellation $\mathcal{A}$. The rate of the code/design[1] is given by K/L symbols/channel use. Every linear design S can be expressed as

$$S = \sum_{k=0}^{K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1} \tag{2.1}$$

where $\{A_k\}_{k=0}^{2K-1}$ is a set of complex matrices called weight matrices of S.

[1]Note that if the signal set is of size 2^b the throughput rate R in bits per second per Hertz is related to the rate of the design $\mathcal{R}$ as $R = \mathcal{R}b$.

Among the linear STBCs we are concerned with STBCs that allow simple decoding similar to ODs i.e. "single-symbol decodable" STBCs which are defined as

Definition 2.1.2 (Single-symbol Decodable (SD) STBC). A single-symbol decodable (SD) $L \times N$ STBC of rate K/L in K complex indeterminates $x_k = x_{kI} + \mathbf{j}x_{kQ}$, $k = 0, \cdots, K-1$ is a linear STBC such that the ML decoding metric given by (1.9) can be written as a square of several terms each depending on at-most one indeterminate only. Such a SD STBC is said to be of Full-rank, abbreviated as FSDD (Full-rank, Single-symbol Decodable Design), if its rank is equal to the minimum of L and N for a chosen constellation for the indeterminates.

Fig.2.1 shows the classes of SD STBCs studied in this chapter. We first characterize all linear STBCs that admit single-symbol ML decoding (not necessarily full-rank) over quasi-static fading channels-called Single-symbol Decodable Designs (SDD). Further, we characterize all full-rank SDD called Full-rank SDD (FSDD). We show that the class of FSDD consists of only

- an extension of GLCOD which we have called Unrestricted Full-rank Single-symbol Decodable Designs (UFSDD) and
- a class of non-UFSDDs called Restricted Full-rank Single-symbol Decodable Designs (RFSDD).

The remaining part of this chapter is organized as follows: A brief presentation of basic, well known results concerning GLCODs along with a generalization [55] is given in Section 2.2. In section 2.3 we characterize the class SDD of all single-symbol decodable (not necessarily full-rank) designs. Within the class of SDD the class FSDD consisting of all Full-rank SDD is characterized in Section 2.4. Section 2.5 deals exclusively with square designs. Finally, Section 2.6 consists of some concluding remarks and a couple of directions for further research.

2.2 Generalized Linear Complex Orthogonal Designs (GLCOD)

In Subsection 1.2.2 the definition of GLCOD and few basic results were mentioned. In this section we collect some more important results on square as well as non-square GLCODs

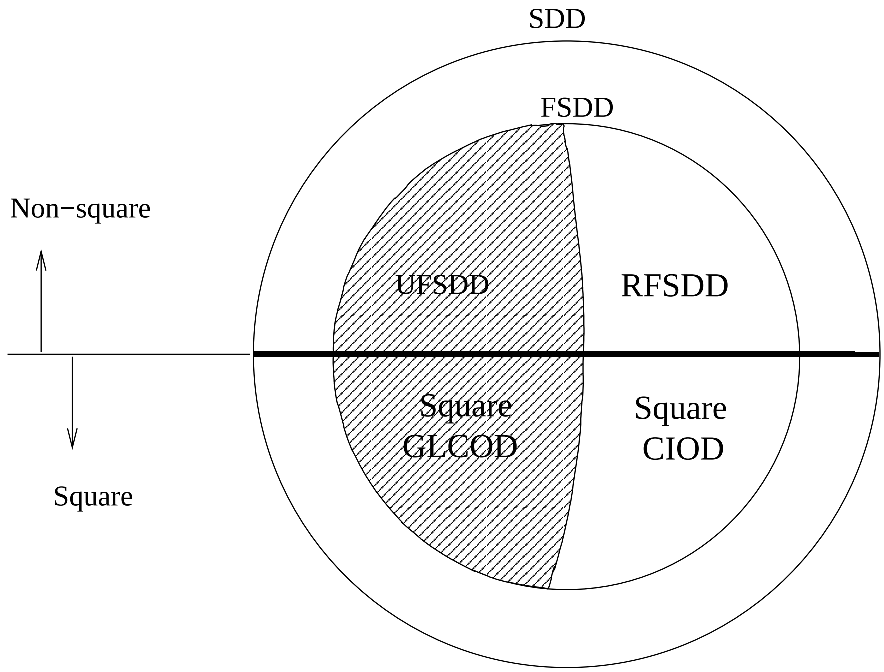


Figure 2.1: The classes of single-symbol decodable (SDD) codes.

from [13, 15, 22] which are presentable in terms of the weight matrices. In the Subsection 2.2.1 we show that some existence results on the ODs for STBCs given in [13] are valid under milder conditions, and are reported in [55].

Consider a square GLCOD[2], $S = \sum_{k=0}^{K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$. The weight matrices satisfy,

$$A_k^{\mathcal{H}} A_k = \hat{\mathcal{D}}_k, \quad k = 0, \cdots, 2K-1 \tag{2.2}$$

$$A_l^{\mathcal{H}} A_k + A_k^{\mathcal{H}} A_l = 0, \quad 0 \le k \ne l \le 2K-1. \tag{2.3}$$

Observe that $\hat{\mathcal{D}}_k$ is of full-rank for all k. Define $B_k = A_k \hat{\mathcal{D}}_k^{-1/2}$. Then the matrices B_k satisfy (using the results shown in [55] and also given in Subsection 2.2.1)

$$B_k^{\mathcal{H}} B_k = I_N, \quad k = 0, \cdots, 2K-1 \tag{2.4}$$

$$B_l^{\mathcal{H}} B_k + B_k^{\mathcal{H}} B_l = 0, \quad 0 \le k \ne l \le 2K-1. \tag{2.5}$$

[2]A rate-1, square GLCOD is referred to as complex linear processing orthogonal design (CLPOD) in [13].

and again defining

$$C_k = B_0^{\mathcal{H}} B_k, \qquad k = 0, \cdots, 2K-1, \tag{2.6}$$

we end up with $C_0 = I_N$ and

$$C_k^{\mathcal{H}} = -C_k, \qquad k = 1, \cdots, 2K-1 \tag{2.7}$$

$$C_l^{\mathcal{H}} C_k + C_k^{\mathcal{H}} C_l = 0, \qquad 1 \leq k \neq l \leq 2K-1. \tag{2.8}$$

The above normalized set of matrices $\{C_1, \cdots, C_{2K-1}\}$ constitute a *Hurwitz family of order N [24].* Let $H(N) - 1$ denote the number of matrices in a Hurwitz family of order N, then the Hurwitz Theorem can be stated as

Theorem 2.2.1 (Hurwitz [24]). *If $N = 2^a b$, b odd and $a, b > 0$ then*

$$H(N) \leq 2a + 2.$$

Observe that $H(N) = 2K$. An immediate consequence of the Hurwitz Theorem are the following results:

Theorem 2.2.2 (Tarokh, Jafarkhani and Calderbank [13]). *A square GLCOD of rate 1 exists iff $N = 2$.*

Theorem 2.2.3 (Trikkonen and Hottinen [15]). *The maximal rate, $\mathcal{R}$ of a square GLCOD of size $N = 2^a b, b$ odd, satisfying equal weight condition is*

$$\mathcal{R} = \frac{a+1}{N}.$$

Remark 2.2.1. Following the results of [55] which is also given in Subsection 2.2.1, we can now say that the Trikkonen and Hottinen theorem holds for all square GLCOD.

An intuitive and simple realization of such GLCODs based on Josefiak's realization of the Hurwitz family, was presented in [22] as

Construction 2.2.4 (Su and Xia [22]). *Let $G_1(x_0) = x_0 I_1$, then the GLCOD of size 2^K, $G_{2^K}(x_0, x_1, \cdots, x_K)$, can be constructed iteratively for $K = 1, 2, 3, \cdots$ as*

$$G_{2^K}(x_0, x_1, \cdots, x_K) = \begin{bmatrix} G_{2^{K-1}}(x_0, x_1, \cdots, x_{K-1}) & x_K I_{2^{K-1}} \\ -x_K^* I_{2^{K-1}} & G_{2^{K-1}}^{\mathcal{H}}(x_0, x_1, \cdots, x_{K-1}) \end{bmatrix}. \tag{2.9}$$

While square GLCODs have been completely characterized non-square GLCODs are not well understood. The main results for non-square GLCODs are due to Liang and Xia. The primary result is

Theorem 2.2.5 (Liang and Xia [20]). *A rate 1 GLCOD exists iff $N = 2$.*

This was further, improved later to,

Theorem 2.2.6 (Su and Xia [22]). *The maximum rate of GCOD (without linear processing) is upper bounded by 3/4.*

The maximal rate and the construction of such maximal rate non-square GLCODs for $N > 2$ remains an open problem. Finally, rate 1/2 constructions for non-square GLCODs were presented in [13] and some constructions for $N = 5, 6$ of rate greater than 1/2 were presented in [21]. The known GLCODs with rate greater than 1/2 are:

1) The rate 3/4 scheme for $N = 4$

$$\Theta_4(x_0, x_1, x_2) = \begin{bmatrix} x_0 & x_1 & x_2 & 0 \\ -x_1^* & x_0^* & 0 & x_2 \\ -x_2^* & 0 & x_0^* & -x_1 \\ 0 & -x_2^* & x_1^* & x_0 \end{bmatrix}. \tag{2.10}$$

Dropping one of the columns, we have a rate 3/4 GLCOD fro three antennas,

2) the rate 7/11=0.6364 design for $N = 5$

$$\Theta_5(x_0, x_1, \cdots, x_6) = \begin{bmatrix} x_0 & x_1 & x_2 & 0 & x_3 \\ -x_1^* & x_0^* & 0 & x_2 & x_4 \\ x_2^* & 0 & -x_0^* & x_1 & x_5 \\ 0 & x_2^* & -x_1^* & -x_0 & x_6 \\ x_3^* & 0 & 0 & -x_6^* & -x_0^* \\ 0 & x_3^* & 0 & x_5^* & -x_1^* \\ 0 & 0 & x_3^* & x_4^* & -x_2^* \\ 0 & -x_4^* & x_5^* & 0 & x_0 \\ x_4^* & 0 & x_6^* & 0 & x_1 \\ -x_5^* & -x_6^* & 0 & 0 & x_2 \\ x_6 & -x_5 & -x_4 & x_3 & 0 \end{bmatrix} \tag{2.11}$$

3) and the rate 3/5 design for $N = 6$

$$\Theta_6(x_0, x_1, x_2, \cdots, x_{17}) = \begin{bmatrix} x_0 & x_1 & x_2 & 0 & x_3 & x_7 \\ -x_1^* & x_0^* & 0 & x_2 & x_4 & x_8 \\ x_2^* & 0 & -x_0^* & x_1 & x_5 & x_9 \\ 0 & x_2^* & -x_1^* & -x_0 & x_6 & x_{10} \\ x_3^* & 0 & 0 & -x_6^* & -x_0^* & x_{11} \\ 0 & x_3^* 0 & x_5^* & -x_1^* & x_{12} & \\ 0 & 0 & x_3^* & x_4^* & -x_2^* & x_{13} \\ 0 & -x_4^* & x_5^* & 0 & x_0 & x_{14} \\ x_4^* & 0 & x_6^* & 0 & x_1 & x_{15} \\ -x_5^* & -x_6^* & 0 & 0 & x_2 & x_{16} \\ x_6 & -x_5 & -x_4 & x_3 & 0 & x_{17} \\ x_7^* & 0 & 0 - x_{10}^* & -x_{14}^* & -x_0^* & \\ 0 & x_7^* & 0 & x_9^* & x_{15}^* & -x_1^* \\ 0 & 0 & x_7^* & x_8^* & -x_{16}^* & -x_2^* \\ 0 & 0 & 0 & x_{17}^* & x_7^* & -x_3^* \\ 0 & 0 & -x_{17}^* & 0 & x_8^* & -x_4^* \\ 0 & -x_{17}^* & 0 & 0 & x_9^* & -x_5^* \\ -x_{17}^* & 0 & 0 & 0 & x_{10}^* & -x_6^* \\ 0 & -x_8^* & x_9^* & 0 & x_{11}^* & x_0 \\ x_8^* & 0 & x_{10}^* & 0 & x_{12}^* & x_1 \\ -x_9^* & -x_{10}^* & 0 & 0 & x_{13}^* & x_2 \\ -x_{11}^* & -x_{12}^* & -x_{13}^* & 0 & 0 & x_3 \\ -x_{15}^* & -x_{14}^* & 0 & -x_{13}^* & 0 & x_4 \\ -x_{16}^* & 0 & x_{14}^* & -x_{12}^* & 0 & x_5 \\ 0 & -x_{16}^* & -x_{15}^* & x_{11}^* & 0 & x_6 \\ 0 & x_{13} & -x_{12} & -x_{14} & x_{10} & 0 \\ x_{13} & 0 & -x_{11} & -x_{15} & x_9 & 0 \\ -x_{12} & x_{11} & 0 & x_{16} & x_8 & 0 \\ x_{14} & -x_{15} & x_{16} & 0 & x_7 & 0 \\ -x_{10} & x_9 & x_8 & -x_7 & x_{17} & 0 \end{bmatrix}. \tag{2.12}$$

2.2.1 Generalization of certain existence results on ODs

In this subsection we show that two theorems regarding the existence of Generalized Linear Orthogonal Designs (GLODs) in the paper [13] are valid under more general conditions than for which they have been stated and proved.

Recalling the definition of GLCOD in [13] along with a correction by the same authors in [14] a Generalized Linear Complex Orthogonal Design (GLCOD)[13] in K complex indeterminates $x_0, x_1, \cdots, x_{K-1}$ of size N and rate $\mathcal{R} = K/p$, $p \geq N$ is a $p \times N$ matrix $\mathcal{E}$, such that

- the entries of $\mathcal{E}$ are complex linear combinations of $0, \pm x_i, \quad i = 0, \cdots, K-1$ and their conjugates.
- $\mathcal{E}^{\mathcal{H}}\mathcal{E} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix with the (i,i)-th diagonal element of the form
$$l_0^{(i)}|x_0|^2 + l_1^{(i)}|x_1|^2 + \cdots + l_{K-1}^{(i)}|x_{K-1}|^2 \tag{2.13}$$
where $l_j^{(i)}, i = 1, 2, \cdots, N, \quad j = 0, 1, \cdots, K-1$ are strictly positive numbers and for all values of i,
$$l_0^{(i)} = l_1^{(i)} = \cdots = l_{K-1}^{(i)}. \tag{2.14}$$

The condition given by (2.14), which we will be henceforth referred as **Equal-Weights condition** has been introduced in [14] as a correction to [13]. When K=N=p, $\mathcal{E}$ is called a Linear Complex Orthogonal Design (LCOD). Furthermore, when the entries are only from $\{\pm x_0, \pm x_1, \cdots, \pm x_{K-1}\}$, their conjugates and multiples of $\mathbf{j}$ then $\mathcal{E}$ is called a Complex Orthogonal Design (COD). If the variables are real and only real linear combinations are used in the above definitions then we get Generalized Linear Real Orthogonal Designs (GLROD), Linear Real Orthogonal Design (LROD) and Real Orthogonal Designs (ROD).

The existence of Orthogonal Designs (OD) is of fundamental importance in the theory of Space-Time Block Codes [13]. In this regard, the paper [13] presents four theorems (Theorems 3.4.1, 4.1.1, 5.4.1 and 5.5.1): Theorem 3.4.1 deals with RODs, Theorem 5.4.1 with CODs and Theorems 4.1.1 and 5.5.1 with GLROD and GLCOD respectively. The proof is given only for Theorem 3.4.1 and the remaining three theorems are stated with the remark that the proofs are similar to that of Theorem 3.4.1.

In what follows, we show that Theorems 3.4.1 and 5.5.1 of [13] are valid under the following two generalizations:

- Dropping the Equal-Weights conditions from the definition of GLCODs.
- Under the condition $N = p$ in contrast to $N = p = K$ as in [13].

Notice that the second generalization above means that as long as the design is a square (not rectangular) then Theorems 3.4.1 and 5.5.1 of [13] are valid without the Equal-Weights condition and moreover the number of variables K need not be equal to $N = p$.

Theorem 3.4.1 of the subject paper - revisited

We begin with analyzing the proof of Theorem 3.4.1 given in the paper [13], which is for real orthogonal designs, without assuming the Equal-Weights condition.

Theorem 2.2.7 (Theorem 3.4.1 of [13]). *A linear processing orthogonal design, $\mathcal{E}$ in variables $x_0, x_1, \cdots, x_{N-1}$ exists iff there exists a linear processing orthogonal design $\mathcal{L}$, such that*

$$\mathcal{L}^{\mathcal{T}}\mathcal{L} = \mathcal{L}\mathcal{L}^{\mathcal{T}} = (x_0^2 + x_1^2 + \cdots + x_{N-1}^2)I. \tag{2.15}$$

The proof given in the subject paper is as follows:
Let $\mathcal{E} = x_0A_0 + \cdots + x_{N-1}A_{N-1}$ be a linear processing orthogonal design and let

$$\mathcal{E}^{\mathcal{T}}\mathcal{E} = x_0^2\mathcal{D}_0 + \cdots + x_{N-1}^2\mathcal{D}_{N-1} \tag{2.16}$$

where the matrices $\mathcal{D}_i$ are diagonal and full-rank. Then, it follows that

$$A_i^{\mathcal{T}}A_i = \mathcal{D}_i, \qquad i = 0, 1, \cdots, N-1 \tag{2.17}$$

$$A_i^{\mathcal{T}}A_j = -A_j^{\mathcal{T}}A_i, \qquad 0 \le i < j \le N-1 \tag{2.18}$$

where $\mathcal{D}_i$ is a full-rank diagonal matrix with positive diagonal entries. Let $\mathcal{D}_i^{1/2}$ be the diagonal matrix having the property that $\mathcal{D}_i^{1/2}\mathcal{D}_i^{1/2} = \mathcal{D}_i$. Define

$$U_i = A_i\mathcal{D}_i^{-1/2}. \tag{2.19}$$

Then the matrices U_i satisfy the following properties:

$$U_i^{\mathcal{T}} U_i = I, \qquad i = 0, 1, \cdots, N-1 \tag{2.20}$$

$$U_i^{\mathcal{T}} U_j = -U_j^{\mathcal{T}} U_i, \qquad 0 \le i < j \le N-1. \tag{2.21}$$

After this step, the proof is completed in [13] with "It follows that $\mathcal{L} = x_0 U_0 + x_1 U_1 + \cdots + x_{N-1} U_{N-1}$ is a linear processing orthogonal design with the desired property". If the Equal-Weights condition is included in the definition of GLCOD then the proof is complete.

However, even without the Equal-Weights condition, (2.21) is valid as shown below: Substituting from(2.19) in (2.21) we have,

$$\mathcal{D}_i^{-1/2} A_i^{\mathcal{T}} A_j \mathcal{D}_j^{-1/2} = -\mathcal{D}_j^{-1/2} A_j^{\mathcal{T}} A_i \mathcal{D}_i^{-1/2}, \quad 0 \le i < j \le N-1, \tag{2.22}$$

$$\mathcal{D}_i^{-1/2} A_i^{\mathcal{T}} A_j \mathcal{D}_j^{-1/2} = \mathcal{D}_j^{-1/2} A_i^{\mathcal{T}} A_j \mathcal{D}_i^{-1/2}, \quad 0 \le i < j \le N-1. \tag{2.23}$$

Equating the (k, l) entries on both sides of (2.23), we have

$$\begin{aligned} d_j^{(k)} a_{kl} d_i^{(l)} &= d_i^{(k)} a_{kl} d_j^{(l)} \\ \Rightarrow d_j^{(k)} d_i^{(l)} &= d_i^{(k)} d_j^{(l)}, a_{kl} \neq 0 \\ \Rightarrow \frac{d_j^{(k)}}{d_j^{(l)}} &= \frac{d_i^{(k)}}{d_i^{(l)}} \quad \forall l, k \text{ such that } a_{kl} \neq 0 \qquad (2.24) \\ \Rightarrow \mathcal{D}_j &= a_j \mathcal{D}_1 \quad \forall j = 1, 2, \cdots, N-1 \qquad (2.25) \end{aligned}$$

where $d_j^{(k)}$ is the k-th diagonal entry of $\mathcal{D}_j^{-1/2}$ and a_{kl} is the (k, l) entry of $A_i^{\mathcal{T}} A_j$.

From the derivation of (2.25) above, it follows that if the set of $\mathcal{D}_i$'s in (2.16) satisfy (2.25) or (2.22) then Theorem 3.4.1 of the paper[13] is correct without the Equal-Weights condition included in the definition of GLROD. For GLCOD, the equivalent of (2.22) is

$$\mathcal{D}_i^{-1/2} A_i^{\mathcal{H}} A_j \mathcal{D}_j^{-1/2} = -\mathcal{D}_j^{-1/2} A_j^{\mathcal{H}} A_i \mathcal{D}_i^{-1/2}, \qquad 0 \le i < j \le N-1, \tag{2.26}$$

and in the following subsection we show that (2.26) is satisfied for **all square designs** real or complex.

A generalization of Theorems 3.4.1 and 5.4.1 of [13]

We now prove a generalization of Theorems 3.4.1 and 5.4.1 of [13]. Note that Theorems 3.4.1 and 5.4.1 of the subject paper assume $N = p = K$ whereas our theorem assumes $N = p$ and there is no restriction on K.

Theorem 2.2.8. *With the Equal-Weights condition removed from the definition of GLCODs, an $N \times N$ square (GLCOD), $\mathcal{E}_c$ in variables $x_0, \cdots, x_{K-1}$ exists iff there exists a GLCOD $\mathcal{L}_c$ such that*

$$\mathcal{L}_c^{\mathcal{H}} \mathcal{L}_c = (|x_0|^2 + \cdots + |x_{K-1}|^2) I \tag{2.27}$$

Proof. Let $\mathcal{E}_c = \sum_{i=1}^{k} x_{iI} A_{2i} + x_{iQ} A_{2i+1}$ where $x_i = x_{iI} + \mathbf{j} x_{iQ}$. The weight matrices $\{A_i\}$ satisfy,

$$A_i^{\mathcal{H}} A_i = \mathcal{D}_i, \qquad i = 0, 1, \cdots, 2K-1 \tag{2.28}$$

$$A_j^{\mathcal{H}} A_i + A_i^{\mathcal{H}} A_j = 0, \qquad 0 \leq i \neq j \leq 2K-1.. \tag{2.29}$$

It is important to observe that $\mathcal{D}_i$ is a diagonal and full-rank matrix for all i. Define $B_i = A_i \mathcal{D}_i^{-1/2}$ and $\mathcal{L}_c = \sum_{i=1}^{k} x_{iI} B_{2i} + x_{iQ} B_{2i+1}$. Then the design $\mathcal{L}_c$ satisfies (2.27) iff the matrices B_i satisfy

$$B_i^{\mathcal{H}} B_i = I_N, \qquad i = 0, 1, \cdots, 2K-1 \tag{2.30}$$

$$B_j^{\mathcal{H}} B_i + B_i^{\mathcal{H}} B_j = 0, \qquad 0 \leq i \neq j \leq 2K-1. \tag{2.31}$$

Substituting $B_i = A_i \mathcal{D}_i^{-1/2}$, while (2.30) is always satisfied, (2.31) is satisfied iff

$$\mathcal{D}_i^{-1/2} A_i^{\mathcal{H}} A_j \mathcal{D}_j^{-1/2} = -\mathcal{D}_j^{-1/2} A_j^{\mathcal{H}} A_i \mathcal{D}_i^{-1/2}, \qquad 0 \leq i \neq j \leq 2K-1. \tag{2.32}$$

Notice that (2.32) reduces to (2.22) for real orthogonal designs and with $K = N$. In what follows we show that for square designs, (2.32) is satisfied without the Equal Weights condition in the definition of GLCODs.

Define $B_i^{(l)} = \mathcal{D}_l^{-1/2} A_l^{\mathcal{H}} A_i \mathcal{D}_l^{-1/2}$ for $0 \leq l, i \leq 2K-1$. Then $B_l^{(l)} = I_N$ and

$$\left[B_i^{(l)}\right]^{\mathcal{H}} = -B_i^{(l)}, \left[B_i^{(l)}\right]^2 = -\mathcal{D}_l^{-1}\mathcal{D}_i \triangleq \hat{\mathcal{D}}_i^{(l)}, \quad 0 \leq l \neq i \leq 2K-1 \tag{2.33}$$

$$B_j^{(l)} B_i^{(l)} + B_i^{(l)} B_j^{(l)} = 0, \quad 0 \leq i \neq l \neq j \leq 2K-1 \tag{2.34}$$

where we have used the fact that $\mathcal{D}_l^{-1} A_l^{\mathcal{H}}$ is the inverse of A_l, so $A_l \mathcal{D}_l^{-1} A_l^{\mathcal{H}} = I$ (which is true only for square A_l). Now the inverse of $B_i^{(l)}, 0 \leq i \neq l \leq 2K-1$ is $B_i^{(l)} \left[\hat{\mathcal{D}}_i^{(l)}\right]^{-1}$ and also $\left[\hat{\mathcal{D}}_i^{(l)}\right]^{-1} B_i^{(l)}$ which can be verified by multiplying with $B_i^{(l)}$ and then using (2.33). Since the inverse is unique, we have

$$B_i^{(l)} \left[\hat{\mathcal{D}}_i^{(l)}\right]^{-1} = \left[\hat{\mathcal{D}}_i^{(l)}\right]^{-1} B_i^{(l)}, \quad 1 \leq i \neq l \leq 2K-1. \tag{2.35}$$

The (r,m)-th entry, where $(1 \leq r, m \leq N)$, of $B_i^{(l)} \left[\hat{\mathcal{D}}_i^{(l)}\right]^{-1}$ is $b_{r,m}^{(i)} d_m^{(i)}$ where $b_{r,m}^{(i)}$ is the (r,m)-th entry of $B_i^{(l)}$ and $d_m^{(i)}$ is the m-th entry of $\left[\hat{\mathcal{D}}_i^{(l)}\right]^{-1}$. Similarly, the (r,m)-th entry of $\left[\hat{\mathcal{D}}_i^{(l)}\right]^{-1} B_i^{(l)}$ is $d_r^{(i)} b_{r,m}^{(i)}$. Equating the (r,m)-th entries on both sides of (2.35), we have

$$b_{r,m}^{(i)} d_m^{(i)} = d_r^{(i)} b_{r,m}^{(i)}, \quad \forall r, m. \tag{2.36}$$

If $b_{r,m}^{(i)} \neq 0$ then $d_m^{(i)} = d_r^{(i)}$, otherwise, both sides of (2.36) is 0. In either case, we can multiply the left hand side term by $[d_m^{(i)}]^{-1/2}$ and the right hand side term by $[d_r^{(i)}]^{-1/2}$ to obtain

$$\begin{aligned} b_{r,m}^{(i)} [d_m^{(i)}]^{1/2} &= [d_r^{(i)}]^{1/2} b_{r,m}^{(i)}, \quad \forall r, m \\ \Rightarrow B_i^{(l)} \left[\hat{\mathcal{D}}_i^{(l)}\right]^{-1/2} &= \left[\hat{\mathcal{D}}_i^{(l)}\right]^{-1/2} B_i^{(l)}, \quad 0 \leq i \neq l \leq 2K-1. \end{aligned} \tag{2.37}$$

Substituting the value of $B_i^{(l)}, \hat{\mathcal{D}}_i^{(l)}$ from (2.33) in (2.37), we have,

$$\begin{aligned} & \mathcal{D}_l^{-1/2} A_l^{\mathcal{H}} A_i \mathcal{D}_l^{-1/2} [-\mathcal{D}_l^{-1}\mathcal{D}_i]^{-1/2} = [-\mathcal{D}_l^{-1}\mathcal{D}_i]^{-1/2} \mathcal{D}_l^{-1/2} A_l^{\mathcal{H}} A_i \mathcal{D}_l^{-1/2} \\ \Rightarrow \quad & \mathcal{D}_l^{-1/2} A_l^{\mathcal{H}} A_i \mathcal{D}_i^{-1/2} = \mathcal{D}_i^{-1/2} A_l^{\mathcal{H}} A_i \mathcal{D}_l^{-1/2} \\ \Rightarrow \quad & \mathcal{D}_l^{-1/2} A_l^{\mathcal{H}} A_i \mathcal{D}_i^{-1/2} = -\mathcal{D}_i^{-1/2} A_i^{\mathcal{H}} A_l \mathcal{D}_l^{-1/2} \end{aligned} \quad (2.38)$$

which is the same as (2.32). □

When $K = N$, Theorem 2.2.8 reduces to Theorem 5.4.1 of the subject paper. Similarly, when x_i's are real, the weight matrices A_i are real matrices and $K = N$ then Theorem 2.2.8 includes Theorem 3.4.1 of the subject paper. The arguments of this correspondence can not be used to prove Theorems 4.1.1 and 5.5.1 in [13] without the Equal-Weights condition. Indeed, the two designs presented in [22] show that Theorems 4.1.1 and 5.5.1 of [13] are not valid without the Equal-Weights condition.

The results of [15] are for square designs satisfying the condition (2.27). By virtue of Theorem 2.2.8 the results of [15, 13] are valid for all square designs without the Equal-Weights conditions and hence we have the following corollary.

Corollary 2.2.9. *Let $N = 2^a b$ where b is an odd integer and $a = 4c + d$, where $0 \leq d < c$ and $c \geq 0$. The maximal rate of size N, square GLROD without the Equal-Weights condition satisfied is $\frac{8c+2^d}{N}$ and of size N, square GLCOD without the Equal-Weights condition satisfied is $\frac{a+1}{N}$.*

2.3 Single-symbol Decodable Designs

In this section we characterize all STBCs that allow single-symbol ML decoding (Subsection 2.3.1) and using this characterization define single-symbol decodable designs (SDD) in terms of the weight matrices and discuss several examples of such designs.

2.3.1 Characterization of Single-symbol Decodable STBCs

Consider the matrix channel model for quasi-static fading channel given in (1.6)

$$\mathbf{V} = \mathbf{S}\mathbf{H} + \mathbf{N} \tag{2.39}$$

where $\mathbf{V} \in \mathbb{C}^{L\times M}$ is the received signal matrix, $\mathbf{S} \in \mathbb{C}^{L\times N}$ is the transmission or codeword matrix, $\mathbf{N} \in \mathbb{C}^{L\times M}$ is the noise matrix and $\mathbf{H} \in \mathbb{C}^{N\times M}$ defines the channel matrix whose ith row and the jth column element is h_{ij}. Assuming that perfect channel state information (CSI) is available at the receiver, the decision rule for ML decoding is: decide in favor of that matrix $\mathbf{S}$ for which the decoding metric given by

$$M(\mathbf{S}) \triangleq \mathrm{tr}\left((\mathbf{V} - \mathbf{S}\mathbf{H})^{\mathcal{H}}(\mathbf{V} - \mathbf{S}\mathbf{H})\right). \tag{2.40}$$

is minimum. This ML metric (2.40) results in exponential decoding complexity with the rate of transmission in bits/sec/Hz.

For a linear STBC with K variables, we are concerned about those STBCs for which the ML metric (2.40) can be written as sum of several terms with each term involving at-most one variable only and hence single-symbol decodable.

The following theorem characterizes **all linear STBCs**, in terms of the weight matrices, that will allow single-symbol decoding.

Proposition 2.3.1. *For a linear STBC in K variables, $S = \sum_{k=0}^{K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$, the ML metric, $M(S)$ defined in (2.40) decomposes as $M(S) = \sum_{k=0}^{K-1} M_k(x_k) + M_c$ where $M_c = -(K-1)\mathrm{tr}\left(V^{\mathcal{H}}V\right)$ is independent of all the variables and $M_k(x_k)$ is a function only of the variable x_k, iff*[3]

$$A_k^{\mathcal{H}}A_l + A_l^{\mathcal{H}}A_k = 0 \quad \begin{cases} \forall l \neq k, k+1 \text{ if } k \text{ is even} \\ \forall l \neq k, k-1 \text{ if } k \text{ is odd} \end{cases}. \tag{2.41}$$

[3] *The condition (2.41) can also be given as*

$$A_kA_l^{\mathcal{H}} + A_lA_k^{\mathcal{H}} = 0 \quad \begin{cases} \forall l \neq k, k+1 \text{ if } k \text{ is even} \\ \forall l \neq k, k-1 \text{ if } k \text{ is odd} \end{cases}$$

due to the identity $tr(\mathbf{V} - \mathbf{S}\mathbf{H})^{\mathcal{H}}(\mathbf{V} - \mathbf{S}\mathbf{H}) = tr(\mathbf{V} - \mathbf{S}\mathbf{H})(\mathbf{V} - \mathbf{S}\mathbf{H})^{\mathcal{H}}$ when S is a square matrix.

Proof. From (2.40) we have

$$M(S) = \text{tr}\left(\mathbf{V}^{\mathcal{H}}\mathbf{V}\right) - \text{tr}\left((S\mathbf{H})^{\mathcal{H}}\mathbf{V}\right) - \text{tr}\left(\mathbf{V}^{\mathcal{H}}S\mathbf{H})\right) + \text{tr}\left(S^{\mathcal{H}}S\mathbf{H}\mathbf{H}^{\mathcal{H}}\right).$$

Observe that $\text{tr}\left(\mathbf{V}^{\mathcal{H}}\mathbf{V}\right)$ is independent of S. The next two terms in $M(S)$ are functions of $S, S^{\mathcal{H}}$ and hence linear in x_{kI}, x_{kQ}. In the last term,

$$\begin{aligned} S^{\mathcal{H}}S &= \sum_{k=0}^{K-1}(A_{2k}^{\mathcal{H}}A_{2k}x_{kI}^2 + A_{2k+1}^{\mathcal{H}}A_{2k+1}x_{kQ}^2) + \sum_{k=0}^{K-1}\sum_{l=k+1}^{K-1}(A_{2k}^{\mathcal{H}}A_{2l} + A_{2l}^{\mathcal{H}}A_{2k})x_{kI}x_{lI} \\ &+ \sum_{k=0}^{K-1}\sum_{l=k+1}^{K-1}(A_{2k+1}^{\mathcal{H}}A_{2l+1} + A_{2l+1}^{\mathcal{H}}A_{2k+1})x_{kQ}x_{lQ} \\ &+ \sum_{k=0}^{K-1}\sum_{l=0}^{K-1}(A_{2k}^{\mathcal{H}}A_{2l+1} + A_{2l+1}^{\mathcal{H}}A_{2k})x_{kI}x_{lQ}. \end{aligned} \tag{2.42}$$

(a) Proof for the "if part": If (2.41) is satisfied then (2.42) reduces to

$$\begin{aligned} S^{\mathcal{H}}S &= \sum_{k=0}^{K-1}\left(A_{2k}^{\mathcal{H}}A_{2k}x_{kI}^2 + A_{2k+1}^{\mathcal{H}}A_{2k+1}x_{kQ}^2 + \left(A_{2k}^{\mathcal{H}}A_{2k+1} + A_{2k+1}^{\mathcal{H}}A_{2k}\right)x_{kI}x_{kQ}\right) \\ &= \sum_{k=0}^{K-1}T^{\mathcal{H}}T, \text{ where} &(2.43) \\ T &= A_{2k}x_{kI} + A_{2k+1}x_{kQ} &(2.44) \end{aligned}$$

and using linearity of the trace operator, $M(S)$ can be written as

$$\begin{aligned} M(S) &= \text{tr}\left(\mathbf{V}^{\mathcal{H}}\mathbf{V}\right) - \sum_{k=0}^{K-1}\left\{\text{tr}\left((T\mathbf{H})^{\mathcal{H}}\mathbf{V}\right) - \text{tr}\left(\mathbf{V}^{\mathcal{H}}T\mathbf{H})\right) + \text{tr}\left(T^{\mathcal{H}}T\mathbf{H}\mathbf{H}^{\mathcal{H}}\right)\right. \\ &= \sum_k \underbrace{\|\mathbf{V} - (A_{2k}x_{kI} + A_{2k+1}x_{kQ})\mathbf{H}\|^2}_{M_k(x_k)} + M_c \end{aligned} \tag{2.45}$$

where $M_c = -(K-1)\text{tr}\left(V^{\mathcal{H}}V\right)$ and $\|.\|$ denotes the Frobenius norm.

(b) Proof for the "only if part": If (2.41) is not satisfied for any $A_{k_1}, A_{l_1}, k_1 \neq l_1$ then

$$M(S) = \sum_k ||\mathbf{V} - (A_{2k}x_{kI} + A_{2k+1}x_{kQ})\mathbf{H}||^2 + \text{tr}\left((A_{k_1}^{\mathcal{H}}A_{l_1} + A_{l_1}^{\mathcal{H}}A_{k_1})\mathbf{H}^{\mathcal{H}}\mathbf{H}\right) y + M_c$$

$$\text{where } y = \begin{cases} x_{(k_1/2)I}x_{(l_1/2)I} & \text{if both } k_1, l_1 \text{ are even} \\ x_{((k_1-1)/2)Q}x_{((l_1-1)/2)Q} & \text{if both } k_1, l_1 \text{ are odd} \\ x_{((k_1-1)/2)Q}x_{(l_1/2)I} & \text{if } k_1 \text{ odd, } l_1 \text{ even.} \end{cases}$$

Now, from the above it is clear that $M(S)$ can not be decomposed into terms involving only one variable. □

It is important to observe that (2.41) implies that it is not necessary for the weight matrices associated with the in-phase and quadrature-phase of a single variable (say k-th) to satisfy the condition $A_{2k+1}^{\mathcal{H}}A_{2k} + A_{2k}^{\mathcal{H}}A_{2k+1} = 0$. Since $A_{2k+1}^{\mathcal{H}}A_{2k} + A_{2k}^{\mathcal{H}}A_{2k+1}$ is indeed the coefficient of $x_{kI}x_{kQ}$ in $S^{\mathcal{H}}S$, this implies that if terms of the form $x_{kI}x_{kQ}$ can appear in $S^{\mathcal{H}}S$ without violating single-symbol decodability. An example of such a STBC is given in Example 2.3.1.

Example 2.3.1. *Consider*

$$S(x_0, x_1) = \begin{bmatrix} x_{0I} + \mathbf{j}x_{1I} & x_{0Q} + \mathbf{j}x_{1Q} \\ x_{0Q} + \mathbf{j}x_{1Q} & x_{0I} + \mathbf{j}x_{1I} \end{bmatrix}. \tag{2.46}$$

The corresponding weight matrices are given by

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{bmatrix}, A_3 = \begin{bmatrix} 0 & \mathbf{j} \\ \mathbf{j} & 0 \end{bmatrix}$$

and it is easily verified that (2.41) is satisfied and $A_{2k+1}^{\mathcal{H}}A_{2k} + A_{2k}^{\mathcal{H}}A_{2k+1} \neq 0$ *for* $k = 0$ *as well as* $k = 1$ *and*

$$\det\left\{(S - \hat{S})^{\mathcal{H}}(S - \hat{S})\right\} = \left[(\triangle x_{0I} - \triangle x_{0Q})^2 + (\triangle x_{1I} - \triangle x_{1Q})^2\right] \left[(\triangle x_{0I} + \triangle x_{0Q})^2 + (\triangle x_{1I} + \triangle x_{1Q})^2\right]$$

where $x_i - \hat{x}_i = \triangle x_{iI} + \mathbf{j}\triangle x_{iQ}$. *If we set* $\triangle x_{1I} = \triangle x_{1Q} = 0$ *we have*

$$\det\left\{(S-\hat{S})^{\mathcal{H}}(S-\hat{S})\right\} = \left[(\triangle^2 x_{0I} - \triangle^2 x_{0Q})^2\right] \tag{2.47}$$

which is maximized when either $\triangle^2 x_{0I} = 0$ *or* $\triangle^2 x_{0Q} = 0$, *i.e. the k-th indeterminate should take values from a constellation that is parallel to a "real axis" or the "imaginary axis".*

As shown in the above example, it is easily seen that whenever the weight matrices corresponding to one indeterminate (say k-th) of SD design does not satisfy $A_{2k+1}^{\mathcal{H}}A_{2k} + A_{2k}^{\mathcal{H}}A_{2k+1} = 0$, then whenever all the indeterminates of the design take value from the same constellation, the coding gain is maximized when the constellation is a one dimensional constellation parallel to the x-axis or the y-axis[4].

Henceforth, we consider only those STBCs $S = \sum_{k=0}^{K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$**, which have the property that the weight matrices of the in-phase and quadrature components of any variable are orthogonal, that is**

$$A_{2k}^{\mathcal{H}}A_{2k+1} + A_{2k+1}^{\mathcal{H}}A_{2k} = 0, \qquad 0 \le k \le K-1 \tag{2.48}$$

for the following reasons: (i) it is customary and also convenient to assume that all indeterminates take values from one and the same complex constellations and and (ii) all known STBCs satisfy (2.48). Proposition 2.3.1 for these cases specializes to:

Theorem 2.3.2. *For a linear STBC in K complex variables,* $S = \sum_{k=0}^{K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$, *the ML metric,* $M(S)$ *defined in (2.40) decomposes as* $M(S) = \sum_{k=0}^{K-1} M_k(x_k) + M_c$ *where* $M_c = -(K-1)\mathrm{tr}\left(V^{\mathcal{H}}V\right)$, *iff*

$$A_k^{\mathcal{H}}A_l + A_l^{\mathcal{H}}A_k = 0, \qquad 0 \le k \ne l \le 2K-1. \tag{2.49}$$

[4]Such codes are closely related to Quasi-Orthogonal Designs (QOD) discussed in Chapter 5 where we show that such codes are constructible from QODs.

Theorem 2.3.2 implies

Corollary 2.3.3. *For a linear STBC in K complex variables, $S = \sum_{k=0}^{K-1} x_{kI} \ A_{2k} + x_{kQ}A_{2k+1}$, the ML metric, $M(S)$ defined in (2.40) decomposes as $M(S) = \sum_{k=0}^{K-1} M_k(x_k) + M_c$ where $M_c = -(K-1)\text{tr}\left(V^{\mathcal{H}}V\right)$, iff*

$$\text{tr}\left(A_k\mathbf{H}\mathbf{H}^{\mathcal{H}}A_l^{\mathcal{H}} + A_l\mathbf{H}\mathbf{H}^{\mathcal{H}}A_k^{\mathcal{H}}\right) = 0, \quad 0 \leq k \neq l \leq 2K-1. \tag{2.50}$$

If in addition S is square ($N = L$), then (2.50) is satisfied if and only if

$$A_kA_l^{\mathcal{H}} + A_lA_k^{\mathcal{H}} = 0, \quad 0 \leq k \neq l \leq 2K-1. \tag{2.51}$$

Proof. Using the identity, $tr(\mathbf{V}-\mathbf{SH})^{\mathcal{H}}(\mathbf{V}-\mathbf{SH}) = tr(\mathbf{V}-\mathbf{SH})(\mathbf{V}-\mathbf{SH})^{\mathcal{H}}$ and proceeding as in the proof of Theorem 2.3.2 we have (2.50). For square S, (2.50) can be written as

$$\text{tr}\left(\mathbf{H}\mathbf{H}^{\mathcal{H}}\left\{A_kA_l^{\mathcal{H}} + A_lA_k^{\mathcal{H}}\right\}\right) = 0, \quad 0 \leq k \neq l \leq 2K-1 \tag{2.52}$$

which is satisfied iff

$$A_kA_l^{\mathcal{H}} + A_lA_k^{\mathcal{H}} = 0, \quad 0 \leq k \neq l \leq 2K-1.$$

□

Examples of SD STBCs are those from OD, in-particular the Alamouti code. The following example gives two STBCs that are not obtainable as STBCs from ODs.

Example 2.3.2. *For $N = K = 2$ consider*

$$S = \begin{bmatrix} x_{0I} + \mathbf{j}x_{1Q} & 0 \\ 0 & x_{1I} + \mathbf{j}x_{0Q} \end{bmatrix}. \tag{2.53}$$

The corresponding weight matrices are given by

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{j} \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} \mathbf{j} & 0 \\ 0 & 0 \end{bmatrix}.$$

Similarly, for $N = K = 4$ *consider the design*

$$S = \begin{bmatrix} x_{0I} + \mathbf{j}x_{2Q} & x_{1I} + \mathbf{j}x_{3Q} & 0 & 0 \\ -x_{1I} + \mathbf{j}x_{3Q} & x_{0I} - \mathbf{j}x_{2Q} & 0 & 0 \\ 0 & 0 & x_{2I} + \mathbf{j}x_{0Q} & x_{3I} + \mathbf{j}x_{1Q} \\ 0 & 0 & -x_{3I} + \mathbf{j}x_{1Q} & x_{2I} - \mathbf{j}x_{0Q} \end{bmatrix}. \quad (2.54)$$

The corresponding weight matrices are

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{j} & 0 \\ 0 & 0 & 0 & -\mathbf{j} \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{j} \\ 0 & 0 & \mathbf{j} & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_5 = \begin{bmatrix} \mathbf{j} & 0 & 0 & 0 \\ 0 & -\mathbf{j} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, A_7 = \begin{bmatrix} 0 & \mathbf{j} & 0 & 0 \\ \mathbf{j} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easily seen that the two codes of the above example are not covered by GLCODs and satisfy the requirements of Theorem 2.3.2 and hence are single-symbol decodable.

These two STBCs are instances of the so called Co-ordinate Interleaved Designs (CIOD), which is discussed in detail in Chapter 3 and a formal definition of which is Definition 3.1.1. These codes apart from being SD can give STBCs with full-rank also when the indeterminates take values from appropriate signal sets- an aspect which is discussed in detail in Chapter 3.

2.4 Full-rank SDD

In this section we identify all full-rank designs with in the class of SDD, i.e., characterize the class of FSDD and classify the same. Towards this end, we have for square ($N = L$) SDD

Proposition 2.4.1. *A square single-symbol decodable design, $S = \sum_{k=0}^{K-1} x_{kI}A_{2k}+x_{kQ}A_{2k+1}$, exists if and only if there exists a square single-symbol decodable design, $\hat{S} = \sum_{k=0}^{K-1} x_{kI}\hat{A}_{2k}+ x_{kQ}\hat{A}_{2k+1}$ such that*

$$\hat{A}_k^{\mathcal{H}}\hat{A}_l + \hat{A}_l^{\mathcal{H}}\hat{A}_k = 0, k \neq l, \text{ and } \hat{A}_k^{\mathcal{H}}\hat{A}_k = \mathcal{D}_k, \forall k,$$

where $\mathcal{D}_k$ is a diagonal matrix.

Proof. Using (2.49) and (2.51) repeatedly we get

$$A_k^{\mathcal{H}}A_kA_l^{\mathcal{H}}A_l = A_k^{\mathcal{H}}(-A_lA_k^{\mathcal{H}})A_l = (A_l^{\mathcal{H}}A_k)A_k^{\mathcal{H}}A_l = A_l^{\mathcal{H}}A_k(-A_l^{\mathcal{H}}A_k) = A_l^{\mathcal{H}}(A_lA_k^{\mathcal{H}})A_k,$$

which implies that the set of matrices $\{A_k^{\mathcal{H}}A_k\}_{k=0}^{2K-1}$ forms a commuting family of Hermitian matrices and hence can be simultaneously diagonalized by a unitary matrix, U. Define $\hat{A}_k = A_kU^{\mathcal{H}}$, then $\hat{S} = \sum_{k=0}^{K-1} x_{kI}\hat{A}_{2k} + x_{kQ}\hat{A}_{2k+1}$ is a linear STBC such that $\hat{A}_k^{\mathcal{H}}\hat{A}_l + \hat{A}_l^{\mathcal{H}}\hat{A}_k = 0, \forall k \neq l, \hat{A}_k^{\mathcal{H}}\hat{A}_k = \mathcal{D}_k, \forall k$, where $\mathcal{D}_k$ is a diagonal matrix. For the converse, given $\hat{S}$, $S = \hat{S}U$ where U is a unitary matrix. □

Therefore for square SDD, we may, without any loss of generality, assume that $S^{\mathcal{H}}S$ is diagonal. One may expect a similar result to hold for $N \neq L$, but the problem is that

$A_kA_l^{\mathcal{H}} + A_lA_k^{\mathcal{H}} = 0, \forall k \neq l$ is not true in this case. Instead we have a condition of the form

$$\mathrm{tr}\left(A_kHA_l^{\mathcal{H}} + A_lHA_k^{\mathcal{H}}\right) = 0,$$

where $H \in \mathbb{C}^{N\times N}$ is any Hermitian matrix. But still we have proposition 2.4.2 for $N \neq L$.

Proposition 2.4.2. *A non-square single-symbol decodable design, $S = \sum_{k=0}^{K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$, exists if and only if there exists a design, $\hat{S} = \sum_{k=0}^{K-1} x_{kI}\ \hat{A}_{2k} + x_{kQ}\hat{A}_{2k+1}$ such that*

$$\hat{A}_k^{\mathcal{H}}\hat{A}_l + \hat{A}_l^{\mathcal{H}}\hat{A}_k = 0, \qquad k \neq l, \text{ and } \hat{S}^{\mathcal{H}}\hat{S} = \mathcal{D},$$

where $\mathcal{D}$ is a diagonal matrix whose entries are linear sum of $x_{kI}^2, x_{kQ}^2, k = 0, \cdots, K-1$ with non-negative coefficients.

Proof. Consider a SDD, $S = \sum_{k=0}^{K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$. Then $S^{\mathcal{H}}S$ is Hermitian, as a result $S^{\mathcal{H}}S = U\mathcal{D}U^{\mathcal{H}}$, where $U \in \mathbb{C}^{N\times N}$ is a unitary matrix and $\mathcal{D}$ is diagonal. Define

$$\hat{S} = SU^{\mathcal{H}} = \sum_{k=0}^{K-1} x_{kI}\hat{A}_{2k} + x_{kQ}\hat{A}_{2k+1} \text{ where } \hat{A}_k = A_kU^{\mathcal{H}}. \tag{2.55}$$

Now

$$\hat{A}_k^{\mathcal{H}}\hat{A}_l + \hat{A}_l^{\mathcal{H}}\hat{A}_k = U\left\{A_k^{\mathcal{H}}A_l + A_l^{\mathcal{H}}A_k\right\} = 0, k \neq l.$$

Consider

$$\begin{aligned}\hat{S}^{\mathcal{H}}\hat{S} = \mathcal{D} &= \sum_{k=0}^{K-1} x_{kI}^2\hat{A}_{2k}^{\mathcal{H}}\hat{A}_{2k} + x_{kQ}^2\hat{A}_{2k+1}^{\mathcal{H}}\hat{A}_{2k+1} \\ &= \sum_{k=0}^{K-1} x_{kI}^2UA_{2k}^{\mathcal{H}}A_{2k}U^{\mathcal{H}} + x_{kQ}^2UA_{2k+1}^{\mathcal{H}}A_{2k+1}U^{\mathcal{H}}. \end{aligned} \tag{2.56}$$

The diagonal entries of $\hat{A}_k^{\mathcal{H}}\hat{A}_k$ are non-negative for all $k = 0, \cdots, 2K-1$ and the non-diagonal entries cancel out in the sum i.e. in $\hat{S}^{\mathcal{H}}\hat{S}$. Therefore, $\mathcal{D}$ is a diagonal matrix whose entries are linear sum of $x_{kI}^2, x_{kQ}^2, k = 0, \cdots, K-1$ with non-negative coefficients. □

Observe that the matrices $\hat{A}_k$ in proposition 2.4.2 need not be constant matrices and accordingly $\hat{S}$ need not be a linear STBC. However, for the characterization of full-rank SDD this will not matter. Also we could not fabricate a single example for which $\hat{S}$ is non-linear. Hence we conjecture

Conjecture 2.4.3. *A non-square single-symbol decodable design,* $S = \sum_{k=0}^{K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$*, exists if and only if there exists a single-symbol decodable design,* $\hat{S} = \sum_{k=0}^{K-1} x_{kI}$ $\hat{A}_{2k} + x_{kQ}\hat{A}_{2k+1}$ *such that*

$$\hat{A}_k^{\mathcal{H}}\hat{A}_l + \hat{A}_l^{\mathcal{H}}\hat{A}_k = 0, \qquad k \neq l, \text{ and } \hat{A}_k^{\mathcal{H}}\hat{A}_k = \mathcal{D}_k,$$

where $\mathcal{D}_k$ *is a diagonal matrix.*

To characterize non-square SDD, use the following property,

Property 2.4.4. *For a SDD,* $S = \sum_{k=0}^{K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$*, the matrix* $S^{\mathcal{H}}S$ *is positive semi-definite and* $A_k^{\mathcal{H}}A_k, \forall k$ *are positive semi-definite.*

Using property 2.4.4, we have the characterization of full-rank SDD in the next proposition which gives the necessary condition for SDD to achieve full-diversity.

Proposition 2.4.5. *A single-symbol decodable design,* $S = \sum_{k=0}^{K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$*, whose weight matrices* A_k *satisfy*

$$A_k^{\mathcal{H}}A_l + A_l^{\mathcal{H}}A_k = 0, \qquad \forall k \neq l \tag{2.57}$$

achieves full-diversity only if $A_{2k}^{\mathcal{H}}A_{2k} + A_{2k+1}^{\mathcal{H}}A_{2k+1}$ *is full-rank for all* $k = 0, 1, \cdots, K-1$*. In addition if* S *is square then the requirement specializes to* $\mathcal{D}_{2k} + \mathcal{D}_{2k+1}$ *being full-rank for all* $k = 0, 1, \cdots, K-1$*.*

Proof. The proof is by contradiction and in two parts corresponding to whether S is square or non-square.

1) Let S be a square SDD then by proposition 2.4.1, without loss of generality, $A_k^{\mathcal{H}}A_k = \mathcal{D}_k, \forall k$. Suppose $\mathcal{D}_{2k} + \mathcal{D}_{2k+1}$, for some $0 \leq k \leq K-1$, is not full-rank. Then $S^{\mathcal{H}}S =$

$\sum_{k=0}^{K-1} \mathcal{D}_{2k}x_{kI}^2 + \mathcal{D}_{2k+1}x_{kQ}^2$. Now for any two transmission matrices S, $\hat{S}$ that differ only in x_k, the difference matrix $B(S,\hat{S}) = S - \hat{S}$, will not be full-rank as $B^{\mathcal{H}}(S,\hat{S})B(S,\hat{S}) = \mathcal{D}_{2k}(x_{kI} - \hat{x}_{kI})^2 + \mathcal{D}_{2k+1}(x_{kQ} - \hat{x}_{kQ})^2$ is not full-rank.

2) The proof for non-square SDD, S, is very similar to the above except that $B^{\mathcal{H}}(S,\hat{S})$ $B(S,\hat{S}) = A_{2k}^{\mathcal{H}}A_{2k}(x_{kI} - \hat{x}_{kI})^2 + A_{2k+1}^{\mathcal{H}}A_{2k+1}(x_{kQ} - \hat{x}_{kQ})^2$ where $A_k^{\mathcal{H}}A_k$ are positive semi-definite. Since a non-negative linear combination of positive semi-definite matrices is positive semi-definite, for full-diversity it is necessary that $A_{2k}^{\mathcal{H}}A_{2k} + A_{2k+1}^{\mathcal{H}}A_{2k+1}$ is full-rank for all $k = 0, 1, \cdots, K-1$. □

Towards obtaining a sufficient condition for full-diversity, we first introduce

Definition 2.4.1 (Coordinate Product Distance (CPD)). The Coordinate Product Distance (CPD) between any two signal points $u = u_I + \mathbf{j}u_Q$ and $v = v_I + \mathbf{j}v_Q$, $u \neq v$, in the signal set $\mathcal{A}$ is defined as

$$CPD(u,v) = |u_I - v_I||u_Q - v_Q| \tag{2.58}$$

and the minimum of this value among all possible pairs is defined as the CPD of $\mathcal{A}$.

Theorem 2.4.6. *A SSD, $S = \sum_{k=0}^{K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$ where x_k take values from a signal set $\mathcal{A}, \forall k$, satisfying the necessary condition of proposition 2.4.5 achieves full-diversity iff*

1. *either $A_k^{\mathcal{H}}A_k$ is of full-rank for all k*

2. *or the CPD of $\mathcal{A} \neq 0$.*

Proof. Let S be a square SDD satisfying the necessary condition given in Theorem 2.4.5. We have $B^{\mathcal{H}}(S,\hat{S})B(S,\hat{S})$=$\sum_{k=0}^{K-1} \mathcal{D}_{2k+1}$ $(x_{kI} - \hat{x}_{kI})^2 + \mathcal{D}_{2k+1}(x_{kQ} - \hat{x}_{kQ})^2$. Observe that under both these conditions the difference matrix $B(S,\hat{S})$ is full-rank for any two distinct S, $\hat{S}$. The proof is similar for When S is a non-square design. □

Remark 2.4.1. From Proposition 2.4.5 and Theorem 2.4.6 a FSDD is a SDD such that $A_{2k}^{\mathcal{H}}A_{2k} + A_{2k+1}^{\mathcal{H}}A_{2k+1}$ is full-rank for all k and $A_k^{\mathcal{H}}A_l + A_l^{\mathcal{H}}A_k = 0; \quad 0 \leq k \neq l \leq 2K-1$.

Examples of FSDD are the GLCODs and the STBCs of Example 2.3.2.

Note that the sufficient condition 1) of Theorem 2.4.6 is an additional condition on the weight matrices whereas the sufficient condition 2) is a restriction on the signal set $\mathcal{A}$ and not on the weight matrices A_k. Also, notice that the FSDD that satisfy the sufficient condition 1) are precisely an extension of GLCODs; GLCODs have an additional constraint that $A_k^{\mathcal{H}} A_k$ be diagonal.

An important consequence of Theorem 2.4.6 is that there can exist designs that are not covered by GLCODs offering full-diversity and single-symbol decoding provided the associated signal set has non-zero CPD. **It is important to note that whenever we have a signal set with CPD equal to zero, by appropriately rotating it we can end with a signal set with non-zero CPD. Indeed, only for a finite set of angles of rotation we will again end up with CPD equal to zero. So, the requirement of non-zero CPD for a signal set is not at all restrictive in real sense.** In the next chapter we find optimum angle(s) of rotation for lattice constellations that maximize the CPD.

For the case of square designs of size N with rate-one it is shown in Section 2.5 that FSDD exist for $N = 2, 4$ and these are precisely the STBCs of example 2.3.2 (to be named as Co-ordinate Interleaved Designs in Chapter 3) and the Alamouti code.

For a SDD, when $A_k^{\mathcal{H}} A_k$ is full-rank for all k, corresponding to Theorem 2.4.6 with the condition (1) for full-diversity satisfied, we have an extension of GLCOD in the sense that the STBC obtained by using the design with **any** complex signal set for the indeterminates results in a FSDD. That is, there is no restriction on the complex signal set that can be used with such designs. So, we define,

Definition 2.4.2 (Unrestricted FSDD (UFSDD)). A FSDD is called an Unrestricted Full-rank Single-symbol Decodable Design (UFSDD) if $A_k^{\mathcal{H}} A_k$ is of full-rank for all $k = 0, \cdots, 2K-1$.

Remark 2.4.2. Observe that for a square UFSDD S, $A_k^{\mathcal{H}} A_k = \mathcal{D}_k$ is diagonal and hence UFSDD reduces to LCOD. For non-square designs GLCOD is a subset of UFSDD. Also the above extension of the definition of GLCODs was hinted in [22] where they observe that $A_k^{\mathcal{H}} A_k$ can be positive definite. However it is clear from our characterization that such a generalization does not result in any gain for square designs. For non-square designs existence of UFSDDs that are not GLCODs or unitarily equivalent to GLCODs is an open problem.

The FSDD that are not UFSDDs are such that $A_{2k}^{\mathcal{H}}A_{2k}$ and/or $A_{2k+1}^{\mathcal{H}}A_{2k+1}$ is not full-rank for at least one k. (The CIOD codes of Example 2.3.2 are such that $\mathcal{D}_{2k}+\mathcal{D}_{2k+1}$ is full-rank $\forall k$ and $\mathcal{D}_k$ is not full-rank for all k.) We call such FSDD codes Restricted Full-rank Single-symbol Decodable Designs (**RFSDD**), since any full-rank design within this class can be there only with a restriction on the complex constellation from which the indeterminates take values, the restriction being that the CPD of the signal set should not be zero. Formally,

Definition 2.4.3 (Restricted FSDD (RFSDD)). A Restricted Full-rank Single-symbol Decodable Designs (RFSDD) is a FSDD such that $A_k^{\mathcal{H}}A_k$ is not full-rank for at least one k where $k = 0, \cdots, 2K-1$ and the signal set, from which the indeterminates take values from, has non-zero CPD.

Observe that the CIODs are a subset of RFSDD. Figure 2.1 shows all the classes discussed so far, viz., SDD, FSDD, RFSDD, UFSDD. In Section 2.5 we focus on the square RFSDDs as square UFSDD have been discussed in Section 2.3.

2.5 Existence of Square RFSDDs

A main result in this section is the proof for the fact that **there exists square RFSDDs with the maximal rate $\frac{2a}{2^a}$ for $N = 2^a$ antennas whereas only rates up to $\frac{a+1}{2^a}$ is possible with square GLCODs with the same number of antennas**. The other results are: (i) rate-one square RFSDD of size N exist, iff $N = 2, 4$ and (ii) a construction of RFSDDs with maximum rate from GLCODs.

Let $S = \sum_{k=0}^{K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$ be a square RFSDD. We have,

$$A_k^{\mathcal{H}}A_k = \mathcal{D}_k, \qquad k = 0, \cdots, 2K-1 \tag{2.59}$$

$$A_l^{\mathcal{H}}A_k + A_k^{\mathcal{H}}A_l = 0, \qquad 0 \le k \ne l \le 2K-1 \tag{2.60}$$

where $\mathcal{D}_k, k = 0, \cdots, 2K-1$ are diagonal matrices with non-negative entries such that $\mathcal{D}_{2k}+\mathcal{D}_{2k+1}$ is full-rank $\forall k$. First we show that rate 1 RFSDD exist only if $N = 2, 4, 8$.

Theorem 2.5.1. *If S is a size N square RFSDD of rate 1, then $N = 2, 4$ or 8.*

Proof. Let

$$B_k = A_{2k} + A_{2k+1}, \quad k = 0, \cdots, K-1$$

then

$$B_k^{\mathcal{H}} B_k = \hat{\mathcal{D}}_k = \mathcal{D}_{2k} + \mathcal{D}_{2k+1}, \quad k = 0, \cdots, K-1 \tag{2.61}$$

$$B_l^{\mathcal{H}} B_k + B_k^{\mathcal{H}} B_l = 0, \quad 0 \le k \ne l \le K-1. \tag{2.62}$$

Observe that $\hat{\mathcal{D}}_k$ is of full-rank for all k. Define $C_k = B_k \hat{\mathcal{D}}_k^{-1/2}$. Then the matrices C_k satisfy

$$C_k^{\mathcal{H}} C_k = I_N, \quad k = 0, \cdots, K-1 \tag{2.63}$$

$$C_l^{\mathcal{H}} C_k + C_k^{\mathcal{H}} C_l = 0, \quad 0 \le k \ne l \le K-1. \tag{2.64}$$

Define

$$\hat{C}_k = C_0^{\mathcal{H}} C_k, \quad k = 0, \cdots, K-1, \tag{2.65}$$

then $\hat{C}_0 = I_N$ and

$$\hat{C}_k^{\mathcal{H}} = -\hat{C}_k, \quad k = 1, \cdots, K-1 \tag{2.66}$$

$$\hat{C}_l^{\mathcal{H}} \hat{C}_k + \hat{C}_k^{\mathcal{H}} \hat{C}_l = 0, \quad 1 \le k \ne l \le K-1. \tag{2.67}$$

The normalized set of matrices $\{\hat{C}_1, \cdots, \hat{C}_{K-1}\}$ constitute a *Hurwitz family of order N [24]* and for $N = 2^a b$, b odd and $a, b > 0$ the number of such matrices $K-1$ is bounded by [24]

$$K \le 2a + 2.$$

For rate 1, RFSDD ($K = N$), the inequality can be satisfied only for $N = 2, 4$ or 8. □

Therefore the search for rate 1, square RFSDDs can be restricted to $N = 2, 4, 8$. The rate 1, RFSDDs for $N = 2, 4$ have been presented in Example 2.3.2. We will now prove that a rate 1, square RFSDD for $N = 8$ does not exist. Towards this end we first derive the maximal rates of square RFSDDs.

Theorem 2.5.2. *The maximal rate, R, achievable by a square RFSDD with $N = 2^a b, b$ odd (where $a, b > 0$) transmit antennas is*

$$R = \frac{2a}{2^a b} \tag{2.68}$$

Proof. Let $S = \sum_{k=0}^{K-1} x_{kI} A_{2k} + x_{kQ} A_{2k+1}$ be a square RFSDD. Define the RFSDD

$$S' = \sum_{k=0}^{K-1} x_{kI} \underbrace{C_0^{\mathcal{H}} A_{2k} \hat{\mathcal{D}}_0^{-1/2}}_{A'_{2k}} + x_{kQ} \underbrace{C_0^{\mathcal{H}} A_{2k+1} \hat{\mathcal{D}}_0^{-1/2}}_{A'_{2k+1}}$$

where C_k and $\hat{\mathcal{D}}_k$ are defined in the proof of the previous theorem. Then the set of matrices $\{C'_k = A'_{2k} + A'_{2k+1}\}$ is such that $C'_0 = I_N$ and $\{C'_k, k = 1, \cdots, K-1\}$ is a family of matrices of order N such that

$$C_k'^{\mathcal{H}} C'_k = \hat{\mathcal{D}}_0^{-1} \hat{\mathcal{D}}_k, \quad 1 \le k \le K-1, \tag{2.69}$$

$$C_l'^{\mathcal{H}} C'_k + C_k'^{\mathcal{H}} C'_l = 0, \quad 0 \le k \neq l \le K-1, \tag{2.70}$$

where $\hat{\mathcal{D}}_0^{-1} \hat{\mathcal{D}}_k$ is diagonal and full-rank for all k. Then we have

$$A'_0 + A'_1 = C'_0 = I_N. \tag{2.71}$$

Recollect that the set of matrices $\{A'_k\}$ satisfy (2.59)- (2.60). Also, at least one A'_k is not full-rank. Without loss of generality we assume that A'_0 is of rank $r < N$ (if this not so then exchange the indeterminates and/or the in-phase and quadrature components so that this is satisfied). As A'_0 is of rank r, due to (2.59), $n - r$ columns of A'_0 are zero vectors. Assume that first r columns of A'_0 are non-zero (If this is not the case, we can always multiply all the weight matrices with a Permutation matrix such that A'_0 is of this form) i.e.

$$A'_0 = \begin{bmatrix} B'_0 & 0 \end{bmatrix} \tag{2.72}$$

where $B'_0 \in \mathbb{C}^{N \times r}$. Applying (2.60) to A'_0, A'_1 and substituting from (2.71),(2.72) we have

$$A'^{\mathcal{H}}_0(I_N - A'_0) + (I_N - A'^{\mathcal{H}}_0)A'_0 = 0 \tag{2.73}$$

$$\Rightarrow \quad A'^{\mathcal{H}}_0 + A'_0 = 2\mathcal{D}'_0 \tag{2.74}$$

$$\Rightarrow \quad \begin{bmatrix} B'^{\mathcal{H}}_0 \\ 0 \end{bmatrix} + \begin{bmatrix} B'_0 & 0 \end{bmatrix} = 2\mathcal{D}'_0 \tag{2.75}$$

$$\Rightarrow \quad B'_0 = \begin{bmatrix} B'_{11} \\ 0 \end{bmatrix} \tag{2.76}$$

where B'_{11} is a $r \times r$ matrix and full-rank and $A'^{\mathcal{H}}_k A'_k = \mathcal{D}'_k, k = 0, \cdots, 2K-1$. Therefore the matrices A'_0, A'_1 are of the form

$$A'_0 = \begin{bmatrix} B'_{11} & 0 \\ 0 & 0 \end{bmatrix}, A'_1 = \begin{bmatrix} I_r - B'_{11} & 0 \\ 0 & I_{N-r} \end{bmatrix}. \tag{2.77}$$

Let

$$D_1 = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$$

be a matrix such that

$$A'^{\mathcal{H}}_i D_1 + D^{\mathcal{H}}_1 A'_i = 0, i = 0, 1 \tag{2.78}$$

where $D_{11} \in \mathbb{C}^{r \times r}, D_{22} \in \mathbb{C}^{N-r \times N-r}$. Substituting the structure of A'_0 we have

$$A'^{\mathcal{H}}_0 D_1 + D^{\mathcal{H}}_1 A'_0 = 0 \tag{2.79}$$

$$\Rightarrow \quad \begin{bmatrix} B'^{\mathcal{H}}_{11} D_{11} + D^{\mathcal{H}}_{11} B'_{11} & B'^{\mathcal{H}}_{11} D_{12} \\ D^{\mathcal{H}}_{12} B'_{11} & 0 \end{bmatrix} = 0. \tag{2.80}$$

As B'_{11} is full-rank it follows that $D_{12} = 0$. Substituting the structure of A'_1 we have

$$\begin{bmatrix} (I_r - B'^{\mathcal{H}}_{11})D_{11} + D^{\mathcal{H}}_{11}(I_r - B'_{11}) & D^{\mathcal{H}}_{21} \\ D_{21} & D_{22} + D^{\mathcal{H}}_{22} \end{bmatrix} = 0 \tag{2.81}$$

$$\Rightarrow D_{21} = 0. \tag{2.82}$$

It follows that D_1 is block diagonal and consequently all the $A'_k, 2 \leq k \leq 2K-1$ are block diagonal of the form D_1 as they satisfy (2.78). Consequently, $C'_k = {A'}_{2k} + {A'}_{2k+1}, k = 1, \cdots, K-1$ are also block diagonal of the form

$$C'_k = \begin{bmatrix} C'_{k1} & 0 \\ 0 & C'_{k2} \end{bmatrix} \text{ where } C'_{k1} \in \mathbb{C}^{r \times r}, C'_{k2} \in \mathbb{C}^{N-r \times N-r}.$$

Also, from (2.80), (2.81) we have

$$D_{11} = -D^{\mathcal{H}}_{11}, D_{22} = -D^{\mathcal{H}}_{22}. \tag{2.83}$$

Now, in addition to this block diagonal structure the matrices $A'_k, 2 < k \leq K-1$ have to satisfy (2.60) among themselves. It follows that the sets of square matrices $\{C'_{k1}, k = 0, \ldots, K-1\}$ and $\{C'_{k2}, k = 0 \ldots, K-1\}$ satisfy

$$C'^2_{ki} = -\mathcal{D}_{ki}, \quad k = 1, \cdots, K-1, \quad i = 1, 2; \tag{2.84}$$

$$C'_{ki}C'_{li} = -C'_{li}C'_{ki}, \quad 1 \leq k \neq l \leq K-1, i = 1, 2, \tag{2.85}$$

where $-\mathcal{D}_{ki}$ are diagonal and full-rank $\forall k, i$. Define

$$\hat{C}_{ki} = C'_{ki}\mathcal{D}^{-1/2}_{ki}, \quad k = 1, \cdots, K-1, \quad i = 1, 2; \tag{2.86}$$

then from Theorem 2.2.8,

$$\hat{C}_{ki}^2 = -I, \quad k = 1, \cdots, K-1, \quad i = 1,2; \tag{2.87}$$

$$\hat{C}_{ki}\hat{C}_{li} = -\hat{C}_{li}\hat{C}_{ki}, \quad 1 \leq k \neq l \leq K-1, i = 1,2. \tag{2.88}$$

and the sets of square matrices $\{\hat{C}_{k1}, k = 1, 2, \ldots, K-1\}$ and $\{\hat{C}_{k2}, k = 1, 2, \ldots, K-1\}$ constitute Hurwitz families of order $r, N-r$ corresponding to $i = 1, 2$ respectively. Let $H(N) - 1$ be the maximum number of matrices in a Hurwitz family of order N, then from the Hurwitz Theorem [24] , $N = 2^a b, b$ odd and

$$H(N) = 2a + 2. \tag{2.89}$$

Observe that due to the block diagonal structure of C'_k, $K = \min\{H(r_i), H(N-r_i)\}$. Following the Hurwitz Theorem it is sufficient to consider both $r, N-r$ to be of the form 2^a, say $2^{a_1}, 2^{a_2}$ respectively. It follows that K is maximized iff $r = N - r = 2^{a'} \Rightarrow N = 2^{a'+1}$. It follows that the maximum rate of RFSDD of size $N = 2^a$ $(a = a' + 1)$ is

$$\mathcal{R} = \frac{2a}{2^a}. \tag{2.90}$$

□

An important observation regarding square RFSDDs is summarized in the following Corollary:

Corollary 2.5.3. *A maximal rate square RFSDD,* $S = \sum_{k=0}^{K-1} x_{kI} A_{2k} + x_{kQ} A_{2k+1}$ *exists iff both* $\mathcal{D}_{2k}, \mathcal{D}_{2k+1}$ *are not full-rank for all* k.

Proof. Immediate from the proof of above theorem. □

An immediate consequence of this characterization of maximal rate RFSDDs is:

Theorem 2.5.4. *A square RFSDD of rate 1, exists iff* $N = 2, 4$.

Proof. From (2.90) $\mathcal{R} = 1$ iff $N = 2, 4$ □

It follows that

Theorem 2.5.5. *The maximal rate, $\mathcal{R}$, achievable by a square FSDD with $N = 2^a b, b$ odd (where $a, b > 0$) transmit antennas is*

$$\mathcal{R} = \frac{2a}{2^a b}. \tag{2.91}$$

Furthermore square GLCODs are not maximal rate FSDD except for $N = 2$.

Next we give a construction of square RFSDD that achieves the maximal rates obtained in Theorem 2.5.2.

Theorem 2.5.6. *A square RFSDD S, of size N, in variables $x_i, i = 0, \cdots, K$ achieving the rate of Theorem 2.5.2 is given by*

$$S = \begin{bmatrix} \underbrace{\Theta(\tilde{x}_0, \cdots, \tilde{x}_{K/2})}_{\Theta_1} & 0 \\ 0 & \underbrace{\Theta(\tilde{x}_{K/2}, \cdots, \tilde{x}_{K-1})}_{\Theta_2} \end{bmatrix} \tag{2.92}$$

where $\Theta(x_0, \cdots, x_{K/2-1})$ is a maximal rate square GLCOD of size $N/2$ [15, 22], $\tilde{x}_i = \mathrm{Re}\{x_i\} + \mathbf{j}\,\mathrm{Im}\{x_{(i+K/2)_K}\}$ and where $(a)_K$ denotes $a \pmod K$.

Proof. The proof is by direct verification. As the maximal rate of square GLCOD of size $N/2$ is $\frac{a}{2^{a-1}b}$ [15, 22] the rate of S in (2.92) is $2\frac{a}{2^a b} = \frac{2a}{2^a b}$ consequently S is maximal rate. Next we show that S is a RFSDD. Consider

$$S^{\mathcal{H}} S = \begin{bmatrix} \Theta_1^{\mathcal{H}}\Theta_1 & 0 \\ 0 & \Theta_2^{\mathcal{H}}\Theta_2 \end{bmatrix},$$

by construction, the sum of weight matrices of x_{kI}^2, x_{kQ}^2 for any symbol x_k is I_N and (2.59)-(2.60) are satisfied as Θ is a GLCOD. Therefore S is a RFSDD. □

Other square RFSDDs can be constructed from (2.92) by applying some of the following

- permuting rows and/or columns of (2.92),
- permuting the real symbols $\{x_{kI}, x_{kQ}\}_{k=0}^{K-1}$,
- multiplying a symbol by -1 or $\pm\mathbf{j}$
- conjugating a symbol in (2.92).

Following [15, Theorem 2] we have

Theorem 2.5.7. *All square RFSDDs can be constructed from RFSDD S of (2.92) by possibly deleting rows from a matrix of the form*

$$S' = USV \tag{2.93}$$

where U, V are unitary matrices, up to permutations and possibly sign change in the set of real and imaginary parts of the symbols.

Proof. This follows from the observation after (2.84) that the pair of sets $\{C'_{ki}\}_{k=1}^{K-1}, i = 1, 2$ constitute a Hurwitz family and Theorem 2 of [15] which applies to Hurwitz families. □

It follows that the CIOD codes presented in Example 2.3.2 are unique up to multiplication by unitary matrices.

Observe that the square RFSDDs of Theorem 2.5.6 can be thought of as designs combining co-ordinate interleaving and GLCODs. We therefore, call such RFSDDs as (instances of) co-ordinate interleaved orthogonal designs (CIOD) and study such designs in detail in the next chapter.

2.6 Discussion

In this chapter we have characterized all linear STBCs from designs called SDD that allow single-symbol decoding. Among these all those that offer full-diversity have been identified. The maximal rates of such designs have been derived and a construction that achieves these maximal rates is presented. Throughout the chapter we have assumed that the indeterminates of a design take values from one complex constellation. All the arguments and results hold even if we allow different indeterminates to take values from different complex constellations. In summary, it has been shown that full-rate, full-diversity and single-symbol decodable designs can exist outside the well-known class of GLCODs. Such codes have been explicitly shown for square designs. Interesting directions for further research are: (i) constructing non-square RFSDDs and (ii) obtaining maximal rate of non-square RFSDDs.

Chapter 3

Co-ordinate Interleaved Orthogonal Designs

In the previous chapter we completely characterized single-symbol decodable designs (SDD) in terms of the weight matrices. Among these we characterized all full-rank SDD called FSDD. This chapter is devoted to an interesting class of FSDD called co-ordinate interleaved orthogonal designs.

We first give the construction of the CIOD for two transmit antennas and then formally define the class of Co-ordinate Interleaved Orthogonal Designs (CIOD) and its generalization, Generalized CIOD (GCIOD) which includes both square and non-square designs in Section 3.1. Also, we show that rate 1 GCIODs exist for 2, 3 and 4 transmit antennas and for all other antenna configurations the rate is strictly less than 1. Rate 6/7 designs for 5 and 6 transmit antennas, rate 7/9 designs for 7 transmit antennas, rate 3/4 designs for 8 transmit antennas, rate 7/11 designs for 9 and 10 transmit antennas and rate 3/5 designs for 11 and 12 transmit antennas are also presented. A construction of rate 2/3 GCIOD for $N \geq 8$ is then presented. In Section 3.2 the signal set expansion associated with the use of STBC from any co-ordinate interleaving when the uninterleaved complex variables take values from a signal set is then brought out and the notion of co-ordinate product distance (CPD) is discussed. The coding gain aspects of the STBC from CIODs constitute Section 3.3 and we show that, for lattice constellations, GCIODs have higher coding gain as compared to GLCODs. Simulation results are presented in Section 3.4. The maximum mutual information (MMI) of GCIODs is discussed in Section 3.5 and is compared with that of GLCODs to show that, except for $N = 2$, CIODs have higher MMI. A brief discussion on the results of this chapter constitute Section 3.6.

3.1 Co-ordinate Interleaved Orthogonal Designs

We begin from an intuitive construction of the CIOD for two transmit antennas before giving a formal definition (Definition 3.1.1). Consider the Alamouti code

$$S = \begin{bmatrix} x_0 & x_1 \\ -x_1^* & -x_0^* \end{bmatrix}.$$

When the number of receive antennas $M = 1$, observe that the diversity gain in the Alamouti code is due to the fact that each symbol sees two different channels h_0 and h_1 and the low ML decoding complexity is due to the use of the orthogonality of columns of signal transmission matrix, by the receiver, over two symbol periods to form an estimate of each symbol.

Alternately, diversity gain may still be achieved by transmitting quadrature components of each symbol separately on different antennas. More explicitly, consider that the in-phase component, x_{0I}, of a symbol, $x_0 = x_{0I} + \mathbf{j}x_{0Q}$, is transmitted on antenna zero and in the next symbol interval the quadrature component, x_{0Q}, is transmitted from antenna one as shown in Table 3.1. It is apparent that this procedure is similar to that of

Table 3.1: The Encoding And Transmission Sequence For N =2, Rate 1/2 CIOD

	antenna 0	antenna 1
time t	x_{0I}	0
time $t+T$	0	x_{0Q}

co-ordinate interleaving[1] and that the symbol has diversity two if the difference of the in-phase and quadrature components is not-zero, but the rate is half. This loss of rate can

[1]The idea of rotating QAM constellation was first presented in [66] and the term “co-ordinate interleaving” was first introduced by J. Roy in [46, 47] in the context of TCM for fading channels. This concept of rotation of QAM constellation was extended to multi-dimensional QAM constellations in [67, 62] at the cost of the decoding complexity. However, for the two-dimensional case there is no increase in the decoding complexity as shown in [44, 45].

be compensated by choosing two symbols and exchanging their quadrature components so that one coordinate of each symbol is transmitted on one of the antennas as shown in Table 3.2. As only one antenna is used at a time for transmission, the only operation re-

Table 3.2: The Encoding And Transmission Sequence For N =2, Rate 1 CIOD

	antenna 0	antenna 1
time t	$x_{0I} + \mathbf{j}x_{1Q}$	0
time $t+T$	0	$x_{1I} + \mathbf{j}x_{0Q}$

quired at the receiver to decouple the symbols is to exchange the quadrature components of the received signals for two symbol periods after phase compensation. That the decoding is single-symbol symbol decoding with the in-phase and quadrature-phase components having got affected by noise components of different variances for any GCIOD is shown in Subsection 3.1.1. In the same subsection the full-rankness of GCIOD is also proved. If we combine, the Alamouti scheme with co-ordinate interleaving we have the scheme for 4 transmit antennas of Example 2.3.2, and whose baseband representation shown in Fig. 3.1 and whose receiver structure is explained in detail in Example 3.1.2. Now, a formal definition of GCIODs follows:

Definition 3.1.1 (Generalized Co-ordinate interleaved design (GCIOD)). A Generalized co-ordinate interleaved design of size $N_1 \times N_2$ in variables $x_i, i = 0, \cdots, K-1$ is a $L \times N$ matrix S, such that

$$S(x_0, \cdots, x_{K-1}) = \begin{bmatrix} \Theta_1(\tilde{x}_0, \cdots, \tilde{x}_{K/2-1}) & 0 \\ 0 & \Theta_2(\tilde{x}_{K/2}, \cdots, \tilde{x}_{K-1}) \end{bmatrix} \quad (3.1)$$

where $\Theta_1(x_0, \cdots, x_{K/2-1})$ and $\Theta_2(x_0, \cdots, x_{K/2-1})$ are GLCODs of size $N_1 \times L_1$ and $N_2 \times L_2$ respectively, and rates $K/2L_1, K/2L_2$ respectively, where $N_1 + N_2 = N$, $L_1 + L_2 = L$, $\tilde{x}_i = \mathrm{Re}\{x_i\} + \mathbf{j}\,\mathrm{Im}\{x_{(i+K/2)_K}\}$ and where $(a)_K$ denotes $a \pmod K$. In addition if $\Theta_1 = \Theta_2$ then we call this design a **Co-ordinate interleaved orthogonal design(CIOD)**[2].

[2]These designs were named as Co-ordinate interleaved orthogonal design (CIOD) in [51, 52] since two different columns are indeed orthogonal. However, the energy of different columns may be different

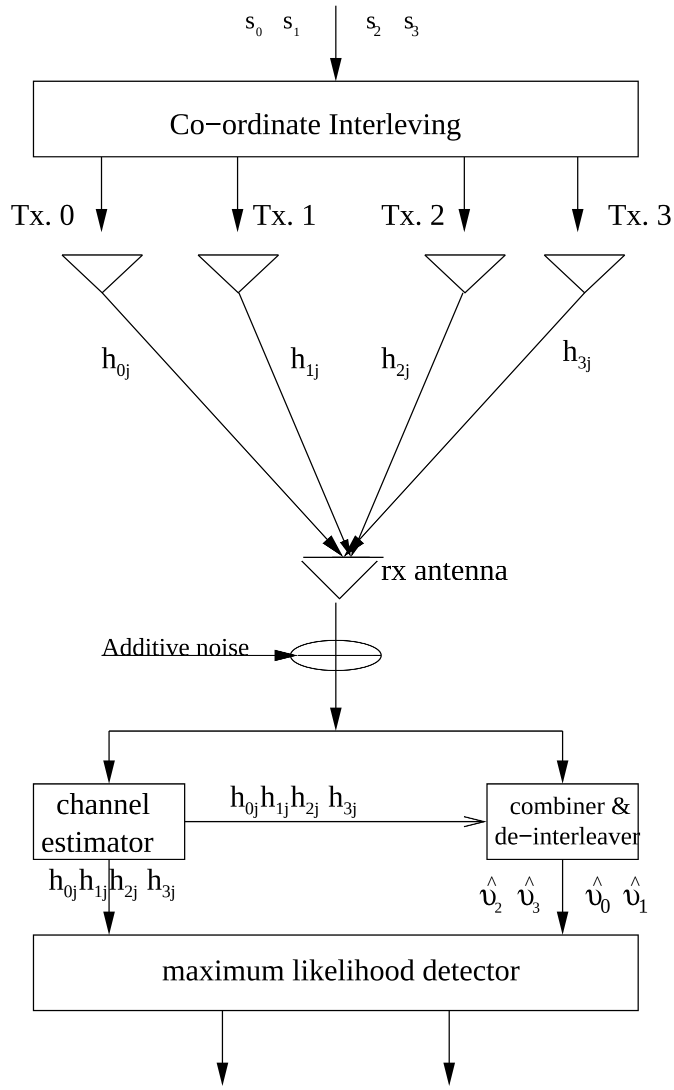


Figure 3.1: Baseband representation of the CIOD for four transmit and the j-th receive antennas.

It turns out that the theory of CIODs is simpler as compared to that of GCIOD. Note that when $\Theta_1 = \Theta_2$ and $N = L$ we have the construction of square RFSDDs given in Theorem 2.5.6. Examples of square CIOD for $N = 2, 4$ were presented in Example 2.3.2. An example of GCIOD, where $\Theta_1 \neq \Theta_2$ is

Example 3.1.1.

$$S(x_0, \cdots, x_3) = \begin{bmatrix} x_{0I} + \mathbf{j}x_{2Q} & x_{1I} + \mathbf{j}x_{3Q} & 0 \\ -x_{1I} + \mathbf{j}x_{3Q} & x_{0I} - \mathbf{j}x_{2Q} & 0 \\ 0 & 0 & x_{2I} + \mathbf{j}x_{0Q} \\ 0 & 0 & -x_{3I} + \mathbf{j}x_{1Q} \end{bmatrix} \qquad (3.2)$$

where Θ_1 is the rate 1 Alamouti code and Θ_2 is the trivial, rate 1, GLCOD for $N = 1$ given by

$$\Theta_2 = \begin{bmatrix} x_0 \\ -x_1' \end{bmatrix}.$$

Observe that S is non-square and rate 1. This code can also be thought of as being obtained by dropping the last column of the CIOD in (2.54). Finally, observe that (3.2) is not unique and we have different designs as we take

$$\Theta_2 = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}, \begin{bmatrix} x_0 \\ -x_1 \end{bmatrix}$$

etc. for the second GLCOD.

3.1.1 Coding and Decoding for STBCs from GCIODs

First, we show that a GCIOD is a RFSDD and hence is single-symbol decodable and achieves full diversity if indeterminates take values from a signal set with non-zero CPD.

Theorem 3.1.1. *A GCIOD is a RFSDD.*

whereas in conventional GLCODs apart from orthogonality or two different columns, all the columns will have the same energy. GLCODs are in fact, "orthonormal".

Proof. The proof is by direct verification. Let S be a GCIOD defined in (3.1). Consider

$$S^{\mathcal{H}}S = \begin{bmatrix} \Theta_1^{\mathcal{H}}\Theta_1 & 0 \\ 0 & \Theta_2^{\mathcal{H}}\Theta_2 \end{bmatrix} \tag{3.3}$$

$$= \begin{bmatrix} \left(\sum_{k=0}^{K/2-1} x_{kI}^2 + x_{(k+K/2)_K Q}^2\right) I_{N_1} & 0 \\ 0 & \left(\sum_{k=K/2}^{K-1} x_{kI}^2 + x_{(k+K/2)_K Q}^2\right) I_{N_2} \end{bmatrix}, \tag{3.4}$$

where $(a)_K$ denotes $a \pmod K$. By construction, the sum of weight matrices of x_{kI}^2, x_{kQ}^2 for any symbol x_k is I_N and (2.59)-(2.60) are satisfied as Θ_1, Θ_2 are GCLODs. Moreover, observe that there are no terms of the form $x_{kI}x_{kQ}, x_{kI}x_{lQ}$ etc. in $S^{\mathcal{H}}S$, and therefore S is a FSDD. Now for any given $0 \leq k \leq K-1$ the weight matrices of both x_{kI}^2, x_{kQ}^2 are not full-rank and therefore, by definition 2.4.3, S is a RFSDD. □

The transmission scheme for a GCIOD, $S(x_0, \cdots, x_{K-1})$ of size N, is as follows: let Kb bits arrive at the encoder in a given time slot. The encoder selects K complex symbols, $s_i, i = 0, \cdots, K-1$ from a complex constellation $\mathcal{A}$ of size $|\mathcal{A}| = 2^b$. Then setting $x_i = s_i, i = 0, \cdots, K-1$, the encoder populates the transmission matrix with the complex symbols for the corresponding number of transmit antennas. The corresponding transmission matrix is given by $S(s_0, \cdots, s_{K-1})$. The received signal matrix (1.6) is given by,

$$\mathbf{V} = S\mathbf{H} + \mathbf{N}. \tag{3.5}$$

Now as GCIODs is a RFSDD (Theorem 3.1.1), it is single-symbol decodable and the receiver uses (2.45) to form an estimate of each s_i. That is the ML rule for each $s_i, i = 0, \cdots, K-1$ is given by

$$\min_{s_i \in \mathcal{A}} M_i(s_i) = \min_{s_i \in \mathcal{A}} \|\mathbf{V} - (A_{2i}s_{iI} + A_{2i+1}s_{iQ})\mathbf{H}\|^2. \tag{3.6}$$

Remark 3.1.1. Note that forming the ML metric for each variable in (3.6), implicitly involves co-ordinate de-interleaving, in the same way as the coding involves co-ordinate interleaving. Also notice that the components s_{iI} and s_{iQ} have been weighted differently- something that does not happen for GLCODs. We elaborate these aspects of decoding

GCIODs by considering the decoding of rate 1, CIOD for $N = 4$ in detail.

Example 3.1.2 (Coding and Decoding for CIOD for $N = 4$). *Consider the CIOD for $N = 4$ given in (2.54). If the signals $s_0, s_1, s_2, s_3 \in \mathcal{A}$ are to be communicated, their interleaved version as given in Definition 3.1.1 are transmitted. The signal transmission matrix is*

$$S = \begin{bmatrix} \underbrace{s_{0I} + \mathbf{j}s_{2Q}}_{\tilde{s}_0} & \underbrace{s_{1I} + \mathbf{j}s_{3Q}}_{\tilde{s}_1} & 0 & 0 \\ -s_{1I} + \mathbf{j}s_{3Q} & s_{0I} - \mathbf{j}s_{2Q} & 0 & 0 \\ 0 & 0 & \underbrace{s_{2I} + \mathbf{j}s_{0Q}}_{\tilde{s}_2} & \underbrace{s_{3I} + \mathbf{j}s_{1Q}}_{\tilde{s}_3} \\ 0 & 0 & -s_{3I} + \mathbf{j}s_{1Q} & s_{2I} - \mathbf{j}s_{0Q} \end{bmatrix} \quad (3.7)$$

which is obtained by replacing x_i in the CIOD by s_i where each s_i, $i = 0, 1, 2, 3$ takes values from a signal set $\mathcal{A}$ with 2^b points.

The received signals at the different time slots, v_{jt}, $t = 0, 1, 2, 3$ and $j = 0, 1, \cdots, M-1$ for the M receive antennas are given by

$$v_{j0} = h_{0j}\tilde{s}_0 + h_{1j}\tilde{s}_1 + n_{j0}; \qquad v_{j1} = -h_{0j}\tilde{s}_1^* + h_{1j}\tilde{s}_0^* + n_{j1};$$
$$v_{j2} = h_{2j}\tilde{s}_2 + h_{3j}\tilde{s}_3 + n_{j2}; \qquad v_{j3} = -h_{2j}\tilde{s}_3^* + h_{3j}\tilde{s}_2^* + n_{j3} \quad (3.8)$$

where n_{ji} , $i = 0, 1, 2, 3$ and $j = 0, \cdots, M-1$ are complex independent Gaussian random variables.

Let $\mathbf{V}_j = [v_{j0},\ v_{j1}^*,\ v_{j2},\ v_{j3}^*]^{\mathcal{T}}$, $\tilde{S} = [\tilde{s}_0,\ \tilde{s}_1,\ \tilde{s}_2,\ \tilde{s}_3]^{\mathcal{T}}$, $\mathbf{N}_j = [n_{j0},\ n_{j1}^*,\ n_{j2},\ n_{j3}^*]^{\mathcal{T}}$ and

$$\mathbf{H}_j = \begin{bmatrix} h_{0j} & h_{1j} & 0 & 0 \\ h_{1j}^* & -h_{0j}^* & 0 & 0 \\ 0 & 0 & h_{2j} & h_{3j} \\ 0 & 0 & h_{3j}^* & -h_{2j}^* \end{bmatrix}$$

where $j = 0, 1, \cdots, M-1$. *Using this notation, (3.8) can be written as*

$$\mathbf{V}_j = \mathbf{H}_j \tilde{S} + \mathbf{N}_j. \tag{3.9}$$

Let

$$\tilde{\mathbf{V}}_j = [\tilde{v}_{j0},\ \tilde{v}_{j1},\ \tilde{v}_{j2},\ \tilde{v}_{j3}]^{\mathcal{T}} = \mathbf{H}_j^{\mathcal{H}} \mathbf{V}_j.$$

Then, we have

$$\tilde{\mathbf{V}}_j = \begin{bmatrix} \left(|h_{0j}|^2 + |h_{1j}|^2\right) I_2 & 0 \\ 0 & \left(|h_{2j}|^2 + |h_{3j}|^2\right) I_2 \end{bmatrix} \tilde{S} + \mathbf{H}_j^{\mathcal{H}} \mathbf{N}_j. \tag{3.10}$$

Rearranging the in-phase and quadrature-phase components of $\tilde{v}_{ji}$*'s, (which corresponds to deinterleaving) define, for* $i = 0, 1$,

$$\hat{v}_i = \sum_{j=0}^{M-1} \tilde{v}_{ji,I} + \mathbf{j}\tilde{v}_{ji+2,Q} = a s_{i,I} + \mathbf{j} b s_{i,Q} + u_{0i} \tag{3.11}$$

$$\hat{v}_{i+2} = \sum_{j=0}^{M-1} \tilde{v}_{ji+2,I} + \mathbf{j}\tilde{v}_{ji,Q} = b s_{i+2,I} + \mathbf{j} a s_{i+2,Q} + u_{1i} \tag{3.12}$$

where $a = \sum_{j=0}^{M-1} \{\alpha_{0j}^2 + \alpha_{1j}^2\}$, $b = \sum_{j=0}^{M-1} \{\alpha_{2j}^2 + \alpha_{3j}^2\}$ *and* u_{0i}, u_{1i} *are complex Gaussian random variables. Note that we have used* $h_{ij} = \alpha_{ij} e^{\mathbf{j}\theta_{ij}}$ *from Chapter 1. Let* $\tilde{\mathbf{N}}_j = [\tilde{n}_{j0}\ \tilde{n}_{j1}\ \tilde{n}_{j2}\ \tilde{n}_{j3}]^T = \mathbf{H}_j^{\mathcal{H}} \mathbf{N}_j$. *Then* $u_{0i} = \sum_{j=0}^{M-1} \tilde{n}_{ji,I} + \mathbf{j}\tilde{n}_{ji+2,Q}$ *and* $u_{1i} = \sum_{j=0}^{M-1} \tilde{n}_{ji+2,I} + \mathbf{j}\tilde{n}_{ji,Q}$ *where* $i = 0, 1$. *Note that* u_{00} *and* u_{01} *have the same variance and similarly* u_{10} *and* u_{11}. *The variance of the in-phase component of* u_{00} *is* a *and that of the quadrature-phase component is* b. *The in-phase component of* u_{10} *has the same variance as that of the quadrature-phase component of* u_{00} *and vice versa. The ML decision rule for such a situation, derived in a general setting is: Consider the received signal* r, *given by*

$$r = c_1 s_I + \mathbf{j} c_2 s_Q + n \tag{3.13}$$

where c_1, c_2 *are real constants and* s_I, s_Q *are in-phase and quadrature-phase components of transmitted signal* s. *The ML decision rule when the in-phase,* n_I, *and quadrature-phase component,* n_Q, *of the Gaussian noise,* n *have different variances* $c_1\sigma^2$ *and* $c_2\sigma^2$ *is derived by considering the pdf of* n, *given by*

$$p_n(n) = \frac{1}{2\pi\sigma^2\sqrt{c_1 c_2}} e^{-\frac{n_I^2}{2c_1\sigma^2}} e^{-\frac{n_Q^2}{2c_2\sigma^2}}. \tag{3.14}$$

The ML rule is: decide in favor of s_i, *if and only if*

$$p_n(r/s_i) \geq p_n(r/s_k), \ \forall \ i \neq k. \tag{3.15}$$

Substituting from (3.13) and (3.14) into (3.15) and simplifying we have

$$c_2|r_I - as_{i,I}|^2 + c_1|r_Q - bs_{i,Q}|^2 \leq c_2|r_I - as_{k,I}|^2 + c_1|r_Q - bs_{k,Q}|^2, \forall \ i \neq k. \tag{3.16}$$

We use this by substituting $c_1 = a$ *and* $c_2 = b$, *to obtain (3.17) and* $c_1 = b$ *and* $c_2 = a$, *to obtain (3.18). For* $\hat{v}_j$, $j = 0, 1$, *choose signal* $s_i \in \mathcal{A}$ *iff*

$$b|\hat{v}_{j,I} - as_{i,I}|^2 + a|\hat{v}_{j,Q} - bs_{i,Q}|^2 \leq b|\hat{v}_{j,I} - as_{k,I}|^2 + a|\hat{v}_{j,Q} - bs_{k,Q}|^2, \quad \forall \ i \neq k \tag{3.17}$$

and for $\hat{v}_j$, $j = 2, 3$, *choose signal* s_i *iff*

$$a|\hat{v}_{j,I} - bs_{i,I}|^2 + b|\hat{v}_{j,Q} - as_{i,Q}|^2 \leq a|\hat{v}_{j,I} - bs_{k,I}|^2 + b|\hat{v}_{j,Q} - as_{k,Q}|^2, \quad \forall \ i \neq k. \tag{3.18}$$

From the above two equations it is clear that decoupling of the variables is achieved by involving the de-interleaving operation at the receiver in (3.11) and (3.12). Remember that the entire decoding operation given in this example is equivalent to using (3.6). We have given this example only to bring out the de-interleaving operation involved in the decoding of GCIODs.

Next we show that rate-1, GCIODs (and hence CIODs) exist for $N = 2, 3, 4$ only.

Theorem 3.1.2. *A rate 1, GCIOD exists iff* $N = 2, 3, 4$.

Proof. First observe from (3.1) that the GCIOD is rate 1 iff the GLCODs Θ_1, Θ_2 are rate 1. Following, Theorem 2.2.5, we have that a rate 1 non-trivial GLCOD exist iff $N = 2$. Including the trivial GLCOD for $N = 1$, we have that rate 1 GCIOD exists iff $N = 1 + 1, 1 + 2, 2 + 2$, i.e. $N = 2, 3, 4$. □

Next we construct GCIODs of rate greater than 1/2 for $N > 4$. Using the rate 3/4 design (2.10) i.e. by substituting $\Theta_1 = \Theta_2$ by the rate 3/4 GLCOD in (3.1), we have rate 3/4 CIOD for 8 transmit antennas which is given by

$$S = \begin{bmatrix} \Theta_4(x_{0I} + \mathbf{j}x_{3Q}, x_{1I} + \mathbf{j}x_{4Q}, x_{2I} + \mathbf{j}x_{5Q}) & 0 \\ 0 & \Theta_4(x_{3I} + \mathbf{j}x_{0Q}, x_{4I} + \mathbf{j}x_{1Q}, x_{5I} + \mathbf{j}x_{2Q}) \end{bmatrix}. \tag{3.19}$$

Deleting one, two and three columns from S we have rate 3/4 GCIODs for $N = 7, 6, 5$ respectively. Observe that by dropping columns of a CIOD we get GCIODs and not CIODs. But the GCIODs for $N = 5, 6, 7$ are not maximal rate designs that can be constructed from the Definition 3.1.1 using known GLCODs.

Towards constructing higher rate GCIODs for $N = 5, 6, 7$, observe that the number of indeterminates of GLCODs Θ_1, Θ_2 in Definition 3.1.1 are equal. This is necessary for full-diversity so that the in-phase or the quadrature component of each indeterminate, each seeing a different channel, together see all the channels. The construction of such GLCODs for $N_1 \neq N_2$, in general, is not immediate. One way is to set some of the indeterminates in the GLCOD with higher number of indeterminates to zero, but this results in loss of rate. We next give the construction of such GLCODs which does not result in loss of rate.

Construction 3.1.3. *Let Θ_1 be a GLCOD of size $L_1 \times N_1$, rate $r_1 = K_1/L_1$ in K_1 indeterminates $x_0, \cdots, x_{K_1-1}$ and similarly let Θ_2 be a GLCOD of size $L_2 \times N_2$, rate $r_2 = K_2/L_2$ in K_2 indeterminates $y_0, \cdots, y_{K_2-1}$. Let $K = \mathrm{lcm}(K_1, K_2)$, $n_1 = K/K_1$ and*

$n_2 = K/K_2$. *Construct*

$$\hat{\Theta}_1 = \begin{bmatrix} \Theta_1(x_0, x_1, \cdots, x_{K_1-1}) \\ \Theta_1(x_{K_1}, x_{K_1+1}, \cdots, x_{2K_1-1}) \\ \Theta_1(x_{2K_1}, x_{2K_1+1}, \cdots, x_{3K_1-1}) \\ \vdots \\ \Theta_1(x_{(n_1-1)K_1}, x_{(n_1-1)K_1+1}, \cdots, x_{n_1K_1-1}) \end{bmatrix} \quad (3.20)$$

and

$$\hat{\Theta}_2 = \begin{bmatrix} \Theta_2(y_0, y_1, \cdots, y_{K_2-1}) \\ \Theta_2(y_{K_2}, y_{K_2+1}, \cdots, y_{2K_2-1}) \\ \Theta_2(y_{2K_2}, y_{2K_2+1}, \cdots, y_{3K_2-1}) \\ \vdots \\ \Theta_2(y_{(n_2-1)K_2}, y_{(n_2-1)K_2+1}, \cdots, y_{n_2K_2-1}) \end{bmatrix}. \quad (3.21)$$

Then $\hat{\Theta}_1$ *of size* $n_1L_1 \times N_1$ *is a GLCOD in indeterminates* $x_0, x_1, \cdots, x_{K-1}$ *and* $\hat{\Theta}_2$ *of size* $n_2L_2 \times N_2$ *is a GLCOD in indeterminates* $y_0, y_1, \cdots, y_{K-1}$. *Substituting these GLCODs in (3.1) we have a GCIOD of rate*

$$\mathcal{R} = \frac{2K}{n_1L_1 + n_2L_2} = \frac{2\mathrm{lcm}(K_1, K_2)}{n_1L_1 + n_2L_2} = \frac{2\mathrm{lcm}(K_1, K_2)}{\mathrm{lcm}(K_1, K_2)(L_1/K_1 + L_2/K_2)} = H(r_1, r_2) \quad (3.22)$$

where $H(r_1, r_2)$ *is the Harmonic mean of* r_1, r_2 *with* $N = N_1 + N_2$ *and delay,* $L = n_1L_1 + n_2L_2$.

We illustrate Construction 3.1.3 by constructing a rate 6/7 GCIOD for six transmit antennas in the following example.

Example 3.1.3. *Let*

$$\Theta_1 = \begin{bmatrix} x_0 & x_1 \\ -x_1^* & x_0^* \end{bmatrix}$$

be the Alamouti code. Then $L_1 = N_1 = K_1 = 2$*. Similarly let*

$$\Theta_2 = \begin{bmatrix} x_0 & x_1 & x_2 & 0 \\ -x_1^* & x_0^* & 0 & x_2 \\ -x_2^* & 0 & x_0^* & -x_1 \\ 0 & -x_2^* & x_1^* & x_0 \end{bmatrix}$$

Then $L_2 = N_2 = 4$*,* $K_2 = 3$ *and the rate is* $3/4$*.* $K = \text{lcm}(K_1, K_2) = 6$*,* $n_1 = K/K_1 = 3$ *and* $n_2 = K/K_2 = 2$*.*

$$\hat{\Theta}_1 = \begin{bmatrix} \Theta_1(x_0, x_1) \\ \Theta_1(x_2, x_3) \\ \Theta_1(x_4, x_5) \end{bmatrix} = \begin{bmatrix} x_0 & x_1 \\ -x_1^* & x_0^* \\ x_2 & x_3 \\ -x_3^* & x_2^* \\ x_4 & x_5 \\ -x_5^* & x_4^* \end{bmatrix}, \tag{3.23}$$

Similarly

$$\hat{\Theta}_2 = \begin{bmatrix} x_0 & x_1 & x_2 & 0 \\ -x_1^* & x_0^* & 0 & x_2 \\ -x_2^* & 0 & x_0^* & -x_1 \\ 0 & -x_2^* & x_1^* & x_0 \\ x_3 & x_4 & x_5 & 0 \\ -x_4^* & x_3^* & 0 & x_5 \\ -x_5^* & 0 & x_3^* & -x_4 \\ 0 & -x_5^* & x_4^* & x_3 \end{bmatrix}. \tag{3.24}$$

The GCIOD for $N = N_1 + N_2 = 6$ is given by

$$S = \begin{bmatrix} x_{0I} + \mathbf{j}x_{6Q} & x_{1I} + \mathbf{j}x_{7Q} & 0 & 0 & 0 & 0 \\ -x_{1I} + \mathbf{j}x_{7Q} & x_{0I} - \mathbf{j}x_{6Q} & 0 & 0 & 0 & 0 \\ x_{2I} + \mathbf{j}x_{8Q} & x_{3I} + \mathbf{j}x_{9Q} & 0 & 0 & 0 & 0 \\ -x_{3I} + \mathbf{j}x_{9Q} & x_{2I} - \mathbf{j}x_{8Q} & 0 & 0 & 0 & 0 \\ x_{4I} + \mathbf{j}x_{10Q} & x_{5I} + \mathbf{j}x_{11Q} & 0 & 0 & 0 & 0 \\ -x_{5I} + \mathbf{j}x_{11Q} & x_{4I} - \mathbf{j}x_{10Q} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{6I} + \mathbf{j}x_{0Q} & x_{7I} + \mathbf{j}x_{1Q} & x_{8I} + \mathbf{j}x_{2Q} & 0 \\ 0 & 0 & -x_{7I} + \mathbf{j}x_{1Q} & x_{6I} - \mathbf{j}x_{0Q} & 0 & x_{8I} + \mathbf{j}x_{2Q} \\ 0 & 0 & -x_{8I} + \mathbf{j}x_{2Q} & 0 & x_{6I} - \mathbf{j}x_{0Q} & -x_{7I} - \mathbf{j}x_{1Q} \\ 0 & 0 & 0 & -x_{8I} + \mathbf{j}x_{2Q} & x_{7I} - \mathbf{j}x_{1Q} & x_{6I} + \mathbf{j}x_{0Q} \\ 0 & 0 & x_{9I} + \mathbf{j}x_{3Q} & x_{10I} + \mathbf{j}x_{4Q} & x_{11I} + \mathbf{j}x_{5Q} & 0 \\ 0 & 0 & -x_{10I} + \mathbf{j}x_{4Q} & x_{9I} - \mathbf{j}x_{3Q} & 0 & x_{11I} + \mathbf{j}x_{5Q} \\ 0 & 0 & -x_{11I} + \mathbf{j}x_{5Q} & 0 & x_{9I} - \mathbf{j}x_{3Q} & -x_{10I} - \mathbf{j}x_{4Q} \\ 0 & 0 & 0 & -x_{11I} + \mathbf{j}x_{5Q} & x_{10I} - \mathbf{j}x_{4Q} & x_{9I} + \mathbf{j}x_{3Q} \end{bmatrix}. \tag{3.25}$$

*The rate of the above design is $\frac{12}{14} = \frac{6}{7} = 0.8571 > 3/4$. This increased rate comes at the cost of additional delay. While the rate 3/4 CIOD for $N = 6$ has a delay of 8 symbol durations, the rate 6/7 GCIOD has a delay of 14 symbol durations. In other words, the rate $3/4$ scheme is **delay-efficient**, while the rate 6/7 scheme is **rate-efficient**[3]. Deleting one of the columns we have a rate 6/7 design for 5 transmit antennas.*

Similarly, taking Θ_1 to be the Alamouti code and Θ_2 to be the rate $7/11$ design of (2.11) in Construction 3.1.3, we have a CIOD for $N = 7$ whose rate is given by

$$\mathcal{R} = \frac{2}{11/7 + 1} = \frac{14}{18} = \frac{7}{9} = 0.777....$$

The delay for this scheme is 36 symbol durations. For, $N = 8$ the maximum rate obtained using known GLCODs is $3/4$. Significantly, **there exist CIOD and GCIOD of rate greater that 3/4 and less than 1, while no such GLCOD is known to exist**. Next, we present the construction of rate $2/3$ GCIOD for all $N > 6$ in the following

[3] *Observe that we are not in a position to comment on the optimality of both the delay and the rate.*

example.

Example 3.1.4. *For a given N, let Θ_1 be the Alamouti code. Then $L_1 = N_1 = K_1 = 2$ and $N_2 = N - 2$. Let Θ_2 be the rate 1/2 GLCOD for $N - 2$ transmit antennas (either using the construction of [13] or [19]). Then $r_2 = 1/2$. The corresponding rate of the GCIOD is given by*

$$\mathcal{R} = \frac{2}{2+1} = \frac{2}{3}.$$

In Table 3.3, we present the rate comparison between GLCODs and CIODs-both rate-efficient and delay efficient; and in Table 3.4, we present the delay comparison.

Table 3.3: Comparison of rates of known GLCODs and GCIODs for all N

Tx. Antennas	GLCODs	GCIOD (rate-efficient)	GCIOD (delay-efficient)
N=2	1	1	1
N=3,4	3/4	1	1
N=5	7/11	6/7	3/4
N=6	3/5	6/7	3/4
N=7	1/2	7/9	3/4
N=8	1/2	3/4	3/4
N=9	1/2	2/3 > 7/11	7/11
N=10	1/2	2/3	7/11
N=11,12	1/2	2/3	3/5
N>12	1/2	2/3	2/3

Table 3.4: Comparison of delays of known GLCODs and GCIODs $N \leq 8$

Tx. Antennas	GLCODs	GCIOD (rate-efficient)	GCIOD (delay-efficient)
N=2	2	2	2
N=3,4	4	4	4
N=5	11	14	8
N=6	30	14	8
N=7	8	36	8
N=8	8	8	8

Observe that both in terms of delay and rate GCIODs are superior to GLCOD.

3.2 GCIODs vs. GLCODs

In this section we make a comparative study of GCIODs and GLCODs with respect to different aspects including signal set expansion, orthogonality and peak to average power ratio (PAPR). Other aspects like Coding gain, performance comparison using simulation results and maximum mutual information are presented in subsequent sections.

As observed earlier, a STBC is obtained from the GCIOD by replacing x_i by s_i and allowing each s_i, $i = 0, 1, \cdots, K-1$, to take values from a signal set $\mathcal{A}$. For notational simplicity we will use only S for $S(x_0, \cdots, x_{K-1})$ dropping the arguments, whenever they are clear from the context.

The following list highlights and compares the salient features of GCIODs and GLCODs:

- Both GCIOD and OD are FSDD and hence STBCs from these designs are single-symbol decodable.

- GCIOD is a RFSDD and hence STBCs from GCIODs achieve full-diversity iff CPD of $\mathcal{A}$ is not equal to zero. In contrast STBCs from GLCODs achieve full-diversity for all $\mathcal{A}$.

- **Signal Set Expansion:** For STBCs from GCIODs, it is important to note that when the variables x_i, $i = 0, 1, \cdots$, take values from a complex signal set $\mathcal{A}$ the transmission matrix have entries which are coordinate interleaved versions of the variables and hence the actual signal points transmitted are not from $\mathcal{A}$ but from an *expanded version of* $\mathcal{A}$ which we denote by $\tilde{\mathcal{A}}$. Figure 3.2(a) shows $\tilde{\mathcal{A}}$ when $\mathcal{A} = \{1, -1, \mathbf{j}, -\mathbf{j}\}$ which is shown in Figure 3.2(c). Notice that $\tilde{\mathcal{A}}$ has 8 signal points whereas $\mathcal{A}$ has 4. Figure 3.2(b) shows $\tilde{\mathcal{A}}'$ where $\mathcal{A}'$ is the four point signal set obtained by rotating $\mathcal{A}$ by 13.2825 degrees counter clockwise i.e., $\mathcal{A}' = \{e^{\mathbf{j}\theta}, -e^{\mathbf{j}\theta}, \mathbf{j}e^{\mathbf{j}\theta}, -\mathbf{j}e^{\mathbf{j}\theta}\}$ where $\theta = 13.2825$ degrees as shown in Figure 3.2(d). Notice that now the expanded signal set has 16 signal points (The value $\theta = 13.2825$ has been chosen so as to maximize the parameter called Coordinate Product Distance of the signal set which is related to diversity and coding gain of the STBCs from GCIODs, discussed in detail in Section 3.3). It is easily seen that $|\mathcal{A}'| \leq |\mathcal{A}|^2$.

  Now for GLCOD, there is an expansion of signal set, but $|\mathcal{A}'| \leq 2|\mathcal{A}|$. For example consider the Alamouti scheme, for the first time interval the symbols are from the signal set $\mathcal{A}$ and for the next time interval symbols are from $\mathcal{A}^*$, the conjugate of symbols of $\mathcal{A}$. But for constellations derived from the square lattice $|\mathcal{A}'| << 2|\mathcal{A}|$

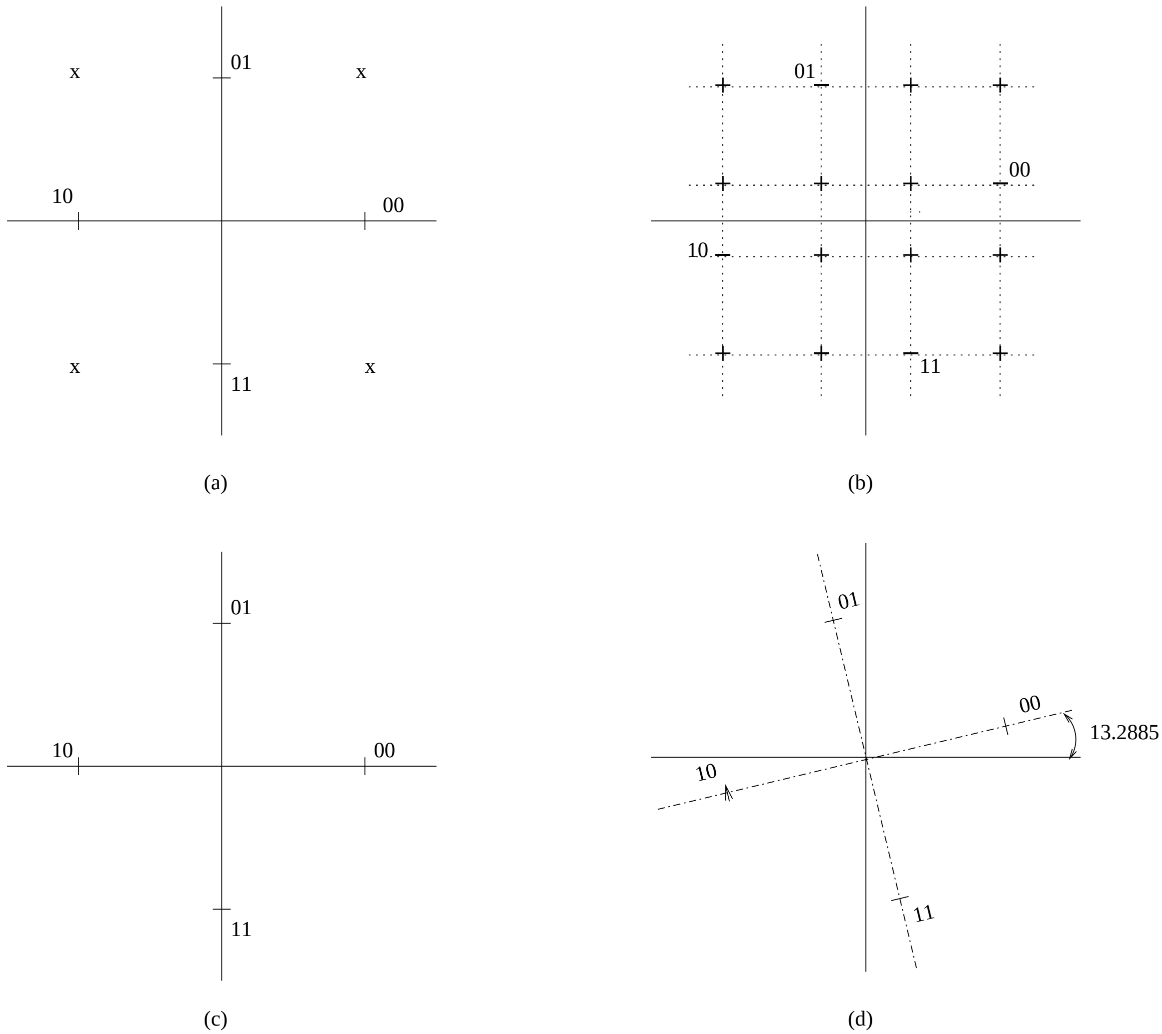


Figure 3.2: Expanded signal sets $\tilde{\mathcal{A}}$ for $\mathcal{A} = \{1, -1, \mathbf{j}, -\mathbf{j}\}$ and a rotated version of it.

and in particular for square QAM $|\mathcal{A}'| = |\mathcal{A}|$. So the transmission is from a larger signal set for GCIODs as compared to GLCODs.

- Another important aspect to notice is that for GCIODs, during the first $L/2$ time intervals $N_1 < N$ of the N antennas transmit and the remaining $N_2 = N - N_1$ antennas transmit nothing and vice versa. So, on an average half of transmit antennas are idle.

- For GCIODs, S, is not an orthonormal matrix but is an orthogonal matrix while for GCLOODs, S, is orthonormal. For example consider the GCIOD for 4 transmit antennas,

$$S^{\mathcal{H}}S = \begin{bmatrix} |\tilde{x}_0|^2+|\tilde{x}_1|^2 & 0 & 0 & 0 \\ 0 & |\tilde{x}_0|^2+|\tilde{x}_1|^2 & 0 & 0 \\ 0 & 0 & |\tilde{x}_2|^2+|\tilde{x}_3|^2 & 0 \\ 0 & 0 & 0 & |\tilde{x}_2|^2+|\tilde{x}_3|^2 \end{bmatrix}. \quad (3.26)$$

- GCIODs out perform GLCODs for $N > 2$ both in terms of rate and delay as shown in Tables 3.3, 3.4.
- Due to the fact that at least half of the entries of GCIOD are zero, the peak-to-average power ratio for any one antenna is high compared to those STBCs obtained from GLCODs. This can be taken care of by "power uniformization" techniques as discussed in [15] for GLCODs with some zero entries.

3.3 Coding Gain and Coordinate Product Distance (CPD)

In this section we derive the conditions under which the coding gain of the STBCs from GCIODs is maximized. Recollect from Chapter 2 that since GCIOD and CIOD are RFSDDs, they achieve full-diversity iff CPD of $\mathcal{A}$ is non-zero. Here, in Subsection 3.3.1 we show that the coding gain defined in (1.12) is equal to a quantity, which we call, the Generalized CPD (GCPD) which a generalization of CPD. In Subsection 3.3.2 we maximize the CPD for lattice constellations by rotating the constellation[4]. Similar results are also obtained for the GCPD for some particular cases. We then compare the coding gains of STBCs from both GCIODs and GLCODs in sub-section 3.3.3 and show that, except for $N = 2$, GCIODs have higher coding gain as compared to GLCODs for lattice constellations at the same spectral efficiency in bits/sec/Hz.

[4]The optimal rotation for 2-D QAM signal sets is derived in [62] using Number theory and Lattice theory. Our proof is simple and does not require mathematical tools from Number theory or Lattice theory.

3.3.1 Coding Gain of GCIODs

Without loss of generality, we assume that the GLCODs Θ_1, Θ_2 of Definition 3.1.1 are such that their weight matrices are unitary. Towards obtaining an expression for the coding gain of CI-STBCs, we first introduce

Definition 3.3.1 (Generalized Coordinate Product Distance (GCPD)). The $GCPD_{N_1,N_2}(u,v)$ between any two signal points $u = u_I + \mathbf{j}u_Q$ and $v = v_I + \mathbf{j}v_Q$, $u \neq v$ of the signal set $\mathcal{A}$ is defined as

$$GCPD_{N_1,N_2}(u,v) = \min\left\{|u_I - v_I|^{2N_1}|u_Q - v_Q|^{2N_2}, |u_I - v_I|^{2N_2}|u_Q - v_Q|^{2N_1}\right\} \tag{3.27}$$

and the minimum of this value among all possible pairs of distinct signal points of the signal set $\mathcal{A}$ is defined as the GCPD of the signal set and will be denoted by $GCPD_{N_1,N_2}(\mathcal{A})$ or simply by $GCPD_{N_1,N_2}$ when the signal set under consideration is clear from the context.

Remark 3.3.1. Observe that

1. When $N_1 = N_2$, the GCPD reduces to the CPD defined in Definition 2.4.1
2. $GCPD_{N_1,N_2}(u,v) = GCPD_{N_2,N_1}(u,v)$ for any two signal points u and v and hence $GCPD_{N_1,N_2}(\mathcal{A}) = GCPD_{N_2,N_1}(\mathcal{A})$.

We have,

Theorem 3.3.1. *The coding gain of a full-rank GCIOD with the variables taking values from a signal set, is equal to the $GCPD_{N_1,N_2}$ of that signal set.*

Proof. For a GCIOD in Definition 3.1.1 we have,

$$S^{\mathcal{H}}S = \begin{bmatrix} I_{N_1}(|\tilde{x}_0|^2 + \cdots + |\tilde{x}_{K/2-1}|^2) & 0 \\ 0 & I_{N_2}(|\tilde{x}_{K/2}|^2 + \cdots + |\tilde{x}_{K-1}|^2) \end{bmatrix} \tag{3.28}$$

where $\tilde{x}_i = \mathrm{Re}\{x_i\} + \mathbf{j}\,\mathrm{Im}\{x_{(i+K/2)_K}\}$ and where $(a)_K$ denotes $a \pmod K$. Consider the codeword difference matrix $B(\mathbf{S}, \mathbf{S}') = \mathbf{S} - \mathbf{S}'$ which is of full-rank for two distinct codeword matrices $\mathbf{S}, \mathbf{S}'$. We have

$$B(\mathbf{S},\mathbf{S}')^{\mathcal{H}}B(\mathbf{S},\mathbf{S}') = \begin{bmatrix} (|\tilde{x}_0 - \tilde{x}'_0|^2 + \cdots + |\tilde{x}_{K/2-1} - \tilde{x}'_{K/2-1}|^2)I_{N_1} & 0 \\ 0 & (|\tilde{x}_{K/2} - \tilde{x}'_{K/2}|^2 + \cdots + |\tilde{x}_{K-1} - \tilde{x}'_{K-1}|^2)I_{N_2} \end{bmatrix} \quad (3.29)$$

where at least one x_k differs from x'_k, $k = 0, \cdots, K-1$. Clearly, the terms $(|\tilde{x}_0 - \tilde{x}'_0|^2 + \cdots + |\tilde{x}_{K/2-1} - \tilde{x}'_{K/2-1}|^2)$ and $(|\tilde{x}_{K/2} - \tilde{x}'_{K/2}|^2 + \cdots + |\tilde{x}_{K-1} - \tilde{x}'_{K-1}|^2)$ are both minimum iff x_k differs from x'_k for only one k. Therefore assume, without loss of generality, that the codeword matrices $\mathbf{S}$ and $\mathbf{S}'$ are such that they differ by only one variable, say x_0 taking different values from the signal set $\mathcal{A}$.

Then, for this case,

$$\Lambda_1 = \det\left\{B^{\mathcal{H}}(\mathbf{S},\mathbf{S}')B(\mathbf{S},\mathbf{S}')\right\}^{1/N} = |x_{0I} - x'_{0I}|^{\frac{2N_1}{N_1+N_2}}|x_{0Q} - x'_{0Q}|^{\frac{2N_2}{N_1+N_2}}.$$

Similarly, when $\mathbf{S}$ and $\mathbf{S}'$ are such that they differ by only in $x_{K/2}$ then

$$\Lambda_2 = \det\left\{B^{\mathcal{H}}(\mathbf{S},\mathbf{S}')B(\mathbf{S},\mathbf{S}')\right\}^{1/N} = |x_{K/2I} - x'_{K/2I}|^{\frac{2N_2}{N_1+N_2}}|x_{K/2Q} - x'_{K/2Q}|^{\frac{2N_1}{N_1+N_2}}$$

and the coding gain is given by

$$\min_{x_0, x_{K/2} \in \mathcal{A}} \{\Lambda_1, \Lambda_2\} = GCPD(N_1, N_2).$$

□

An important implication of the above result is,

Corollary 3.3.2. *The coding gain of a full-rank STBC from a CIOD with the variables taking values from a signal set, is equal to the CPD of that signal set.*

Remark 3.3.2. Observe that as the CPD is independent of the constructional details of GCIOD like the parameters N_1, N_2 and is dependent only on the elements of the signal set, it becomes very amenable to maximization techniques. Therefore the coding gain of STBC from CIOD is independent of the CIOD. In contrast, for GCIOD the coding gain is a function of N_1, N_2.

The full-rank condition of RFSDD i.e. $CPD \neq 0$ can be restated for GCIOD as

Theorem 3.3.3. *The STBC from GCIOD with variables taking values from a signal set achieves full-diversity iff the* $GCPD_{N_1,N_2}$ *of that signal set is non-zero.*

It is important to note that the GCPD (N_1, N_2) is non-zero iff the CPD is non-zero and consequently, **this is not at all a restrictive condition, since given any signal set $\mathcal{A}$ one can always get the above condition satisfied by rotating it. In fact, there are infinitely many angles of rotations that will satisfy the required condition and only finitely many which will not. Moreover, appropriate rotation leads to more coding gain also.**

From the above results it follows that signal constellations with $CPD = 0$ and hence $GCPD = 0$ like regular $M - ary\ QAM$, symmetric $M - ary\ PSK$ will not achieve full-diversity. But the situation gets salvaged by simply rotating the signal set to get this condition satisfied as also indicated in [46, 47, 66]. This result is similar to the ones on co-ordinate interleaved schemes like co-ordinate interleaved trellis coded modulation [46, 47] and bit and co-ordinate interleaved coded modulation [44]-[49], [58] for single antenna transmit systems.

3.3.2 Maximizing CPD and GCPD for Lattice constellations

In this subsection we derive the optimal angle of rotation for QAM constellation so that the CPD and hence the coding gain of CIOD is maximized. We then generalize the derivation so as to present a method to maximize the $GCPD_{N_1,N_2}$.

Maximizing CPD

In the previous section we showed that the coding gain of CIOD is equal to the CPD and that constellations with non-zero CPD can be obtained by rotating the constellations with zero CPD. Here we obtain the optimal angle of rotation for lattice constellations analytically. It is noteworthy that the optimal performance of co-ordinate interleaved TCM for the 2-D QAM constellations considered [46, 47], using simulation results was observed at

32°; analytically, the optimal angle of rotation derived herein is $\theta = \tan(2)/2 = 31.7175°$ for 2-D QAM constellations. The error is probably due to the incremental angle being greater than or equal to 0.5. We first derive the result for square QAM

Theorem 3.3.4. *Consider a square QAM constellation $\mathcal{A}$, with signal points from the square lattice $(2k-1-Q)d + \mathbf{j}(2l-1-Q)d$ where $k, l \in [1, Q]$ and d is chosen so that the average energy of the QAM constellation is 1. Let θ be the angle of rotation. The CPD of $\mathcal{A}$ is maximized at $\theta = \frac{\arctan(2)}{2} = 31.7175°$ and is given by*

$$CPD_{opt} = \frac{4d^2}{\sqrt{5}}. \tag{3.30}$$

Proof. The proof is in three steps. First we derive the optimum value of θ for 4-QAM, denoted as θ_{opt}(the corresponding CPD is denoted as CPD_{opt}). Second, we show that at θ_{opt}, CPD_{opt} is in-fact the CPD for all other (square) QAM. Finally, we show that for any other value of $\theta \in [0, \pi/2]$, $CPD < CPD_{opt}$ completing the proof.

Step 1: Any point $\mathrm{P}(x, y) \in \mathbb{R}^2$ rotated by an angle $\theta \in [0, 90°]$ can be written as

$$\begin{bmatrix} x_R \\ y_R \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}}_{R} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{3.31}$$

Let $P_1(x_1, y_1), P_2(x_2, y_2)$ be two distinct points in $\mathcal{A}$ such that $\triangle x = x_1 - x_2, \triangle y = y_1 - y_2$. Observe that $\triangle x, \triangle y = 0, \pm 2d, \cdots, \pm 2(Q-1)d$. We may write $\triangle x = \pm 2md, \triangle y = \pm 2nd, m, n \in [0, Q-1]$ but both $\triangle x, \triangle y$ cannot be zero simultaneously, as P_1, P_2 are distinct points in $\mathcal{A}$. Since, rotation is a linear operation,

$$\begin{bmatrix} \triangle x_r \\ \triangle y_r \end{bmatrix} = R \begin{bmatrix} \triangle x \\ \triangle y \end{bmatrix}, \tag{3.32}$$

where $\triangle x_r = x_{1R} - x_{2R}, \triangle y_R = y_{1R} - y_{2R}$. The $CPD(P_1, P_2)$ is then given by

$$CPD(P_1, P_2) = |\triangle x_r||\triangle y_r| = \left|\triangle x \triangle y \cos(2\theta) + \frac{(\triangle x)^2 - (\triangle y)^2}{2} \sin(2\theta)\right|. \quad (3.33)$$

For 4-QAM, possible values of $CPD(P_1, P_2)$ are

$$CPD_1(P_1, P_2) = 2d^2|\sin(2\theta)| \text{ and } CPD_2(P_1, P_2) = 4d^2|\cos(2\theta)|. \quad (3.34)$$

As sine is an increasing function and cosine a decreasing function of θ in the first quadrant, equating CPD_1, CPD_2 gives the optimal angle of rotation, θ_{opt}. Let $CPD(\theta)$ be the CPD at angle θ and $CPD_{opt} = \max_\theta CPD(\theta)$. It follows that $\theta_{opt} = \frac{\arctan(\pm 2)}{2} = 31.7175°, 58.285°$ and $CPD_{opt} = 2d^2 \sin(2\theta_{opt}) = 4d^2 \cos(2\theta_{opt})$.

Step 2: Substituting the optimal values of $\sin(2\theta_{opt}), \cos(2\theta_{opt})$ in (3.33) we have

$$CPD(P_1, P_2) = \frac{4d^2}{\sqrt{5}} \left|\pm nm + n^2 - m^2\right| \text{ where } n, m \in \mathbb{Z} \quad (3.35)$$

and both n, m are not simultaneously zero and $\mathbb{Z}$ is the set of integers. It suffice to show that

$$| \pm nm + n^2 - m^2| \geq 1 \forall n, m$$

provided both n, m are not simultaneously zero, completing the proof. We consider the $\pm$ case separately. We have

$$\begin{aligned} |nm + n^2 - m^2| &= \left|\left(n + \frac{m}{2}\right)^2 - \left(1 + \frac{1}{4}\right) m^2\right| \\ &= \left|\left(n + \frac{m}{2}\{1 + \sqrt{5}\}\right)\left(n + \frac{m}{2}\{1 - \sqrt{5}\}\right)\right| \\ ||^{ly}, \quad |-nm + n^2 - m^2| &= \left|\left(n - \frac{m}{2}\{1 - \sqrt{5}\}\right)\left(n - \frac{m}{2}\{1 - \sqrt{5}\}\right)\right|. \end{aligned}$$

The quadratic equations in $n, |\pm nm + n^2 - m^2|$ has roots

$$n = \frac{m}{2}\{\pm 1 \pm \sqrt{5}\}.$$

Since $n, m \in \mathbb{Z}$, $|\pm nm + n^2 - m^2| \in \mathbb{Z}$ and is equal to zero only if $n = 0, \frac{m}{2}\{\pm 1 \pm \sqrt{5}\}$. Necessarily, $|\pm nm + n^2 - m^2| \geq 1$ for $n, m \in \mathbb{Z}$ and both n, m are not simultaneously zero. Therefore at θ_{opt} the $CPD(\theta_{opt}) = CPD_{opt}$.

Step 3: Next we prove that for all other values of $\theta \in [0, \frac{\pi}{2}]$, $CPD(\theta) < CPD_{opt}$. To this end, observe that for any value of θ other than θ_{opt} either CPD_1 or CPD_2 is less than CPD_{opt} (see the attached plot of CPD_1, CPD_2 in fig. 3.3). It follows that

$$CPD(\theta) \leq CPD_{opt}$$

with equality iff $\theta = \theta_{opt}$. □

Observe that Theorem 3.3.4 has application in all schemes where the performance depends on the CPD such as those in [53], [48], [49], [46, 47], etc. and the references therein.

Remark: The 4 QAM constellation in Fig. 3.2 is a rotated version (45°) of the QAM signal set considered in Theorem 3.3.4.

Next we generalize Theorem 3.3.4 to all lattice constellations. We first find constellations that have the same CPD as the square QAM of which it is a subset. Towards that end we define,

Definition 3.3.2 (non-reducible lattice constellation (NLC)). A non-reducible lattice constellation is a finite subset of the square lattice, $(2k - 1 - Q)d + \mathbf{j}(2l - 1 - Q)d$ where $k, l \in \mathbb{Z}$, such that there exists at least a pair of signal points $p_1 = (2k_1 - 1 - Q)d + \mathbf{j}(2l_1 - 1 - Q)d$ and $p_2 = (2k_2 - 1 - Q)d + \mathbf{j}(2l_2 - 1 - Q)d$ such that either $|k_1 - k_2|$ or $|l_1 - l_2|$ is equal to 1 and correspondingly either $|l_1 - l_2|$ or $|k_1 - k_2|$ is equal to 0.

We have,

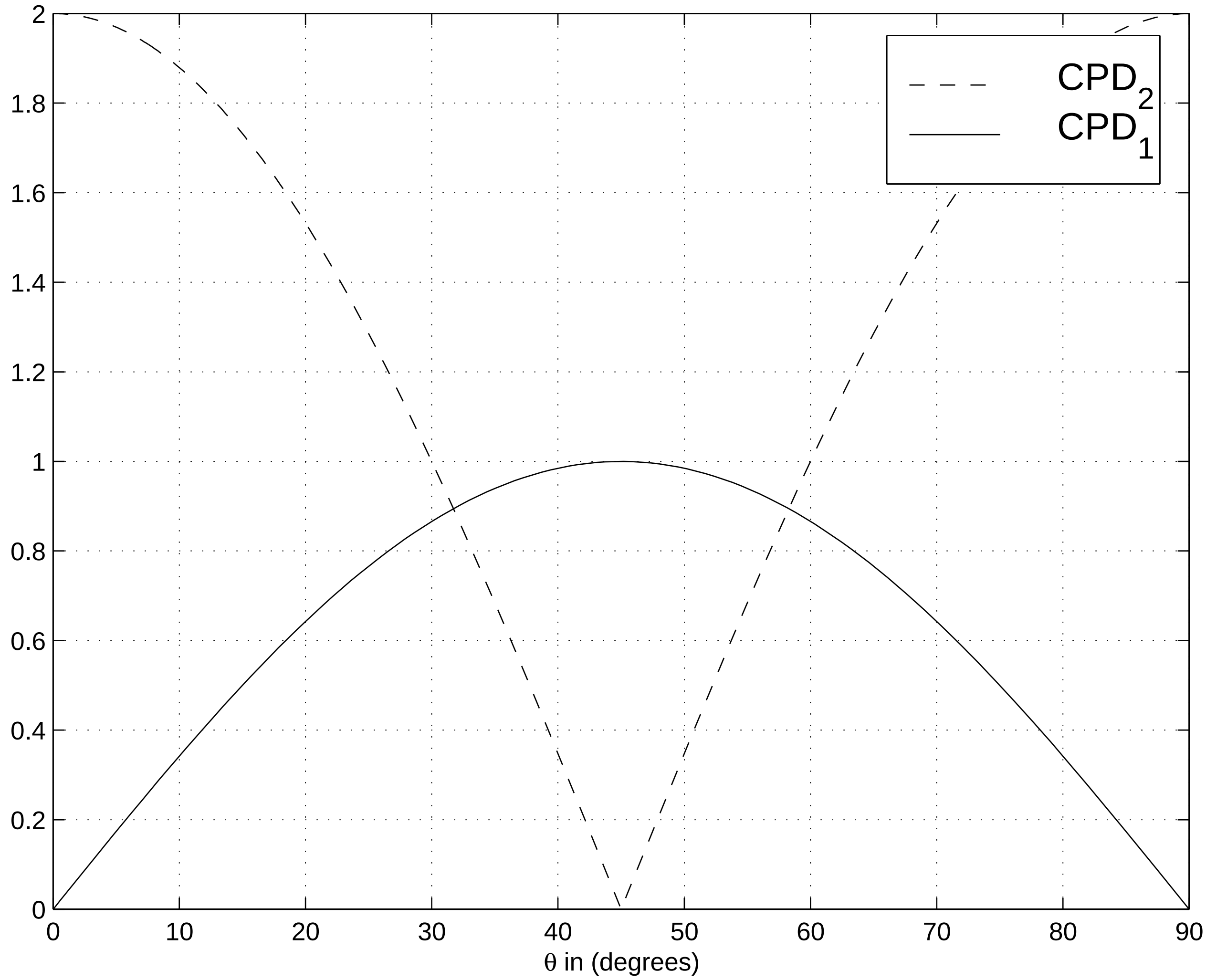


Figure 3.3: The plots of CPD_1, CPD_2 for $\theta \in [0\ 90°]$.

Corollary 3.3.5. *The CPD of a non-reducible lattice constellation, $\mathcal{A}$, rotated by an angle θ, is maximized at $\theta = \frac{\arctan(2)}{2} = 31.7175°$ and is given by*

$$CPD_{opt} = \frac{4d^2}{\sqrt{5}}. \tag{3.36}$$

Proof. Since $\mathcal{A}$ is a subset of an appropriate square QAM constellation, we immediately have from Theorem 3.3.4

$$CPD_{opt} \geq \frac{4d^2}{\sqrt{5}}. \tag{3.37}$$

We only need to prove the equality condition. The CPD between any two points in NLC at θ_{opt} is given by (3.35)

$$CPD(P_1, P_2) = \frac{4d^2}{\sqrt{5}} \left| \pm nm + n^2 - m^2 \right| \text{ where } n, m \in \mathbb{Z}. \tag{3.38}$$

Since for NLC there exists at least a pair of signal points $p_1 = (2k_1 - 1 - Q)d + \mathbf{j}(2l_1 - 1 - Q)d$ and $p_2 = (2k_2 - 1 - Q)d + \mathbf{j}(2l_2 - 1 - Q)d$ such that either $|k_1 - k_2|$ or $|l_1 - l_2|$ is equal to 1 and correspondingly either $|l_1 - l_2|$ or $|k_1 - k_2|$ is equal to 0, we have

$$CPD(p_1, p_2) = \frac{4d^2}{\sqrt{5}}. \tag{3.39}$$

□

In addition to the NLCs, the lattice constellations that are a proper subset of the scaled rectangular lattices,$(2k-1-Q)2d + \mathbf{j}(2l-1-Q)d$ and $(2k-1-Q)d + \mathbf{j}(2l-1-Q)2d$ where $k, l \in \mathbb{Z}$ have CPD equal to $\frac{4d^2}{\sqrt{5}}$. All other lattice constellations have $CPD > \frac{4d^2}{\sqrt{5}}$.

Maximizing the GCPD of the QPSK signal set

To derive the optimal angles of rotation for mazimizing the GCPD we consider only QPSK, since the optimal angle varies with the constellation, unlike CPD.

Theorem 3.3.6. *Consider a QPSK constellation $\mathcal{A}$, with signal points $(2k-3)d + \mathbf{j}(2l-3)d$ where $k, l \in [1, 2]$ and $d = 1/\sqrt{2}$, rotated by an angle θ so as to maximize the $GCPD_{N_1,N_2}$.*

The $GCPD_{N_1,N_2}(\mathcal{A})$ *is maximized at* $\theta_{opt} = \arctan(x_0)$ *where* x_0 *is the positive root of the equation*

$$\left(1 - \frac{1}{x}\right)^{2N_1} (1+x)^{2N_2} = 1 \tag{3.40}$$

where $N_1 > N_2$ *and the corresponding* $GCPD_{N_1,N_2}(\mathcal{A})$ *is* $4d^2\left(\frac{x_0^{\frac{2N_1}{N_1+N_2}}}{1+x_0^2}\right)$.

Proof. Any point $\mathrm{P}(x, y) \in \mathbb{R}^2$ rotated by an angle $\theta \in [0, 90^\circ]$ can be written as

$$\begin{bmatrix} x_R \\ y_R \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}}_{R} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{3.41}$$

Let $P_1(x_1, y_1), P_2(x_2, y_2)$ be two distinct points in $\mathcal{A}$ such that $\triangle x = x_1 - x_2$ and $\triangle y = y_1 - y_2$. Observe that $\triangle x, \triangle y \in \{0, \pm 1d\}$. We may write $\triangle x = \pm 2md, \triangle y = \pm 2nd, m, n \in [-1, 0, 1]$ but both $\triangle x, \triangle y$ cannot be zero simultaneously, as P_1, P_2 are distinct points in $\mathcal{A}$. Since, rotation is a linear operation,

$$\begin{bmatrix} \triangle x_r \\ \triangle y_r \end{bmatrix} = R \begin{bmatrix} \triangle x \\ \triangle y \end{bmatrix}, \tag{3.42}$$

where $\triangle x_r = x_{1R} - x_{2R}, \triangle y_r = y_{1R} - y_{2R}$. Then, we have

$$\begin{aligned} |\triangle x_r|^{\frac{2N_1}{N_1+N_2}} |\triangle y_r|^{\frac{2N_2}{N_1+N_2}} &= |2dm\cos(\theta) + 2dn\sin(\theta)|^{\frac{2N_1}{N_1+N_2}} \\ &\quad |-2dm\sin(\theta) + 2dn\cos(\theta)|^{\frac{2N_2}{N_1+N_2}}. \end{aligned} \tag{3.43}$$

The possible values of $GCPD_{(N_1,N_2)}(P_1, P_2)$ are

$$GCPD_1 = 4d^2 \left|\sin(\theta) - \cos(\theta)\right|^{\frac{2N_1}{N_1+N_2}} \left|\sin(\theta) + \cos(\theta)\right|^{\frac{2N_2}{N_1+N_2}} \tag{3.44}$$

$$GCPD_2 = 4d^2 \left|\sin(\theta) + \cos(\theta)\right|^{\frac{2N_1}{N_1+N_2}} \left|\sin(\theta) - \cos(\theta)\right|^{\frac{2N_2}{N_1+N_2}} \tag{3.45}$$

$$GCPD_3 = 4d^2 \left|\sin(\theta)\right|^{\frac{2N_1}{N_1+N_2}} \left|\cos(\theta)\right|^{\frac{2N_2}{N_1+N_2}} \tag{3.46}$$

$$GCPD_4 = 4d^2 \left|\cos(\theta)\right|^{\frac{2N_1}{N_1+N_2}} \left|\sin(\theta)\right|^{\frac{2N_2}{N_1+N_2}}. \tag{3.47}$$

Now by symmetry it is sufficient to consider $\theta \in [0, \pi/4)$. In this range $\sin(\theta) < \cos(\theta) \leq 1$ and accordingly, if $N_1 > N_2$ then $GCPD_3 < GCPD_4$ and similarly $GCPD_1 < GCPD_2$. Equating $GCPD_1, GCPD_3$ gives the optimal angle of rotation, θ_{opt}. We have

$$\begin{aligned}
GCPD_1 &= GCPD_3 \\
(\sin(\theta_{opt}) - \cos(\theta_{opt}))^{\frac{2N_1}{N_1+N_2}} (\sin(\theta_{opt}) + \cos(\theta_{opt}))^{\frac{2N_2}{N_1+N_2}} &= (\sin(\theta_{opt}))^{\frac{2N_1}{N_1+N_2}} (\cos(\theta_{opt}))^{\frac{2N_2}{N_1+N_2}} \\
(1 - \cot(\theta_{opt}))^{\frac{2N_1}{N_1+N_2}} (1 + \tan(\theta_{opt}))^{\frac{2N_2}{N_1+N_2}} &= 1.
\end{aligned}$$

Substituting $\tan(\theta_{opt}) = x$ we have that x is the root of the equation (3.40). The $GCPD_1$ and hence the GCPD at this value is

$$\begin{aligned}
GCPD_1 &= 4d^2 \left|\sin(\theta_{opt}) - \cos(\theta_{opt})\right|^{\frac{2N_1}{N_1+N_2}} \left|\sin(\theta_{opt}) + \cos(\theta_{opt})\right|^{\frac{2N_2}{N_1+N_2}} \\
&= 4d^2 \frac{(x_0 - 1)^{\frac{2N_1}{N_1+N_2}} (x_0 + 1)^{\frac{2N_2}{N_1+N_2}}}{1 + x_0^2} \\
&= 4d^2 \left(\frac{x_0^{\frac{2N_1}{N_1+N_2}}}{1 + x_0^2} \right).
\end{aligned} \tag{3.48}$$

□

Table 3.5 gives the optimal angle of rotation for various values of $N = N_1 + N_2$ along with the normalized $GCPD_{N_1,N_2}$ $(GCPD_{N_1,N_2})/4d^2)$. Observe that for any given N the coding gain is large if N_1, N_2 are of the same size i.e nearly equal. Also observe that the optimal angle of rotation lies in the range (26.656, 31.7175] and the corresponding normalized coding gain varies from (0.2,0.4472]. Note that the infimum corresponds

Table 3.5: The optimal angle of rotation for QPSK and normalized $GCPD_{N_1,N_2}$ for various values of $N = N_1 + N_2$.

N	N_1	N_2	x_0	θ_{opt}	$GCPD_{N_1,N_2}/4d^2$
3	2	1	0.555	29°	0.3487
5	4	1	0.5246	27.76°	0.28
	3	2	0.5751	29.9°	0.3869
6	4	2	0.555	29.9°	0.3487
	3	3	0.61	31.7175°	0.4472
7	5	2	0.543	28.51°	0.3229
	4	3	0.5856	30.35°	0.40
9	7	2	0.53	27.94°	0.29
	5	4	0.591	30.622°	0.4135
10	8	2	0.526	27.76°	0.3487
	5	5	0.61	31.7175°	0.4472
12	10	2	0.52	27.5°	0.265
	6	6	0.61	31.7175°	0.4472
N	$N-2$	2	> 0.5	$> 26.5656°$	> 0.2

to the limit where $N_1 = N$, $N_2 = 0$ and the maximum corresponds to $N_1 = N_2 = N/2$. unfortunately, the optimal angle varies with the constellation size, unlike CPD. In the next proposition we find upper and lower bounds on $GCPD_{N_1,N_2}$ for rotated lattice constellations.

Proposition 3.3.7. *The $GCPD_{N_1,N_2}$ for rotated NLC is bounded as*

$$CPD^{\frac{2N_2}{N_1+N_2}} \leq GCPD_{N_1,N_2} \leq CPD, N_2 > N_1$$

with equality iff $N_1 = N_2$.

Proof. From Definition 3.3.1 we have for a given signal set $\mathcal{A}$

$$GCPD_{N_1,N_2} = \min_{u \neq v \in \mathcal{A}} \left\{ |u_I - v_I|^{\frac{2N_1}{N_1+N_2}} |u_Q - v_Q|^{\frac{2N_2}{N_1+N_2}}, |u_I - v_I|^{\frac{2N_2}{N_1+N_2}} |u_Q - v_Q|^{\frac{2N_1}{N_1+N_2}} \right\}. \tag{3.49}$$

Let p, q be two signal points such that

$$GCPD_{N_1,N_2} = GCPD_{N_1,N_2}(p,q). \tag{3.50}$$

When $N_1 = N_2$ or $\triangle x = \triangle y$ there is nothing to prove as the inequality is satisfied.

Therefore let $N_1 \neq N_2$ and $\triangle x \neq \triangle y$. When the signal points from a the square lattice $(2k-1-Q)d + \mathbf{j}(2l-1-Q)d$ where $k, l \in [1Q]$ and d is chosen so that the average energy of the QAM constellation is 1, rotated by an angle θ then

$$\begin{aligned} GCPD_{(N_1,N_2)}(p,q) &= \left\{ |\triangle x_r|^{\frac{2N_1}{N_1+N_2}} |\triangle y_r|^{\frac{2N_2}{N_1+N_2}}, |\triangle x_r|^{\frac{2N_2}{N_1+N_2}} |\triangle y_r|^{\frac{2N_1}{N_1+N_2}} \right\} \\ &= 4d^2 \left| m\cos(\theta) + n\sin(\theta) \right|^{\frac{2N_1}{N_1+N_2}} \\ &\quad \left| -m\sin(\theta) + n\cos(\theta) \right|^{\frac{2N_2}{N_1+N_2}}, m, n \in \mathbb{Z}. \end{aligned} \tag{3.51}$$

For a NLC the $GCPD_{N_1,N_2}$ is bounded by the $GCPD_{N_1,N_2}$ for QPSK and is given by (3.48). Now the root of (3.40), x_0, is such that $x_0 \in (0.5, 1)$ and $N_2 > N/2$ and we immediately have

$$4d^2 \frac{x_0^{\frac{2N_2}{N_1+N_2}}}{(1+x_0^2)^2} < 4d^2 \frac{x_0}{(1+x_0^2)^2} \tag{3.52}$$

completing $GCPD_{N_1,N_2} \leq CPD$. For the second part observe that, for $N_2 > N_1$, $|m\cos(\theta) + n\sin(\theta)|^{N_2} < |m\cos(\theta) + n\sin(\theta)|^{N_1}$ as $|m\cos(\theta) + n\sin(\theta)| < 1$. Substituting this in (3.51) we have the lower bound. □

In Proposition 3.3.7, if we use $\theta = \arctan(2)$ for rotating the NLC then the GCPD is bounded as

$$CPD_{opt}^{\frac{2N_2}{N_1+N_2}} \leq GCPD_{N_1,N_2} \leq CPD_{opt}, N_2 > N_1, \tag{3.53}$$

$$\Rightarrow \left(\frac{4d^2}{\sqrt{5}}\right)^{\frac{2N_2}{N_1+N_2}} \leq GCPD_{N_1,N_2} \leq \left(\frac{4d^2}{\sqrt{5}}\right), N_2 > N_1. \tag{3.54}$$

Remark 3.3.3. It is clear from Table 3.5 and the above inequalities on GCPD that the value of GCPD decreases as the QAM constellation size increases and also as the difference between N_1, N_2 increases. Therefore, while Construction 3.1.3 gives high-rate designs, the coding gain decreases for QAM constellations. In Appendix A, we present another class of non-square RFSDDs whose coding gain is greater than CPD and a construction derived from Construction 3.1.3 where the coding gain is still given by GCPD but the difference between N_1, N_2 is smaller. These codes do not belong to the class of GCIODs considered in this chapter.

3.3.3 Coding gain of GCIOD vs that of GLCOD

In this subsection we compare the coding gains of GCIOD and GLCOD for the same number of transmit antennas and the same spectral efficiency in bits/sec/Hz-for same total transmit power. For sake of simplicity we assume that both GCIOD and GLCOD use square QAM constellations.

The number of transmit antennas N=2

The total transmit power constraint is given by $\mathrm{tr}\left(S^{\mathcal{H}}S\right) = L = 2$. If the signal set has unit average energy then the Alamouti code transmitted is

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} x_0 & x_1 \\ -x_1^* & x_0^* \end{bmatrix}$$

where the multiplication factor is for power normalization. For the same transmit power the rate 1 CIOD is

$$S = \begin{bmatrix} x_{0I} + \mathbf{j}x_{1Q} & 0 \\ 0 & x_{1I} + \mathbf{j}x_{0Q} \end{bmatrix}.$$

Therefore the coding gain of the Alamouti code for NLC is given by $\frac{4d^2}{2}$ and that of CIOD is given by Theorem 3.3.4 as $\frac{4d^2}{\sqrt{5}}$. Therefore the coding gain of the CIOD for N=2 is inferior to the Alamouti code by a factor of $\frac{2}{\sqrt{5}} = \frac{2}{2.23} = 0.894$, which corresponds to a coding gain of 0.4 dB for the Alamouti code[5].

[5]In Chapter 6, we revisit these codes for their use in fast-fading channels where we show that this loss of coding gain vanishes and the CIOD for $N = 2$ is single-symbol decodable while the Alamouti code is not.

The number of transmit antennas N=4

The total transmit power constraint is given by $\text{tr}\left(S^{\mathcal{H}}S\right) = L = 4$. If the signal set has unit average energy then the rate 3/4 COD code transmitted is

$$S = \frac{1}{\sqrt{3}} \begin{bmatrix} x_0 & x_1 & x_2 & 0 \\ -x_1^* & x_0^* & 0 & x_2 \\ -x_2^* & 0 & x_0^* & -x_1 \\ 0 & -x_2^* & x_1^* & x_0 \end{bmatrix}$$

where the multiplication factor is for power normalization. For the same transmit power, the rate 1 CIOD is

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} x_{0I} + \mathbf{j}x_{2Q} & x_{1I} + \mathbf{j}x_{3Q} & 0 & 0 \\ -x_{1I} + \mathbf{j}x_{3Q} & x_{0I} - \mathbf{j}x_{2Q} & 0 & 0 \\ 0 & 0 & x_{2I} + \mathbf{j}x_{0Q} & x_{3I} + \mathbf{j}x_{1Q} \\ 0 & 0 & -x_{3I} + \mathbf{j}x_{1Q} & x_{2I} - \mathbf{j}x_{0Q} \end{bmatrix}.$$

If the rate 3/4 code uses a 2^n square QAM and the rate 1 CIOD uses a $2^{\frac{3n}{4}}$ square QAM, then they have same spectral efficiency in bits/sec/Hz, and the possible values of n for realizable square constellations is $n = 8i, i \in \mathbb{Z}^+$. Let d_1, d_2 be the values of d so that the average energy of 2^n square QAM and $2^{\frac{3n}{4}}$ square QAM is 1. Therefore the coding gain of rate 3/4 COD for NLC is given by $\Lambda_{COD} = \frac{4d_1^2}{3}$ and that of CIOD is given by Theorem 3.3.4 as $\Lambda_{CIOD} = \frac{4d_2^2}{2\sqrt{5}}$. Using the fact that for unit energy M-QAM square constellations $d = \sqrt{\frac{6}{M-1}}$, we have

$$\Lambda_{COD} = \frac{8}{(2^{8i} - 1)} \text{ and } \Lambda_{CIOD} = \frac{12}{\sqrt{5}(2^{6i} - 1)} \text{ where } i \in \mathbb{Z}^+$$

for a spectral efficiency of $6i$ bits/sec/Hz. For $i = 1, 2, 3$ we have $\Lambda_{COD} = 0.0314$, 1.2207e-004, 4.7684e-007 and $\Lambda_{CIOD} = 0.0422$, 6.5517e-004, 1.0236e-005 respectively, corresponding to a coding gain of $1.29, 7.29, 13.318$ dB for the CIOD code. Observe that in contrast to the coding gain for $N = 2$ which is independent of the spectral efficiency, the coding gain for $N = 4$ appreciates with spectral efficiency.

The number of transmit antennas N=8

The total transmit power constraint is given by $\mathrm{tr}\left(S^{\mathcal{H}}S\right) = L = 8$. If the signal set has unit average energy then the rate 1/2 COD code has a multiplication factor of 1/2 and for the same transmit power, the rate 3/4 CIOD has a multiplication factor of $1/\sqrt{3}$. The rate 1/2 COD code uses a 2^n square QAM and the rate 3/4 CIOD uses a $2^{\frac{3n}{2}}$ square QAM, then they have same spectral efficiency in bits/sec/Hz, and the possible values of n for realizable square constellations is $n = 4i, i \in \mathbb{Z}^+$. Let d_1, d_2 be the values of d so that the average energy of 2^n square QAM and $2^{\frac{3n}{2}}$ square QAM is 1. Therefore the coding gain of rate 1/2 COD for NLC is given by $\Lambda_{COD} = \frac{4d_1^2}{4}$ and that of CIOD is given by Theorem 3.3.4 as $\Lambda_{CIOD} = \frac{4d_2^2}{3\sqrt{5}}$. Using the fact that for unit energy M-QAM square constellations $d = \sqrt{\frac{6}{M-1}}$, we have

$$\Lambda_{COD} = \frac{6}{(2^{4i}-1)} \text{ and } \Lambda_{CIOD} = \frac{8}{\sqrt{5}(2^{3i}-1)} \text{ where } i \in \mathbb{Z}^+$$

for a spectral efficiency of $3i$ bits/sec/Hz. For $i = 1, 2, 3$ we have Λ_{COD}= 0.4, 0.0235, 0.0015 and Λ_{CIOD} = 0.4737, 0.0563, 0.007 respectively, corresponding to a coding gain of $0.734, 3.789, 6.788$ dB for the CIOD code. Observe that as in the case of $N = 4$ the coding gain appreciates with spectral efficiency.

Next we compare the coding gains of some GCIODs.

The number of transmit antennas N=3

Both the GCIOD and GCOD for $N = 3$ is obtained from the $N = 4$ codes by dropping one of the columns, consequently the rates and the total transmit power constraint are same as for $N = 4$. Accordingly, the rate 3/4 GCOD code uses a 2^n square QAM and the rate 1 GCIOD uses a $2^{\frac{3n}{4}}$ square QAM where $n = 8i, i \in \mathbb{Z}^+$. The coding gain for the rate 3/4 GCOD for NLC is given by $\Lambda_{GCOD} = \frac{4d_1^2}{3}$ and that of GCIOD is lower bounded by Proposition 3.3.7 as $\Lambda_{GCIOD} > \left(\frac{4d_2^2}{2\sqrt{5}}\right)^{\frac{4}{3}}$. Using the fact that for unit energy M-QAM square constellations $d = \sqrt{\frac{6}{M-1}}$, we have

$$\Lambda_{GCOD} = \frac{8}{(2^{8i}-1)} \text{ and } \Lambda_{GCIOD} > \left(\frac{12}{\sqrt{5}(2^{6i}-1)}\right)^{\frac{4}{3}} \text{ where } i \in \mathbb{Z}^+$$

for a spectral efficiency of $6i$ bits/sec/Hz. For $i = 1,2,3$ we have $\Lambda_{GCOD} = 0.0314$, $1.2207e^{-4}$, $4.7684e^{-7}$ and $\Lambda_{GCIOD} > 0.0147$, $5.69e^{-5}$, $2.22e^{-7}$ respectively.

Observe that at high spectral rates, even the lower bound is larger than the coding gain of GCOD. In practice, however, the GCIOD performs better than GCOD at all spectral rates.

3.4 Simulation Results

In this section we present simulation results for 4-QAM and 16-QAM modulation over a quasi-static fading channel. The fading is assumed to be constant over a fade length of 120 symbol durations.

First, we compare the CIOD for $N = 4$, with (i) the STBC (denoted by STBC-CR in Fig.3.4 and 3.5) of [62], (ii) rate 1/2, COD and (iii) rate 3/4 COD for four transmit antennas for the **identical throughput** of 2 bits/sec/Hz. *The comparison with quasi-orthogonal design for $N = 4$ is given in Chapter 5.* For CIOD the transmitter chooses symbols from a QPSK signal set rotated by an angle of 13.2825° so as to maximize the CPD. For STBC-CR the symbols are from a QPSK signal set and rate 1/2 COD from 16-QAM signal set. For rate 3/4 COD, the symbols are chosen from 6-PSK for a throughput of 1.94 bits/sec/Hz which is close to 2 bits/sec/Hz. The average transmitted power is equal in all the cases i.e. $\mathrm{E}\{tr(S^{\mathcal{H}}S)\} = 16$, so that average energy per bit using the channel model of (1.7) is equal. Fig. 3.4. shows the BER performance for these schemes. Observe that the scheme of this paper outperforms rate 1/2 COD by 3.0 dB, rate 3/4 COD by 1.3 dB and STBC-CR by 1.2 dB at $P_b = 10^{-5}$. A comparison of the coding gain, Λ, of these schemes is given in tabular form in Table 3.6. For CIOD, $\Lambda_{CIOD} = 1.788$

Table 3.6: The coding gains of CIOD, STBC-CR, rate 3/4 COD and rate 1/2 COD for 4 tx. antennas and QAM constellations

R (bits/sec/Hz)	Λ_{CIOD}	$\Lambda_{STBC-CR}$	$\Lambda_{rate\ 3/4\ COD}$	$\Lambda_{rate\ 1/2\ COD}$
2	1.788	2	1.333	0.8
3	0.5963	0.66	0.5333	0.1905
4	0.3587	0.4	-	0.0471

while for STBC-CR $\Lambda_{STBC-CR} = 2$ at $R = 2$ bits/sec/Hz, but still CIOD out-performs STBC-CR because the coding gain is derived on the basis of an upper bound. If we take

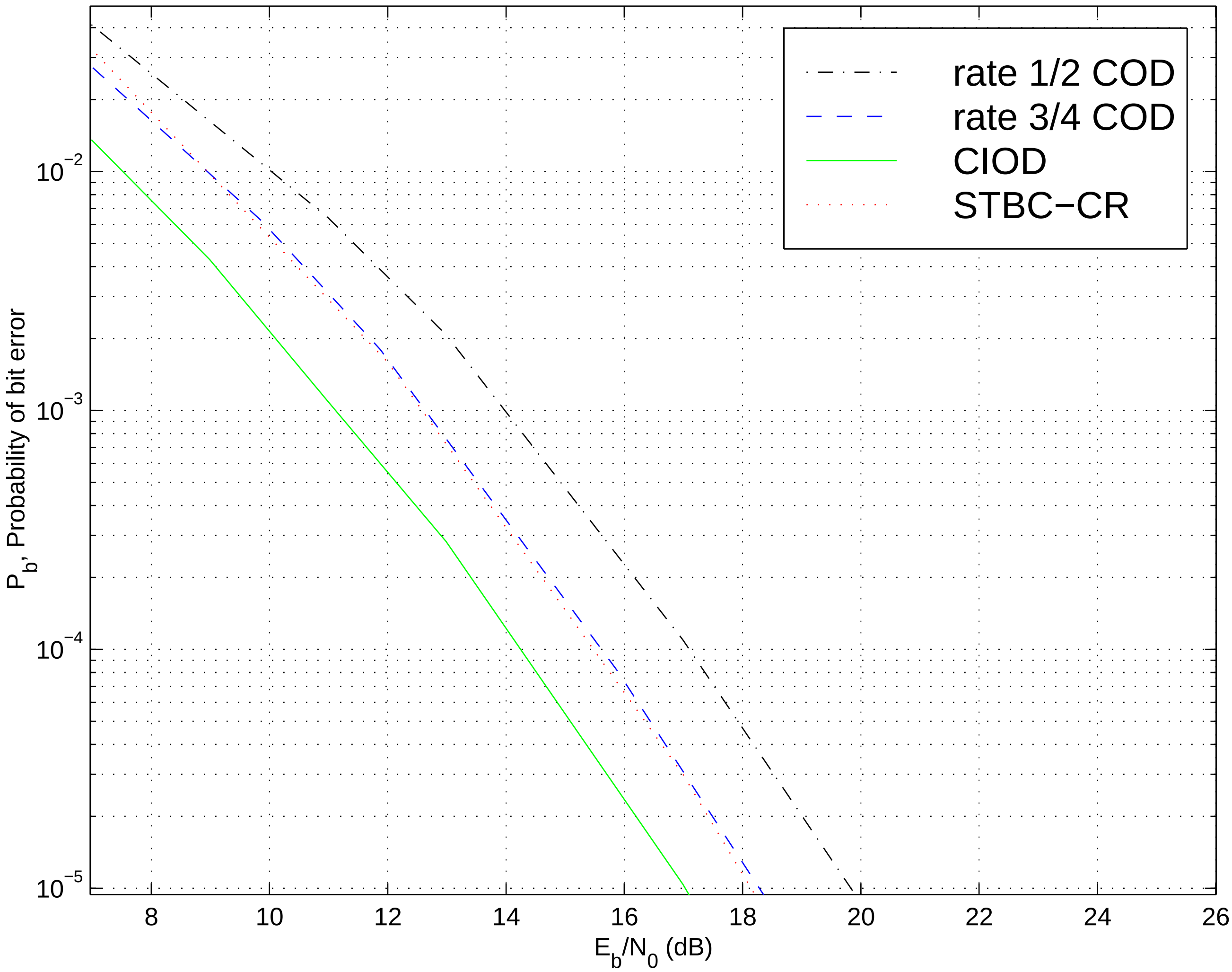


Figure 3.4: The BER performance of coherent QPSK rotated by an angle of 13.2825° (Fig.3.2) used by the CIOD scheme for 4 transmit and 1 receive antenna compared with STBC-CR, rate 1/2 COD and rate 3/4 COD at a throughout of 2 bits/sec/Hz in Rayleigh fading for the same number of transmit and receive antennas.

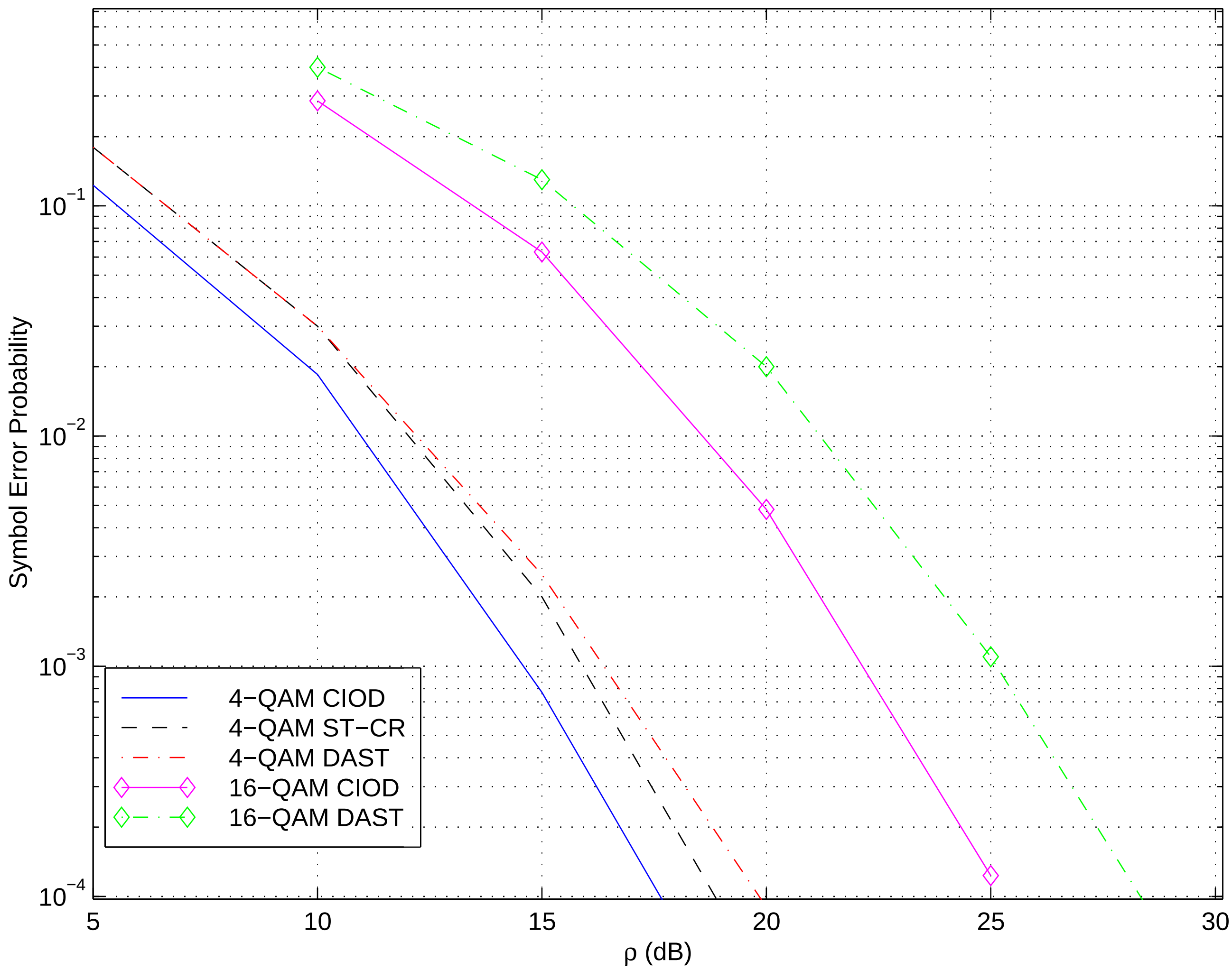


Figure 3.5: The BER performance of the CI-STBC with 4- and 16-QAM modulations and comparison with ST-CR and DAST schemes

into consideration the *kissing number* i.e. the number of codewords at the given minimum coding gain, then we clearly see that though both STBC-CR and STBC-QOD have higher coding gains, both have more than double the Kissing number of CIOD. The results for rest of the schemes are in accordance with their coding gains;

$$10\log_{10}\left(\frac{\Lambda_{CIOD}}{\Lambda_{rate\ 1/2\ COD}}\right) = 3.5 \text{ and } 10\log_{10}\left(\frac{\Lambda_{CIOD}}{\Lambda_{rate\ 3/4\ COD}}\right) = 1.3.$$

Observe that rate 3/4 COD and STBC-CR have almost similar performance at 2 bits/sec/Hz, and around 1.6dB coding gain over rate 1/2 COD. A possible apparent inconsistency of these with the results in [35, 36], which report coding gain of over 2 dB, is due to the fact that symbol error rate (SER) vs. ρ is plotted in [35, 36]. As rate 1/2 COD chooses symbols from 16 QAM and STBC-CR from 4 QAM, SER vs. ρ plot gives an overestimate of the errors for STBC-OD as compared to STBC-CR and therefore bit error rate (BER) vs. E_b/N_0 is a more appropriate plot for comparison at the same through put (2 bits/sec/Hz).

From the Table 3.6, which gives the coding gains of various schemes at spectral efficiencies of 2,3,4 bits/sec/Hz, we see that the coding gain of STBC-CR and CIOD are nearly equal (differ by a factor of 1.11) and significantly greater than other schemes. But, the main factor in favor of CIOD as compared to STBC-CR (as also any STBC other than STBC-OD) is that CIOD allows linear complexity ML decoding while STBC-CR has exponential ML decoding complexity. At a modest rate of 4 bits/sec/Hz, CIOD requires 64 metric computations while STBC-CR requires $16^4 = 65,536$ metric computations. Even the sphere-decoding algorithm is quite complex requiring exponential complexity when $M < N$ and polynomial otherwise [42].

For 4-QAM and 16-QAM constellations, Fig. 3.5 shows the performance for CIOD, STBC-CR and Diagonal Algebraic Space Time (DAST) codes of [37]. As expected CIOD shows better performance.

3.5 Maximum Mutual Information (MMI) of CIODs

In this section we analyze the maximum mutual information (MMI) that can be attained by GCIOD schemes presented in this chapter. We show that except for the Alamouti scheme all other GLCOD have lower MMI than the corresponding GCIOD. We also compare the MMI of rate 1 STBC-CR with that of GCIOD to show that GCIOD have higher MMI. Comparison with QOD is presented in Chapter 5.

It is very clear from the number of zeros in the transmission matrices of GCIODs, presented in the previous sections, that these schemes do not achieve capacity. This is because the emphasis is on low decoding complexity rather than attaining capacity. Nevertheless we intend to quantify the loss in capacity due to the presense of zeros in GCIODs.

We first consider the $N = 2, M = 1$ CIOD. Equation (1.6), for the CIOD code given in (2.53) with power normalization, can be written as

$$V = \sqrt{\rho}\mathcal{H}s + N \tag{3.55}$$

where

$$\mathcal{H} = \begin{bmatrix} h_0 & 0 \\ 0 & h_1 \end{bmatrix}$$

and $s = [\tilde{s}_0 \quad \tilde{s}_1]^T$, and where $\tilde{s}_0 = s_{0I} + \mathbf{j}s_{1Q}, \tilde{s}_1 = s_{1I} + \mathbf{j}s_{0Q}, s_0, s_1 \in \mathcal{A}$. If we define $C_D(N, M, \rho)$ as the maximum mutual information of the GCIOD for N transmit and M receive antennas at SNR ρ then

$$\begin{aligned} C_D(2,1,\rho) &= \frac{1}{2}E(\log\det(I_2 + \rho\mathcal{H}^*\mathcal{H})) = \frac{1}{2}E\log\{(1+\rho|h_{00}|^2)(1+\rho|h_{10}|^2)\} \\ &= \frac{1}{2}E\log\{1+\rho|h_{00}|^2\} + \frac{1}{2}E\log\{1+\rho|h_{10}|^2\} \\ &= C(1,1,\rho) < C(2,1,\rho). \end{aligned} \tag{3.56}$$

It is similarly seen for CIOD code for $N = 4$ given in (2.54) that

$$\begin{aligned} C_D(4,1,\rho) &= \frac{1}{4}E(\log\det(I_4 + \frac{\rho}{2}\mathcal{H}^\dagger\mathcal{H})) \\ &= \frac{1}{2}E\log\{(1+\frac{\rho}{2}(|h_{00}|^2+|h_{10}|^2))(1+\frac{\rho}{2}(|h_{20}|^2+|h_{30}|^2))\} \\ &= \frac{1}{2}E\log\{1+\frac{\rho}{2}(|h_{00}|^2+|h_{10}|^2)\} + \frac{1}{2}E\log\{1+\frac{\rho}{2}(|h_{20}|^2+|h_{30}|^2)\} \\ &= C(2,1,\rho) < C(4,1,\rho) \end{aligned} \tag{3.57}$$

and

$$C_D(3,1,\rho) = \frac{1}{2}\{C(2,1,\rho) + C(1,1,\rho)\} < C(3,1,\rho). \tag{3.58}$$

Therefore CIODs do not achieve full channel capacity even for one receive antenna. The capacity loss is negligible for one receiver as is seen from Fig. 3.6 and Fig. 3.7; this is because the increase in capacity is small from two to four transmitters in this case. The capacity loss is substantial when the number of receivers is more than one, as these schemes achieve capacity that could be attained with half the number of transmit antennas. This is because half of the antennas are not used during any given frame length. The capacity loss when $M > 2$ is even more.

Another important aspect is the comparison of MMI of CODs for three and four transmit antennas with the capacity of CIOD and GCIOD for similar antenna configuration-we already know that for two transmit antennas complex orthogonal designs have higher capacity. It is shown in [39] that

$$C_O(3, M, \rho) = \frac{3}{4}C(3M, 1, M\rho) \tag{3.59}$$

where $C_O(N, M, \rho)$ is the MMI of GLCOD for N transmit and M receive antennas at a SNR of ρ. Using similar procedure as given in [39] we found that

$$C_O(4, M, \rho) = \frac{3}{4}C(4M, 1, M\rho). \tag{3.60}$$

Equation (3.60) is plotted for , $M = 1, 2$, in Fig. 3.6 and (3.58) is plotted in Fig. 3.7

along with the corresponding plots for CIOD derived from (3.57) and (3.58). We see from these plots that the capacity of CIOD is just less than the actual capacity when there is only one receiver and is considerably greater than the capacity of code rate 3/4 complex orthogonal designs for four transmitters. When there are two receivers the capacity of CIOD is less than the actual capacity but is considerably greater than the capacity of code rate 3/4 complex orthogonal designs four transmitters.

Next we present the comparison of GCOD and GCIOD for $N > 4$. Consider the MMI of GLCOD of rate K/L. The effective channel induced by the GLCOD is given by [40]

$$\mathbf{y} = \frac{L\rho}{KN}\|\mathbf{H}\|^2\mathbf{x} + \mathbf{n} \tag{3.61}$$

where $\mathbf{y}$ is a $2K \times 1$ vector after linear processing of the received matrix $\mathbf{Y}$, $\mathbf{x}$ is a $2K \times 1$ vector consisting of the in-phase and quadrature components of the K indeterminates $x_0, \cdots, x_{K-1}$ and $\mathbf{n}$ is the noise vector with Gaussian iid entries with zero mean and

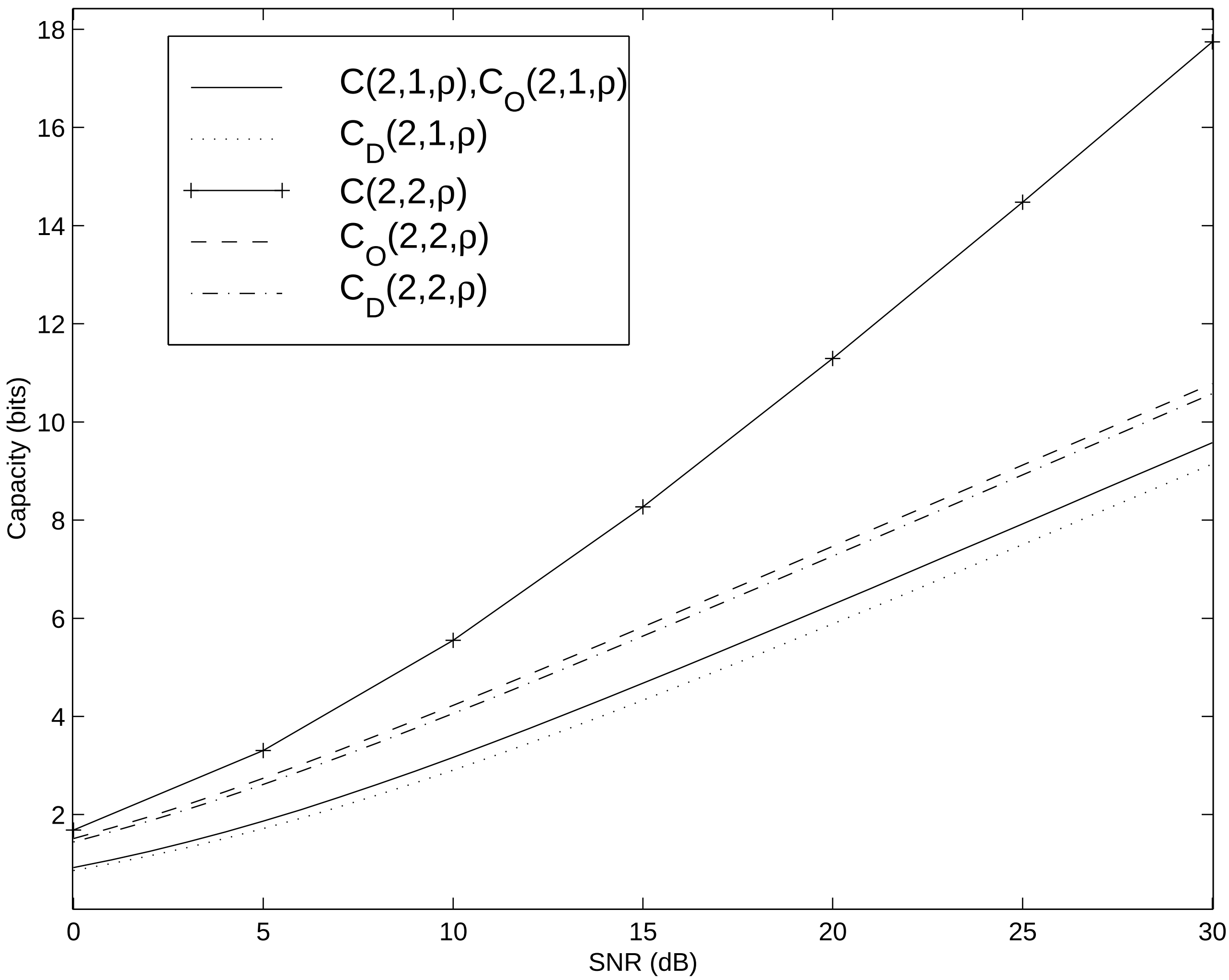


Figure 3.6: The maximum mutual information of CIOD code for two transmitters and one, two receiver compared with that of complex orthogonal design (Alamouti scheme) and the actual channel capacity.

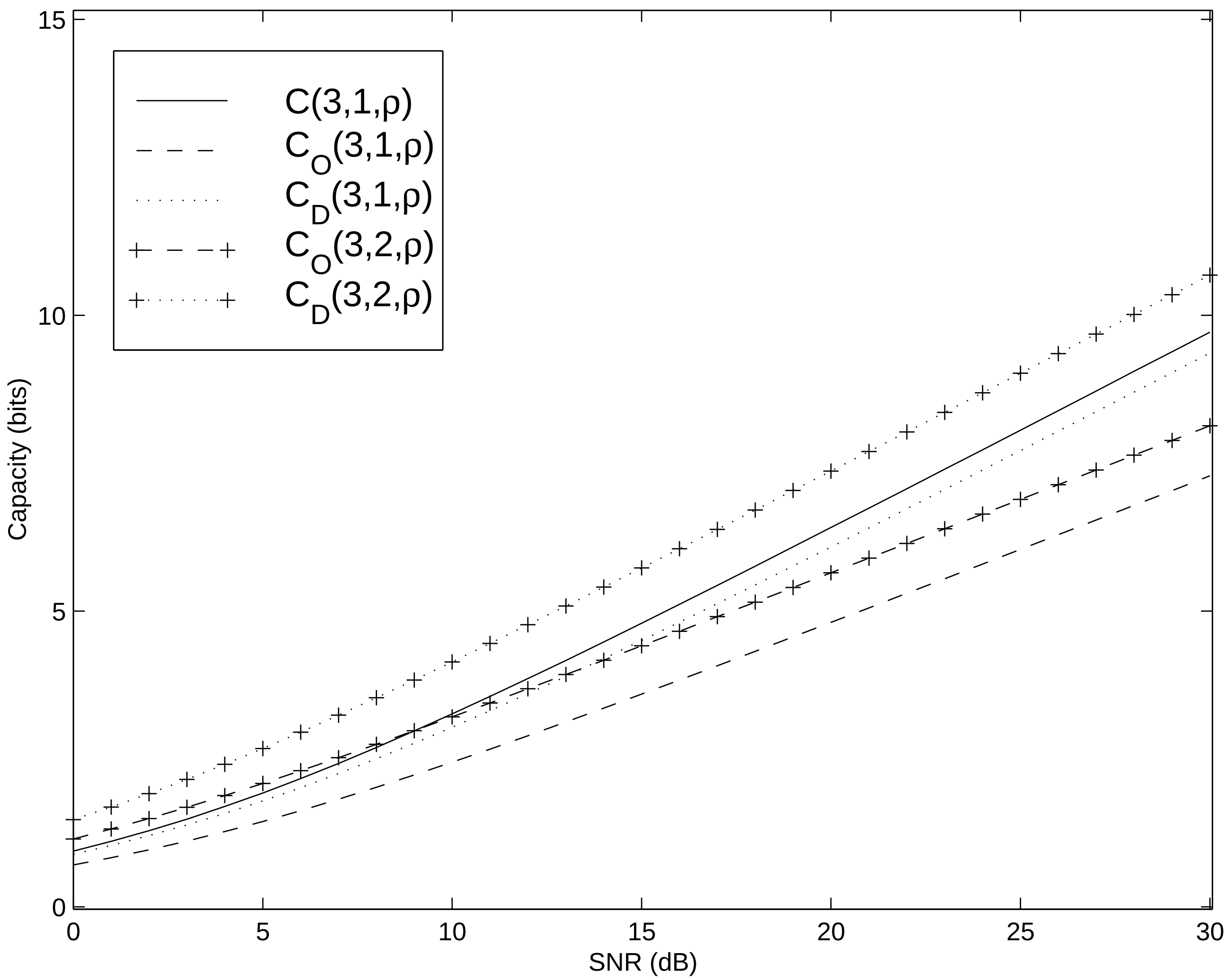


Figure 3.7: The maximum mutual information of GCIOD code for three transmitters and one, two receiver compared with that of code rate 3/4 complex orthogonal design for three transmitters and the actual channel capacity.

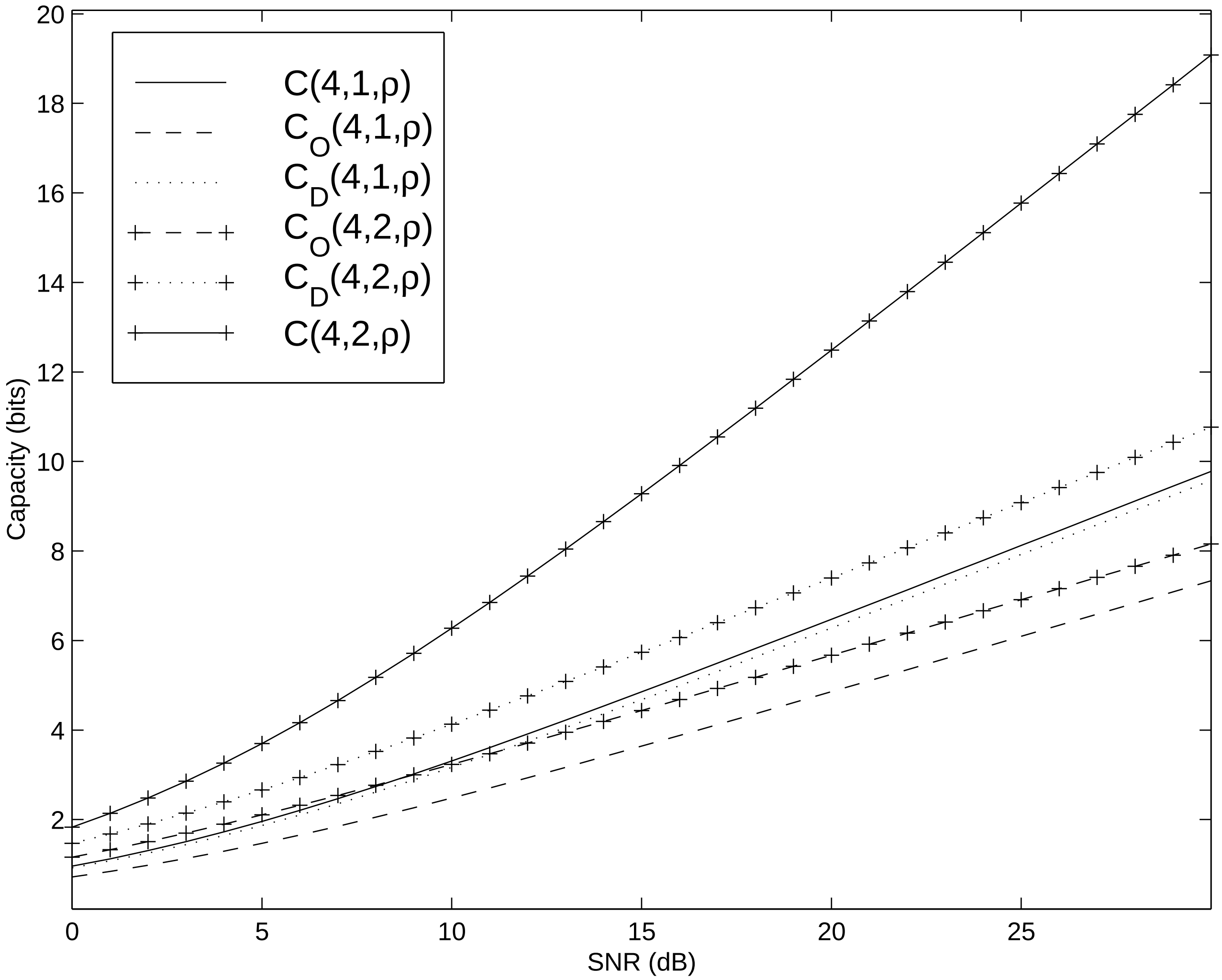


Figure 3.8: The maximum mutual information of CIOD code for four transmitters and one, two receiver compared with that of code rate 3/4 complex orthogonal design for four transmitters and the actual channel capacity.

variance $\|\mathbf{H}\|^2/2$. Since (3.61) is a scaled AWGN channel with $SNR = \frac{L\rho}{KN}\|\mathbf{H}\|^2$ and rate K/L, the average MMI in bits per channel use of GLCOD can be written as [63]

$$C_O(N, M, \rho) = \frac{K}{L}\mathrm{E}\left[\log_2\left(1 + \frac{L\rho}{KN}\|\mathbf{H}\|^2\right)\right] \tag{3.62}$$

observe that $\mathbf{H}$ is a $N \times M$ matrix. Since $\|\mathbf{H}\|^2 = \vec{\mathbf{H}}^{\mathcal{H}}\vec{\mathbf{H}}$ where $\vec{\mathbf{H}}$ is the $NM \times 1$ vector formed by stacking the columns of $\mathbf{H}$, we have

$$\begin{aligned} C_O(N, M, \rho) &= \frac{K}{L}C(MN, 1, \frac{ML}{K}\rho) && (3.63)\\ &= \frac{K}{L}\left(\frac{1}{\Gamma(MN)}\int_0^\infty \log\left(1 + \frac{L\rho\lambda}{KN}\right)\lambda^{MN-1}\mathrm{e}^{-\lambda}\mathrm{d}\lambda\right) && (3.64)\end{aligned}$$

where (3.64) follows from [6, eqn. (10)]. For GCIOD, recollect that it consists of two GLCODs, Θ_1, Θ_2 of rate $K/2L_1, K/2L_2$ as defined in (3.1). Let $C_{1,O}, C_{2,O}$ be the MMI of Θ_1, Θ_2 respectively. Then the MMI of GCIOD is given by

$$\begin{aligned} C_D(N, M, \rho) &= \frac{1}{L}\left\{L_1C_{1,O} + L_2C_{2,O}\right\} && (3.65)\\ &= \frac{1}{L}\left\{L_1C_O(N_1, M, c\rho) + L_2C_O(N_2, M, c\rho)\right\} && (3.66)\\ &= \frac{K}{2L}\left\{C(MN_1, 1, \frac{2L_1Mc\rho}{K}) + C(MN_2, 1, \frac{2L_2cM\rho}{K})\right\} && (3.67)\end{aligned}$$

where c is a constant so as to normalize the average transmit power i.e. $(N_1+N_2)cK/2 = L$ $\Rightarrow c = \frac{2L}{KN}$. The above result follows from the fact that the GCIOD is block diagonal with each block being a GLCOD. When $L_1 = L_2$ i.e. $\Theta_1 = \Theta_2$ we have

$$C_D(N, M, \rho) = \frac{K}{L}C(\frac{MN}{2}, 1, \frac{2LM\rho}{K}) \tag{3.68}$$

as we have already seen for $N = 2, 4$.

Let $\triangle C = C_D - C_O$. For square designs ($N = L = 2^a b, b$ odd) we have

$$\triangle C = \frac{2a}{N}C(M2^{a-1}b, 1, 2^{a+1}M\rho) - \frac{a+1}{N}C(M2^a b, 1, 2^a M\rho). \tag{3.69}$$

It is sufficient to consider $b = 1$. When $N = 2$, $\frac{2a}{N} = \frac{a+1}{N} = 1$ and $\triangle C = C(M, 1, M\rho) -$

$C(2M, 1, M\rho) < 0$, as seen from [6, Figure 3: and Table 2]. When $N > 2$, $2a > a + 1$ and $\lim_{a\to\infty} \frac{2a}{a+1} = 2$. Also $C(M2^{a-1}, 1, M\rho)$ is marginally smaller than $C(M2^{a}, 1, M\rho)$ for $M > 1, a > 1$ as can be seen from [6, Figure 3: and Table 2]. It therefore follows that

Theorem 3.5.1. *The MMI of square CIOD is greater than MMI of square GLCOD except when $N = 2$.*

It can be shown that a similar result holds for GCIOD also, by carrying out the analysis for each N. We are omitting $N = 5, 6, 7$. For $N \geq 8$ we compare rate 2/3 GCIOD with the rate 1/2 GLCODs. The MMI of rate 1/2 GCIOD is given by

$$C_O(N, M, \rho) \quad = \quad \frac{1}{2}C(MN, 1, M\rho). \tag{3.70}$$

The MMI of rate 2/3 GCIOD is given by,

$$C_D(N, M, \rho) = \frac{1}{3}\left\{C(2M, 1, M\rho) + C(M(N-2), 1, M\rho)\right\}. \tag{3.71}$$

For reasonable values of N that is $N > 8$, $C(MN, 1, M\rho) \approx C(M(N-2), 1, M\rho)$ and $C(2M, 1, M\rho) \approx C(MN, 1, M\rho)$ and it follows that

$$C_D(N, M, \rho) \approx \frac{2}{3}C(MN, 1, M\rho). \tag{3.72}$$

Figure 3.9 shows the capacity plots for $N = 8$, observe that the capacity of rate 2/3 GCIOD is considerably greater than that of rate 1/2 GLCOD. At a capacity of 7 bits the gain is around 10 dB for $M = 8$. Similar plots are obtained for all $N > 8$ with increasing coding gains and have been omitted.

3.6 Discussion

In this chapter we have presented a class of FSDD called GCIOD of which CIOD is a subclass. Construction of fractional rate GCIODs have been dealt with thoroughly resulting in construction of various high rate GCIODs. In particular a rate 6/7 GCIOD for $N = 5, 6$, rate 7/9 GCIOD for $N = 7$ and rate 2/3 GCIOD for $N \geq 6$ have been presented. The expansion of signal constellation due to co-ordinate interleaving has been brought

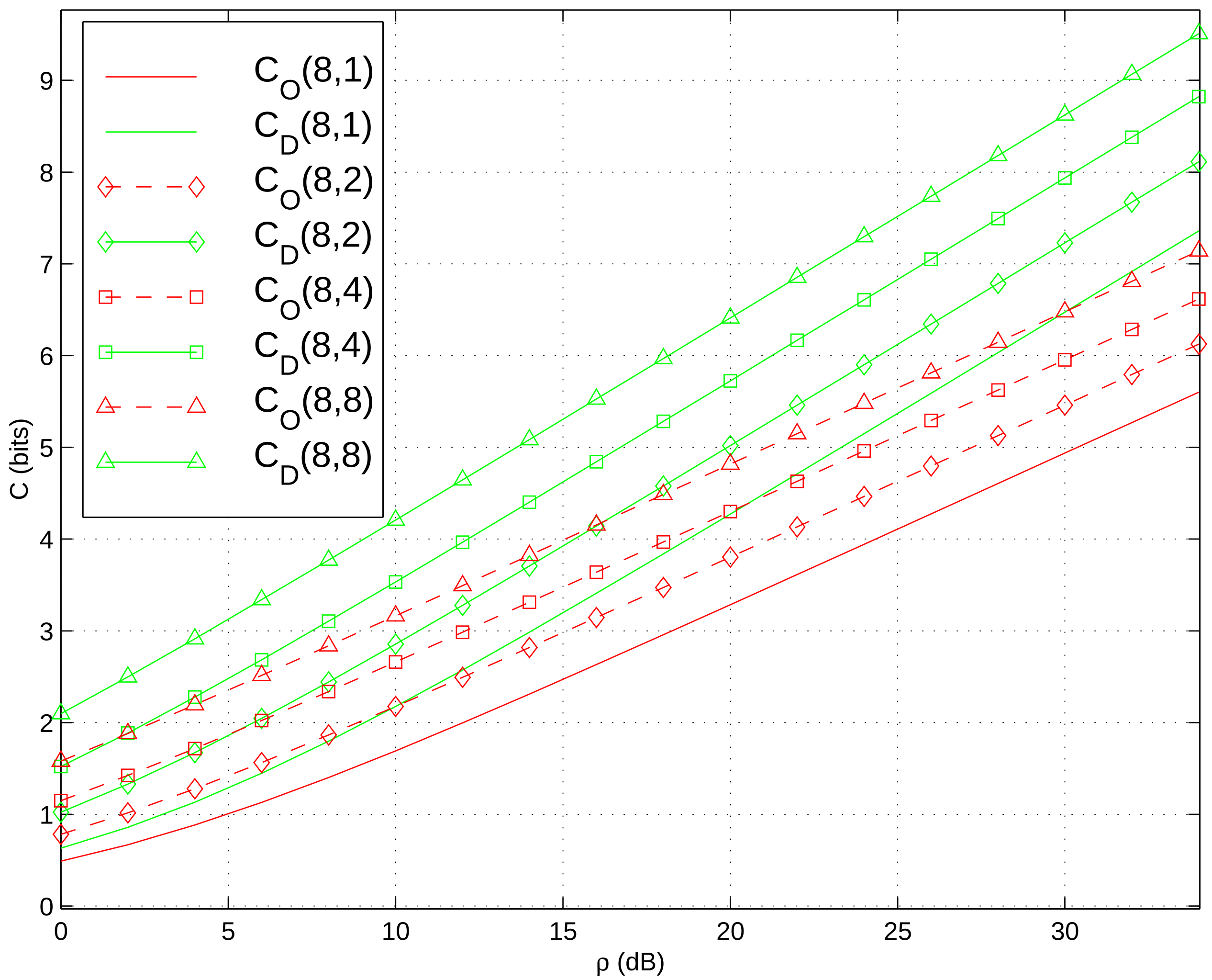


Figure 3.9: The maximum mutual information (average) of rate 2/3 GCIOD code for eight transmitters and one, two, four and eight receivers compared with that of code rate 1/2 complex orthogonal design for eight transmitters over Rayleigh fading channels.

out. The coding gain of GCIOD is linked to a new distance called generalized co-ordinate product distance (GCPD) as a consequence the coding gain of CIOD is linked to CPD. Both the GCPD and the CPD for signal constellations derived from the square lattice have been investigated. Simulation results are then presented for $N = 4$ to substantiate the theoretical analysis and finally the maximum mutual information for GCIOD has been derived and compared with GLCOD. It is interesting to note that except for $N = 2$, the GCIOD turns out to be superior to GLCOD in terms of rate, coding gain and MMI. A significant drawback of GCIOD schemes is that half of the antennas are idle,as a result these schemes have higher peak-to-average ratio (PAR) compared to the ones using Orthogonal Designs. This problem can be solved by pre-multiplying with a Hadamard matrix as is done for DAST codes in [37]. This pre-multiplication by a Hadamard matrix will not change the decoding complexity while more evenly distributing the transmitted power across space and time and achieve full-diversity over fast-fading channels resulting in 'smart and greedy' GCIOD codes.

In a nutshell this chapter shows that, except for $N = 2$ (the Alamouti code), CIODs are better than GLCODs in terms of rate, coding gain, MMI and BER.

Few possible directions for further research are listed below.

- While we know that no square RFSDD exists other than square GCIOD the same is not known for non-square RFSDD. If this is true then the GCIOD codes presented here complete the theory of FSDD, if not then there exists a class of codes other than GCIODs that are also FSDD.
- Maximal rates for non-square GCIOD is also an open problem?
- The CPD of non-square lattice constellations and the GCPD for both square and non-square lattice constellations needs to be quantified.

Chapter 4

Characterization of Optimal SNR STBCs

In a recent work [19], space-time block codes (STBC) from Orthogonal designs (OD) were shown to maximize the signal to noise ratio (SNR) and also it was shown that for a linear STBC the maximum SNR is achieved when the weight matrices are unitary. In this chapter we show that STBCs from ODs are not the only codes that maximize SNR; we characterize all linear STBCs that maximize SNR thereby showing that maximum SNR can be achieved with non-unitary weight matrices also subject to a constraint on the transmitted symbols (which is that the in-phase and quadrature component are of equal energy). This constraint is satisfied by some known signal sets like BPSK rotated by an angle of 45°, QPSK and X-constellations. It is then shown that the Generalized Co-ordinate Interleaved orthogonal Designs (GCIOD) presented in the previous chapter achieve maximum SNR and corresponds to a generalized maximal ratio combiner under this constraint. This result is a generalization to a previous result on maximum SNR presented in [17] because the necessary conditions for maximal SNR derived therein for real symbols are not necessary for the complex symbol case as shown in this chapter. Also we show that though GCIOD maximizes SNR for QPSK of Fig. 4.1 (b), the same GCIOD with the rotated QPSK of Fig. 4.1 (c) performs better in terms of probability of error. This is due to the fact that the maximum SNR approach does not necessarily maximize the coding gain also.

4.1 Background

In a recent work [19], space-time block codes were constructed from an optimal SNR perspective. The approach adopted therein is:(i) obtain a linear filter, Z, at the receiver, for a real symbol weighed by a matrix A, that maximizes SNR, (this maximized SNR is a function of A) (ii) for the case when channel state information is not known at the receiver, obtain A that maximizes SNR and (iii) add additional symbols with weight matrices that maximize SNR and are orthogonal to each other so as to increase data rate.

Although, this strategy leads to an alternate construction of STBC from ODs, it is deficient in that it dose not characterize all STBCs that maximize SNR. For example

Example 4.1.1. *Consider the CIOD for 2 transmit antennas given by,*

$$S = \begin{bmatrix} \tilde{x}_0 & 0 \\ 0 & \tilde{x}_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{A_0} x_{0I} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \mathbf{j} \end{bmatrix}}_{A_1} x_{0Q} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{A_2} x_{1I} + \underbrace{\begin{bmatrix} \mathbf{j} & 0 \\ 0 & 0 \end{bmatrix}}_{A_3} x_{1Q} \quad (4.1)$$

where $x_i = x_{iI} + \mathbf{j}x_{iQ}, i = 0, 1$ are symbols from the QPSK signal set shown in Fig. 4.1 (b). The received signal matrix is given by $V = SH + W$, where H denotes the channel matrix and W the noise matrix. Let $\hat{V} = Z_0^{\mathcal{H}}V = Z_0SH + Z_0^{\mathcal{H}}W$ where $Z_0 = A_0 + A_1$ is the matrix used linear processing at the receiver for detecting x_0. Then the symbol x_0 is detected based on $D_0 = \mathrm{tr}\left(\hat{V}\right)$. The SNR of this code for symbols transmitted from the QPSK signal set shown in Fig. 4.1 (b), under the total transmit power constraint $\mathrm{tr}\left(S^{\mathcal{H}}S\right) = 2E_s$, is

$$\mathrm{SNR} = \frac{|\mathrm{tr}\,(ZSH)\,|^2}{E\left\{|\mathrm{tr}\,(Z^{\mathcal{H}}W)\,|^2\right\}} = \frac{\mathrm{tr}\left(HH^{\mathcal{H}}\left\{A_0^{\mathcal{H}}A_0E_{x_{0I}} + A_1^{\mathcal{H}}A_1E_{x_{0Q}}\right\}\right)}{2\sigma^2} = \frac{\mathrm{tr}\left(H^{\mathcal{H}}H\right)E_s}{2\sigma^2} \quad (4.2)$$

where $E_{x_{iI}} = x_{iI}^2, E_{x_{iQ}} = x_{iQ}^2, i = 0, 1$ and are equal for QPSK such that $E_{x_{0I}} = E_{x_{0Q}} = E_{x_{1I}} = E_{x_{1Q}} = E_s/2$. When we set $Z = A_2 + A_3$ we again get (4.2). Now the SNR of Alamouti code, $S = \frac{1}{\sqrt{2}}\begin{bmatrix} x_0 & x_1 \\ -x_1^* & x_0^* \end{bmatrix}$, for symbols transmitted from the same signal set, under the same total transmit power constraint i.e. $\mathrm{tr}\left(S^{\mathcal{H}}S\right) = 2E_s$, is given by [23, eqn.

2.18]

$$\mathrm{SNR}_{\max} = \frac{\mathrm{tr}\left(H^{\mathcal{H}} H\right) E_s}{2\sigma^2}. \tag{4.3}$$

Since (4.3) and (4.2) are equal, this CIOD achieves the maximal SNR when the symbols x_0, x_1 *are from the QPSK shown in Fig. 4.1 (b). This implies that there are STBCs other than STBCs from ODs that maximize SNR.*

In this chapter we characterize all linear STBCs that maximize SNR. Towards this end, we first obtain a linear filter, Z, at the receiver, *for a complex symbol whose in-phase and quadrature components are weighed by different weight matrices*, that maximizes SNR. This is the basic difference between our approach and the approach of [17, 18, 19]. As a result of this maximization, we get the conditions that SNR is maximized only if restrictions are placed on the weight matrices or on the signal set. We then optimize weight matrices under a total transmit power constraint to show that if there is no-restriction on the signal set then the weight matrices are necessarily orthogonal (unitary) and if there is a restriction on the signal set then the sum of weight matrices of the in-phase and quadrature components is unitary. It is then shown that the first case leads to ODs and the later case to CIODs, if the data rate is maximized. We then show that maximum SNR approach is deficient in that it dose not necessarily result in optimal performance due to possible smaller coding gain compared to a case where the SNR is not maximized but has larger coding gain.

4.2 Optimal Linear Filter

As in [17, 18, 19], we start by considering that the transmission from all the antennas N for L symbol durations depends on a complex symbol s. The dependence is as follows: let $s = s_I + \mathbf{j} s_Q$ denote the complex transmitted symbol and let $a_{k,1}, a_{k,2} \in \mathbb{C}^{1\times N}$ denote the transmission weight vectors for s_I, s_Q respectively during the kth symbol period, then the transmission from the ith antenna during the kth symbol duration is given by

$$a_{k,1}^{i} s_I + a_{k,2}^{i} s_Q$$

where $a^i_{k,p}$ is the ith element of $a_{k,p}, \quad p = 1, 2$. Assuming that the receiver uses a matched filter, the output of the matched filter is the $1 \times M$ vector

$$v_k = [s_I a_{k,1} + s_Q a_{k,2}]H + w_k, k = 1, \cdots, N \tag{4.4}$$

where w_k is a complex Gaussian noise vector that is spatially and temporally white and the variance of each entry is σ^2. Let z_k denote the $\mathbb{C}^{1\times M}$ vector used at the receiver for linear processing to detect s, then detection of s is based on

$$D = \frac{1}{L}\sum_{k=1}^{L}\{[s_I a_{k,1} + s_Q a_{k,2}]H + w_k\}z_k^{\mathcal{H}}. \tag{4.5}$$

In matrix notation (4.4) can be written as

$$V = [A_1 s_I + A_2 s_Q]H + W \tag{4.6}$$

where $V \in \mathbb{C}^{L\times M}$ is the received matrix whose kth row is given by v_k, $A_1, A_2 \in \mathbb{C}^{L\times N}$ are the weight matrices of s_I, s_Q respectively whose kth row is given by $a_{k,1}, a_{k,2}$. Note that the transmission matrix is given by $S = A_1 s_I + A_2 s_Q$. Correspondingly (4.5) can be written as,

$$\begin{aligned} D = \frac{1}{L}\mathrm{tr}\left(V Z_1^{\mathcal{H}}\right) &= \frac{1}{L}\mathrm{Real}\left\{\mathrm{tr}\left(\{[A_1 s_I + A_2 s_Q]H + W\}Z^{\mathcal{H}}\right)\right\} \\ &= \frac{1}{L}\left\{\mathrm{tr}\left(A_1 H Z^{\mathcal{H}}\right)s_I + \mathrm{tr}\left(A_2 H Z^{\mathcal{H}}\right)s_Q + \mathrm{tr}\left(W Z^{\mathcal{H}}\right)\right\}. \end{aligned} \tag{4.7}$$

The decision on the symbol s is based on D and the SNR in (4.7) is given by,

$$\mathrm{SNR} = \frac{\left|\frac{1}{L}\mathrm{tr}\left(\{A_1 s_I + A_2 s_Q\} H Z^{\mathcal{H}}\right)\right|^2}{E\left|\frac{1}{L}\mathrm{tr}\left(W Z^{\mathcal{H}}\right)\right|^2} \tag{4.8}$$

$$\begin{aligned} &= \frac{\left|\frac{1}{L}\mathrm{tr}\left(A_1 H Z^{\mathcal{H}}\right)\right|^2 E_{s_I} + \left|\frac{1}{L}\mathrm{tr}\left(A_2 H Z^{\mathcal{H}}\right)\right|^2 E_{s_Q}}{\left(\frac{\sigma^2}{L}\right)\mathrm{tr}\left(Z Z^{\mathcal{H}}\right)} \\ &+ \frac{\frac{1}{L^2}\mathrm{tr}\left(Z H^{\mathcal{H}}\left\{A_1^{\mathcal{H}} A_2 + A_2^{\mathcal{H}} A_1\right\} H Z^{\mathcal{H}} s_I s_Q\right)}{\left(\frac{\sigma^2}{L}\right)\mathrm{tr}\left(Z Z^{\mathcal{H}}\right)} \end{aligned} \tag{4.9}$$

where $E\{W^{\mathcal{H}}W\} = L\sigma^2 I_M$. Since $s_I s_Q$ can be either positive or negative and is unknown at the receiver, necessarily

$$A_1^{\mathcal{H}} A_2 + A_2^{\mathcal{H}} A_1 = 0 \tag{4.10}$$

and

$$\text{SNR} = \left(\frac{L}{\sigma^2}\right) \frac{\left|\frac{1}{L}\text{tr}\left(A_1 H Z^{\mathcal{H}}\right)\right|^2 E_{s_I} + \left|\frac{1}{L}\text{tr}\left(A_2 H Z^{\mathcal{H}}\right)\right|^2 E_{s_Q}}{\text{tr}\left(Z Z^{\mathcal{H}}\right)} \tag{4.11}$$

where $E_{s_I} = s_I^2, E_{s_Q} = s_Q^2$. Define

$$\tilde{E}\{a^{\mathcal{H}}, b\} = 1/L \sum_{k=1}^{L} (a^{\mathcal{H}} b),$$

then (4.11) can be written as

$$\text{SNR} = \frac{L}{\sigma^2} \frac{\left|\tilde{E}\{u_{k,1}, z_k^{\mathcal{H}}\}\right|^2 E_{s_I} + \left|\tilde{E}\{u_{k,2}, z_k^{\mathcal{H}}\}\right|^2 E_{s_Q}}{\tilde{E}\{z_k, z_k^{\mathcal{H}}\}} \tag{4.12}$$

where

$$u_{k,p} = a_{k,p} H, p = 1, 2, k = 1, \cdots, L.$$

Observe that SNR expression for a complex symbol whose in-phase and quadrature components are weighed by different weight matrices (A_1, A_2), given by (4.12), is different from the corresponding expression when s is real and is given by

$$\text{SNR} = \frac{L}{\sigma^2} \frac{\left|\tilde{E}\{u_{k,1}, z_k^{\mathcal{H}}\}\right|^2 E_s}{\tilde{E}\{z_k, z_k^{\mathcal{H}}\}}. \tag{4.13}$$

From (4.12) the maximum SNR is obtained by choosing A_1, A_2, Z such that

$$\frac{L}{\sigma^2} \frac{\left|\tilde{E}\{u_{k,1}, z_k^{\mathcal{H}}\}\right|^2 E_{s_I} + \left|\tilde{E}\{u_{k,2}, z_k^{\mathcal{H}}\}\right|^2 E_{s_Q}}{\tilde{E}\{z_k, z_k^{\mathcal{H}}\}} \text{ is maximum.} \tag{4.14}$$

Using

$$\max_{x,y} \{a(x,y) + b(x,y)\} \leq \max_{x,y}(a(x,y)) + \max_{x,y}(b(x,y)) \tag{4.15}$$

and the generalized Cauchy-Schwartz inequality given in [18, 19, *Theorem 1*]

$$|\tilde{E}\{u_k, z_k^{\mathcal{H}}\}|^2 \leq \tilde{E}\{u_k, u_k^{\mathcal{H}}\}\tilde{E}\{z_k, z_k^{\mathcal{H}}\} \tag{4.16}$$

with equality when

$$z_k = \alpha u_k, \alpha \neq 0 \in \Re, \tag{4.17}$$

we have

$$\begin{aligned}
\max_{z_k} \text{SNR} &= \max_{z_k} \frac{L}{\sigma^2} \frac{\left|\tilde{E}\{u_{k,1}, z_k^{\mathcal{H}}\}\right|^2 E_{s_I} + \left|\tilde{E}\{u_{k,2}, z_k^{\mathcal{H}}\}\right|^2 E_{s_Q}}{\tilde{E}\{z_k, z_k^{\mathcal{H}}\}} \\
&\leq \max_{z_k} \frac{L}{\sigma^2} \frac{\left|\tilde{E}\{u_{k,1}, z_k^{\mathcal{H}}\}\right|^2 E_{s_I}}{\tilde{E}\{z_k, z_k^{\mathcal{H}}\}} + \max_{z_k} \frac{L}{\sigma^2} \frac{\left|\tilde{E}\{u_{k,2}, z_k^{\mathcal{H}}\}\right|^2 E_{s_Q}}{\tilde{E}\{z_k, z_k^{\mathcal{H}}\}} &(4.18) \\
&\leq \frac{1}{\sigma^2} \left\{\tilde{E}\{u_{k,1}, u_{k,1}^{\mathcal{H}}\}E_{s_I} + \tilde{E}\{u_{k,2}, u_{k,2\cdot}^{\mathcal{H}}\}E_{s_Q}\right\}. &(4.19)
\end{aligned}$$

Theorem 4.2.1.

$$\max_{z_k} \text{SNR} \leq \frac{1}{\sigma^2} \left\{\tilde{E}\{u_{k,1}, u_{k,1}^{\mathcal{H}}\}E_{s_I} + \tilde{E}\{u_{k,2}, u_{k,2\cdot}^{\mathcal{H}}\}E_{s_Q}\right\}$$

with equality iff $Z = \alpha(A_1 + A_2)H$ *and*

i) $\tilde{E}\{u_{k,1}, u_{k,1}^{\mathcal{H}}\} = \tilde{E}\{u_{k,2}, u_{k,2}^{\mathcal{H}}\}$

or

ii) $E_{s_I} = E_{s_Q}$

Proof. The "if part" is proved by substituting these conditions in (4.12). For the "only if part," if, $Z \neq \alpha(A_1 + A_2)H$ then applying (4.16) to (4.12)

$$|\tilde{E}\{u_{k,1}, z_k^{\mathcal{H}}\}|^2 E_{s_I} + |\tilde{E}\{u_{k,2}, z_k^{\mathcal{H}}\}|^2 E_{s_Q} \leq \left\{\tilde{E}\{u_{k,1}, u_{k,1}^{\mathcal{H}}\}E_{s_I} + \tilde{E}\{u_{k,2}, u_{k,2}^{\mathcal{H}}\}E_{s_Q}\right\} \tilde{E}\{z_k, z_k^{\mathcal{H}}\} \tag{4.20}$$

with equality iff $Z = \alpha(A_1 + A_2)H$ as $A_1^{\mathcal{H}}A_2 + A_2^{\mathcal{H}}A_1 = 0$. The corresponding value of

SNR is

$$\mathrm{SNR} = \frac{L}{\sigma^2} \frac{\left|(\mathrm{tr}\left(HH^{\mathcal{H}}A_1^{\mathcal{H}}A_1\right)\right|^2 |\alpha_1|^2 \hat{E}_{s_I} + \left|(\mathrm{tr}\left(HH^{\mathcal{H}}A_2^{\mathcal{H}}A_2\right)\right|^2 |\alpha_2|^2 \hat{E}_{s_Q}}{\mathrm{tr}\left(HH^{\mathcal{H}}A_1^{\mathcal{H}}A_1\right)|\alpha_1|^2 + \mathrm{tr}\left(HH^{\mathcal{H}}A_2^{\mathcal{H}}A_2\right)|\alpha_2|^2} \tag{4.21}$$

and the proof is completed for $\alpha_1 = \alpha_2$ by verifying that the bound of (4.19) is achieved only if the conditions are satisfied. □

Remark 4.2.1. In Theorem 4.2.1 we have obtained the optimal receiver weight matrix Z in terms of the channel matrix H and the weight vectors A_1, A_2. Further it shows that the SNR is maximized iff constraints are placed on the weight matrices A_1, A_2 ($\tilde{E}\{u_{k,1}, u_{k,1}^{\mathcal{H}}\} = \tilde{E}\{u_{k,2}, u_{k,2}^{\mathcal{H}}\}$ which corresponds to $\mathrm{tr}\left(HH^{\mathcal{H}}A_1^{\mathcal{H}}A_1\right) = \mathrm{tr}\left(HH^{\mathcal{H}}A_2^{\mathcal{H}}A_2\right)$) or the transmitted symbol ($E_{s_I} = E_{s_Q}$). Also observe that when the symbol is real then no constraints are imposed on the weight matrix or the signal set which is the case considered in [17, 18, 19].

We define the constellation that has the property $E_{s_I} = E_{s_Q}$ as

Definition 4.2.1 (X-constellation). A signal set is called an X-constellation if all its elements on the x and y axes after rotated by an angle of 45°.

Observe that rotated BPSK (45°), QPSK are X-constellations and also any signal set that has the property that $E_{s_I} = E_{s_Q}$. Examples of such signal constellations are presented in Fig. 4.1.

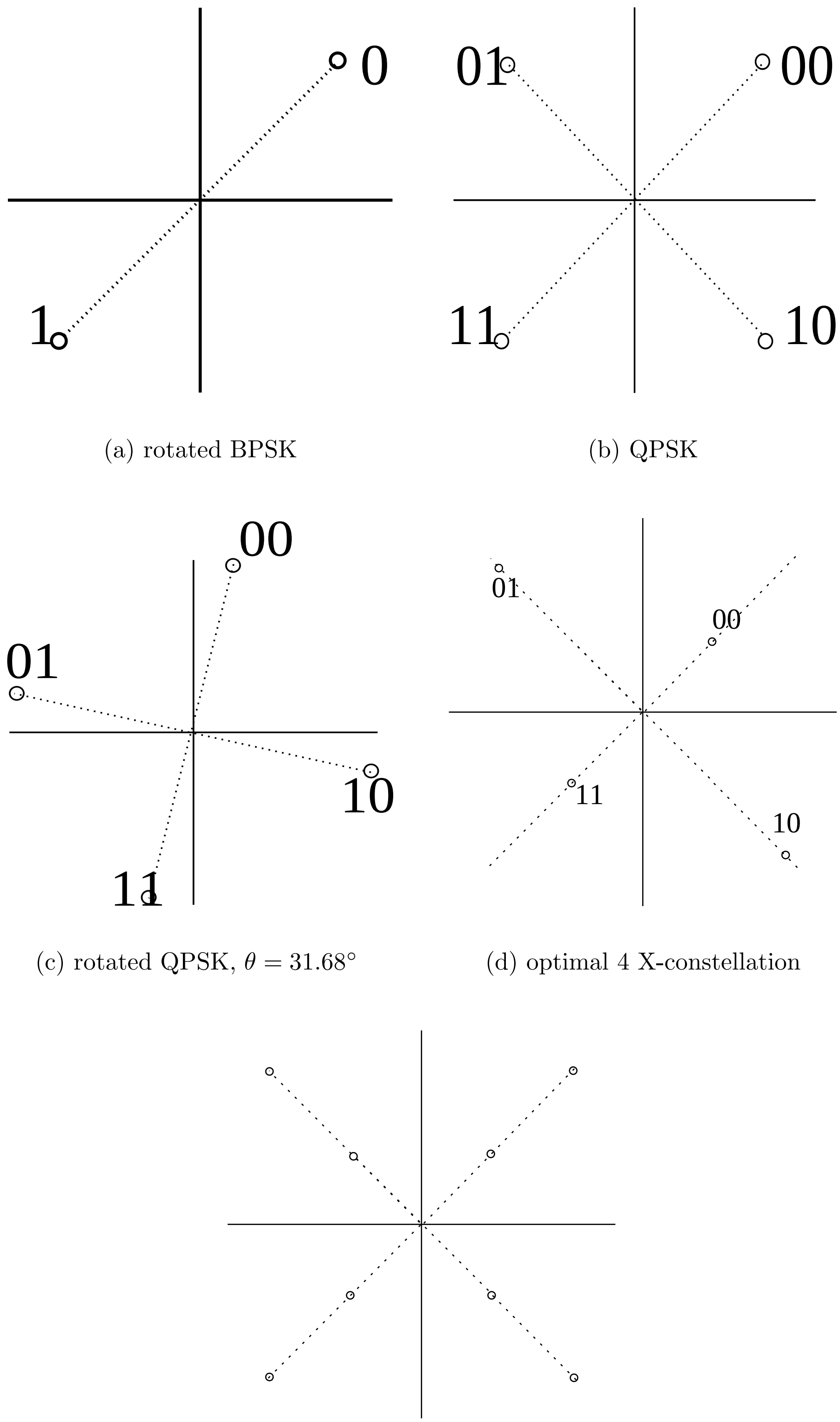
4.3 Design of Transmit Weight Vector

Now, consider the design of the weight matrices $A_1, A_2, p = 1, 2$ subject to total transmit power constraint given by

$$\mathrm{tr}\left(S^{\mathcal{H}}S\right) = LE_s \Rightarrow \mathrm{tr}\left(A_1^{\mathcal{H}}A_1\right)E_{s_I} + \mathrm{tr}\left(A_2^{\mathcal{H}}A_2\right)E_{s_Q} = E_s \tag{4.22}$$

and the constraints imposed by Theorem 4.2.1. We consider the two cases of Theorem 4.2.1 separately:

Case 1: $\tilde{E}\{u_{k,1}, u_{k,1}^{\mathcal{H}}\} = \tilde{E}\{u_{k,2}, u_{k,2}^{\mathcal{H}}\}$.



(a) rotated BPSK (b) QPSK

(c) rotated QPSK, $\theta = 31.68°$ (d) optimal 4 X-constellation

(e) 8 X-constellation

Figure 4.1: The signal constellations considered in Chapter 4. Except (c) all others are examples of X-constellations that maximize SNR for CIODs.

Using, an identity from [64],

$$\mathrm{tr}\,(AB) \leq \mathrm{tr}\,(A)\,\lambda_{\max}(B) \text{ with equality when } B = \lambda I \tag{4.23}$$

where A, B are hermitian semi-definite matrices and $\lambda_{\max}(B)$ is the largest eigen value of B, we have

$$\begin{aligned} \left\{\tilde{E}\{u_{k,1}, u_{k,1}^{\mathcal{H}}\}E_{s_I} + \tilde{E}\{u_{k,2}, u_{k,2}^{\mathcal{H}}.\}E_{s_Q}\right\} &= \mathrm{tr}\left(HH^{\mathcal{H}}A_1^{\mathcal{H}}A_1\right)E_{s_I} + \mathrm{tr}\left(HH^{\mathcal{H}}A_2^{\mathcal{H}}A_2\right)E_{s_Q} \\ &\leq \mathrm{tr}\left(HH^{\mathcal{H}}\right)\left\{E_{s_I}\lambda_{A_1} + E_{s_Q}\lambda_{A_2}\right\} \end{aligned} \tag{4.24}$$

with equality iff $A_i^{\mathcal{H}}A_i = \lambda_{A_i}I, i = 1, 2$. On substituting the total power constraint we have $\lambda_{A_i} = \frac{1}{\sqrt{N}}$. Substituting these condition in (4.22) and substituting back in (4.24) we have

$$\mathrm{SNR}_{\max} = \frac{\mathrm{tr}\left(H^{\mathcal{H}}H\right)}{N\sigma^2}E_s \tag{4.25}$$

and the optimal weight matrices A_1, A_2 are such that $A_i^{\mathcal{H}}A_i = I$, $A_1^{\mathcal{H}}A_2 + A_2^{\mathcal{H}}A_1 = 0$. Observe that Theorem 4.2.1 requires $\lambda_{A_1} = \lambda_{A_2}$. Also observe that (4.25) is same as [23, eqn. 2.18].

Case 2: $E_{s_I} = E_{s_Q}$

Using (4.23), we have

$$\begin{aligned} \left\{\tilde{E}\{u_{k,1}, u_{k,1}^{\mathcal{H}}\}E_{s_I} + \tilde{E}\{u_{k,2}, u_{k,2}^{\mathcal{H}}.\}E_{s_Q}\right\} &= \mathrm{tr}\left(HH^{\mathcal{H}}(A_1 + A_2)^{\mathcal{H}}(A_1 + A_2)\right)E_{s_I} \\ &\leq \mathrm{tr}\left(HH^{\mathcal{H}}\right)\{E_{s_I}\lambda_{A_1+A_2}\} \end{aligned} \tag{4.26}$$

with equality iff $(A_1 + A_2)^{\mathcal{H}}(A_1 + A_2) = \lambda_{A_1+A_2}I$. Substituting these condition in (4.22) and substituting back in (4.26) we have (4.25), and the optimal weight matrices A_1, A_2 are such that $(A_1 + A_2)^{\mathcal{H}}(A_1 + A_2) = \frac{1}{\sqrt{N}}I$, $A_1^{\mathcal{H}}A_2 + A_2^{\mathcal{H}}A_1 = 0$ after normalization.

Observe that Case 1, was considered in [18, 19]. Proceeding as in [19], the rate of the scheme considered so far is $1/L$; if we can add $2K - 2$, $\{A_2, \cdots, A_{2K-1}\}$ weight vectors such that

$$A_k^{\mathcal{H}} A_l + A_l^{\mathcal{H}} A_k = 0, 0 \leq k \neq l \leq 2K-1 \text{ and} \tag{4.27}$$

$$A_k^{\mathcal{H}} A_k = I, k = 0, \cdots, 2K-1 \text{ for Case 1:} \tag{4.28}$$

$$A_{2k}^{\mathcal{H}} A_{2k} + A_{2k+1}^{\mathcal{H}} A_{2k+1} = I, k = 0, \cdots, K-1 \text{ for case 2:} \tag{4.29}$$

then we have a rate K/L STBC that maximizes SNR.

Remark 4.3.1. A natural question arises as to why not start with K complex variables and then find the conditions on the weight matrices that will lead to maximization of SNR. This approach is not fruitful because in such a case maximization of SNR can be done only by coding across the variables, in general, because we would require the energy of each of the $2K$ real variables to be equal. Only exception to this is when all the K complex variables take value from the rotated BPSK signal set of Fig. 4.1 (a). For this case the conditions on the weight matrices that maximize SNR are,

$$A_k^{\mathcal{H}} A_l + A_l^{\mathcal{H}} A_k = 0, 0 \leq k \neq l \leq 2K-1 \text{ and} \tag{4.30}$$

$$\sum_{k=0}^{2K-1} A_k^{\mathcal{H}} A_k = KI. \tag{4.31}$$

Codes that satisfy the conditions of Case 1 are precisely the generalized complex orthogonal designs (GCOD) defined in Definition **??**, as pointed out in [19] and their design is considered in [16, 18, 17, 19, 15, 13, 20, 33] and reviewed in Chapter 2. The Generalized Co-ordinate Interleaved orthogonal designs (GCIOD) of Definition **??** (constructed in Chapter 3) satisfy the criteria of case 2, as can be easily worked out. To put this formally, we have

Theorem 4.3.1. *STBCs from GCIODs are maximal SNR codes iff the transmitted signals constitute an X-constellation.*

The problem of characterization of all codes that satisfy (4.27)-(4.29) has been already discussed in a more general setting in Chapters 2, 3.

4.4 Performance of Maximal SNR STBCs

In this section we show that the codes obtained by maximum SNR approach need not necessarily be the best in terms of largest coding gain. This is done by presenting an example of the performance of CIOD (case 2 codes) when the symbols are from the X-constellation, which maximizes SNR, and also from rotated QPSK, which dose not maximize SNR.

Consider the CIOD for 4 transmit antennas given by

$$S = \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 & 0 & 0 \\ -\tilde{x}_2^* & \tilde{x}_1^* & 0 & 0 \\ 0 & 0 & \tilde{x}_3 & \tilde{x}_4 \\ 0 & 0 & -\tilde{x}_4^* & \tilde{x}_3^* \end{bmatrix}. \tag{4.32}$$

The maximal SNR of this CIOD when the symbols are from the X-constellation, $\text{SNR}_{\max}$ $(M = 1)$, is given by

$$\text{SNR}_{\max} = \frac{\text{tr}\left(H^{\mathcal{H}}H\right)}{4\sigma^2}E_s \tag{4.33}$$

In contrast when the symbols are from the rotated QPSK constellation (Fig. 4.1 (c)) is given by

$$\text{SNR} = \frac{1}{16\sigma^2}\text{tr}\left(H^{\mathcal{H}}H\left\{A_{2i}^{\mathcal{H}}A_{2i}E_{s_{iI}} + A_{2i+1}^{\mathcal{H}}A_{2i+1}E_{s_{iQ}}\right\}\right) \leq \text{SNR}_{\max} \tag{4.34}$$

with equality when $H^{\mathcal{H}}H = \alpha I, \alpha \in \Re, i = 0, \cdots, 3$. Fig. 4.2 gives the SNR performance for both the cases discussed above. While the QPSK is an optimal SNR signal set, it shows a diversity loss while compared to the QPSK signal set rotated by an angle of $\theta = 31.7175°$. This is explained by the pair-wise error probability expression for CIOD [51]

$$P(S \to \hat{S}) \leq \left(\frac{1}{CPD\frac{E_s}{4\sigma^2}}\right)^{4M} \tag{4.35}$$

where CPD denotes the Coordinate Product Distance. Observe that for full-diversity CPD is required to be non-zero for any two distinct points in the signal set. When $\theta = 0°$, the $CPD = 0$ and hence there is a loss of diversity. When $\theta = 31.7175°$, the CPD is maximized ($CPD = 1.788$) [51] and hence the performance of this signal set is better

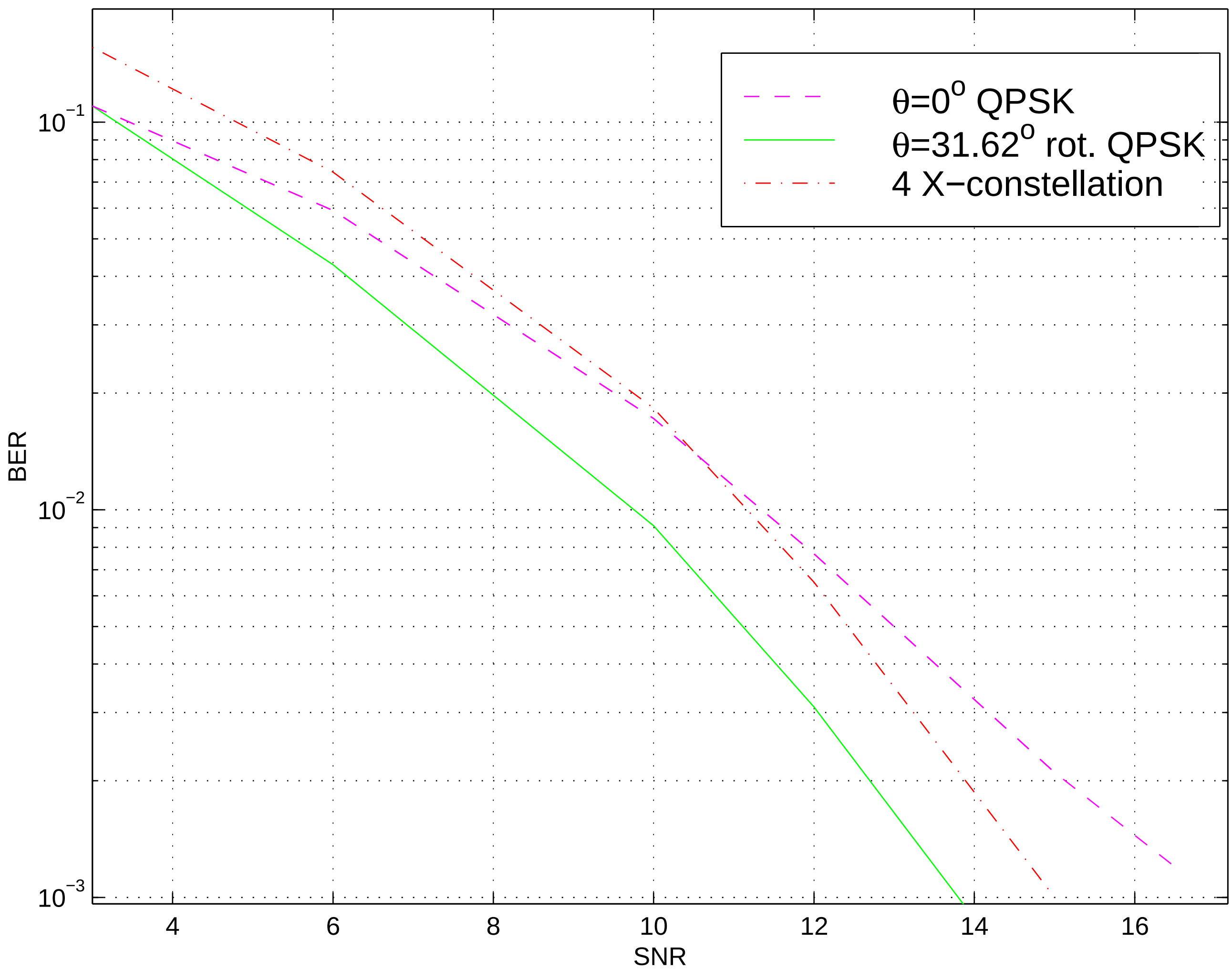


Figure 4.2: The BER performance of CIOD for 4 Tx, 1 Rx. for rotated QPSK when the angle of rotation $\theta = 0, 31.68^\circ$ and optimal X-constellation in quasi-static Rayleigh fading channel.

than the case when $\theta = 0^\circ$. Now we can construct a X-constellation with non-zero CPD by moving the points of QPSK as shown Fig. 4.1 (d). In-fact Fig. 4.1 (d) corresponds to the 4 X-constellation with maximum CPD under unit average energy constraint as shown below: **Construction of optimal 4 X-constellation:** We construct optimal 4 X-constellation that maximizes CPD by scaling the signals of QPSK. Let the signal points 00,11 in Fig. 4.2 (b) be scaled by a and the other two points by b. Then for the average energy of the constellation to be 1 we require $a^2 + b^2 = 2$. The CPD between 00,11 is $CPD_1 = 2a^2$ and between 00,01 or 10 is $CPD_2 = \frac{b^2 - a^2}{2}$. Solving $CPD_1 = CPD_2$ and $a^2 + b^2 = 2$ we have $a = \frac{1}{\sqrt{3}}, b = \sqrt{\frac{5}{3}}$. The optimal value of CPD is given by $CPD = 2/3$, for unit energy X-constellation. For CIOD we require the constellation energy to be 2, so as to satisfy the total transmit power constraint of (4.22).

This maximum value is $CPD = 4/3$ and is considerably less than the maximum value of CPD when the QPSK is rotated. As a result the performance of this optimal 4 X-constellation is also inferior to the optimal rotated QPSK as shown in Fig. 4.2. The rotated QPSK performs better than X-constellation that maximizes SNR because the performance of CIOD dose not depend on average SNR as in the case of COD. Therefore we conclude that maximizing SNR dose not necessarily lead to optimal performance.

4.5 Discussion

In this chapter we have characterized all STBCs that maximize SNR. Using this characterization we showed that there exist codes other than STBCs form ODs that maximize SNR subject to a constraint on the signal constellation with the constraint being: the in-phase and quadrature components of all signals are of equal energy. We then showed that STBCs from CIODs maximize SNR under this constraint. Finally, we showed that optimal SNR codes need not have optimal performance in the sense of giving the largest coding gain.

Chapter 5

Double-symbol Decodable Designs

Space-Time block codes (STBC) from Quasi-Orthogonal designs (QOD) have been attracting wider attention due to their amenability to fast ML decoding (double-symbol decoding) and full-rank along with better performance than those obtained from Orthogonal Designs over quasi-static fading channels for both low and high SNRs [28]-[34]. But the QODs in literature are instances of *Double-symbol Decodable Designs (DSDD)* i.e. designs that allow double-symbol ML decoding and it is worthwhile to characterize all Double-symbol Decodable Designs, as Single-symbol Decodable Designs were characterized in Chapter 2 and whose instances were developed as GCIOD in Chapter 3.

In this chapter we first characterize all linear STBCs, that allow double-symbol ML decoding (not necessarily full-diversity) over quasi-static fading channels-calling them double symbol decodable designs (DSDD). The class DSDD includes QOD as a proper subclass. Among the DSDDs those that offer full-diversity are called Full-rank DSDDs (FDSDD). We show that the class of FDSDD consists of only (i) Generalized QOD (GQOD) and (ii) a class of non-GQODs called Generalized Quasi Complex Restricted Designs (GQCRD). Among GQCRDs we identify those that offer full-rank and call them FGQCRD. These full-rank GQCRDs along with GLCODs constitute the class of FDSDD.

We then upper bound the rates of square GQODs and show that rate 1 GQODs exist for 2 and 4 transmit antennas. Construction of maximal rate square GQODs are then presented to show that the square QODs of [33, 34] are optimal in terms of rate and coding gain. A relation is established between GQODs and GCIODs which leads to the construction of various high rate non-square QODs not obtainable by the constructions of [28]-[34]. The coding gain of GQODs is analyzed which leads to generalization of results of [33, 34].

Next we upper bound the rates of square GQGCRDs and show that rate 1 GQCRDs exist for 2, 4 and 8 transmit antennas. Construction of maximal rate square FGQCRDs are then presented. Simulation results for 8 FGQCRD is then presented to show that although the rate is 1, the performance is poorer that rate 3/4 QOD as the coding gain is less.

5.1 Introduction

Quasi-Orthogonal Designs leading to STBCs that admit Double-symbol decoding were introduced in [28, 29] and later studied in [30, 34] with improvements on the achievable diversity. The designs of [33, 34] are constructed as follows: Let $G_N(x_0, x_1, \cdots, x_{K-1})$ be a $p \times N$ OD then, the QOD $Q_{2N}(x_0, x_1, \cdots, x_{2K-1})$ of size $2p \times 2N$ is constructed as

$$Q_{2N} = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \tag{5.1}$$

where $A = G_N(x_0, x_1, \cdots, x_{K-1})$ and $B = G_N(x_K, x_{K+1}, \cdots, x_{2K-1})$. The variables $x_i, \quad i = 0, 1, \cdots, K-1$ take values from a complex signal set $\mathcal{A}$ and the variables $x_{K+i}, \quad i = 0, 1, \cdots, K-1$ from a rotated version of $\mathcal{A}$. By a proper choice of the angle of rotation one gets a full-rank code with rate K/p that is double-symbol decodable. The construction of QOD in [28] is

$$Q_{2N} = \begin{bmatrix} A & B \\ -B^* & A^* \end{bmatrix} \tag{5.2}$$

Where A^* stands for the matrix A with all entries replaced by their complex conjugates. The construction in [30, 31] also leads to STBCs with similar characteristics as those [33, 34] and we call these also QODs. In-fact these designs are not unique and many such constructions can be presented. For example consider the following construction:

A new construction of a QOD:

Let A, B be as defined in the construction leading to (5.1), then the design

$$Q_{2N} = \begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix} \tag{5.3}$$

is also a QOD, with the additional property that $Q_{2N}^{\mathcal{H}} Q_{2N}$ is diagonal.

Clearly, there is a need for systematic consideration of all such STBCs. Also, in contrast to ODs, the primary question of the maximal rate of QODs was left unanswered even for square QODs. In addition as in the case of ODs, QODs are a class of double-symbol decodable designs and a complete characterization of such codes is very useful. This chapter intends to answer these shortcomings. Recollect from Chapter 1 that:

Definition 5.1.1 (Double-symbol Decodable (DSD) STBC). A double-symbol decodable STBC of rate K/L in K complex indeterminates $x_k = x_{kI} + \mathbf{j}x_{kQ}$, $k = 0, \cdots, K-1$ is a linear STBC such that the ML decoding metric given by (1.9) can be written as a square of several terms each depending on at most two indeterminates only.

5.2 Double-symbol Decodable Designs

In this section we characterize all STBCs that allow double-symbol ML decoding in Subsection 5.2.1. The formal definition in terms of weight matrices and some examples of double-symbol decodable designs (DSDD) are then presented in sub-section 5.2.2.

5.2.1 Characterization of Double-symbol Decodable Linear STBCs

Consider the matrix channel model for quasi-static fading channel given in (1.6)

$$\mathbf{V} = \mathbf{SH} + \mathbf{N} \tag{5.4}$$

where $\mathbf{V} \in \mathbb{C}^{L\times M}$ ($\mathbb{C}$ denotes the complex field) is the received signal matrix, the transmission matrix (also referred as codeword matrix) $\mathbf{S} \in \mathbb{C}^{L\times N}$, $\mathbf{N} \in \mathbb{C}^{L\times M}$ is the noise matrix and $\mathbf{H} \in \mathbb{C}^{N\times M}$ defines the channel matrix, such that the element in the ith row and the jth column is h_{ij}.

Assuming that perfect channel state information (CSI) is available at the receiver, the decision rule for ML decoding is

$$M(\mathbf{S}) \triangleq \mathrm{tr}\left((\mathbf{V}-\mathbf{SH})^{\mathcal{H}}(\mathbf{V}-\mathbf{SH})\right). \tag{5.5}$$

This ML metric (5.5) results in exponential decoding complexity with the rate of transmission in bits/sec/Hz.

For a linear STBC with K variables, we are concerned about those STBCs for whom the ML metric (5.5) can be written as sum of several terms with each term involving only two distinct variables and hence double-symbol decodable. The following proposition characterizes all S, in terms of the weight matrices, that will allow double-symbol decoding.

Proposition 5.2.1. *For a complex linear STBC in $2K$ variables, $S = \sum_{k=0}^{2K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$, the ML metric, $M(S)$ defined in (5.5) decomposes as $M(S) = \sum_{k=0}^{K-1} M_k(x_k, x_{k+K}) + M_C$ where $M_C = -(K-1)\mathrm{tr}\left(V^{\mathcal{H}}V\right)$, iff*

$$A_k^{\mathcal{H}}A_l + A_l^{\mathcal{H}}A_k = 0 \quad \forall l \neq k, (k+2K)_{4K} \tag{5.6}$$

where $(k+2K)_{4K} = (k+2K) \mod 4K$.

Proof. Recollect from Theorem (2.3.2) we have that $S' = \sum_{k=0}^{K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$ is single-symbol decodable design iff

$$A_k^{\mathcal{H}}A_l + A_l^{\mathcal{H}}A_k = 0 \quad 0 \leq \forall l \neq k \leq 2K-1. \tag{5.7}$$

Let S be a double-symbol decodable design such that $x_k, x_{k+K}, k = 0, \cdots, K-1$ are decoded together (for every double-symbol decodable $\hat{S}$ we can obtain such a STBC by re-arranging variables). Now

$$M(S) = \mathrm{tr}\left(\mathbf{V}^{\mathcal{H}}\mathbf{V}\right) - \mathrm{tr}\left((S\mathbf{H})^{\mathcal{H}}\mathbf{V}\right) - \mathrm{tr}\left(\mathbf{V}^{\mathcal{H}}S\mathbf{H}\right) + \mathrm{tr}\left(S^{\mathcal{H}}S\mathbf{H}\mathbf{H}^{\mathcal{H}}\right).$$

Observe that $\mathrm{tr}\left(\mathbf{V}^{\mathcal{H}}\mathbf{V}\right)$ is independent of S. The next two terms in $M(S)$ are functions of $S, S^{\mathcal{H}}$ and hence linear in x_{kI}, x_{kQ}. In the last term,

$$\begin{aligned} S^{\mathcal{H}}S &= \sum_{k=0}^{2K-1}(A_{2k}^{\mathcal{H}}A_{2k}x_{kI}^2 + A_{2k+1}^{\mathcal{H}}A_{2k+1}x_{kQ}^2) + \sum_{k=0}^{2K-1}\sum_{l=k+1}^{2K-1}(A_{2k}^{\mathcal{H}}A_{2l} + A_{2l}^{\mathcal{H}}A_{2k})x_{kI}x_{lI} \\ &+ \sum_{k=0}^{2K-1}\sum_{l=k+1}^{2K-1}(A_{2k+1}^{\mathcal{H}}A_{2l+1} + A_{2l+1}^{\mathcal{H}}A_{2k+1})x_{kQ}x_{lQ} \\ &+ \sum_{k=0}^{2K-1}\sum_{l=0}^{2K-1}(A_{2k}^{\mathcal{H}}A_{2l+1} + A_{2l+1}^{\mathcal{H}}A_{2k})x_{kI}x_{lQ}. \end{aligned} \tag{5.8}$$

(a) Proof for the "if part": If (5.6) is satisfied then (5.8) reduces to

$$\begin{aligned} S^{\mathcal{H}}S &= \sum_{k=0}^{K-1}\Big\{A_{2k}^{\mathcal{H}}A_{2k}x_{kI}^2 + A_{2(k+K)}^{\mathcal{H}}A_{2(k+K)}x_{k+KI}^2 + A_{2k+1}^{\mathcal{H}}A_{2k+1}x_{kQ}^2 \\ &\quad + A_{2(k+K)+1}^{\mathcal{H}}A_{2(k+K)+1}x_{k+KQ}^2 + \left(A_{2k}^{\mathcal{H}}A_{2(k+K)} + A_{2(k+K)}^{\mathcal{H}}A_{2k}\right)x_{kI}x_{k+KI} \\ &\quad \left(A_{2(k+K)+1}^{\mathcal{H}}A_{2k+1} + A_{2k+1}^{\mathcal{H}}A_{2(k+K)+1}\right)x_{k+KQ}x_{kQ}\Big\} \\ &= \sum_{k=0}^{K-1}T^{\mathcal{H}}T, \text{ where} \qquad (5.9) \\ T &= A_{2k}x_{kI} + A_{2k+1}x_{kQ} + A_{2(k+K)}x_{k+KI} + A_{2(k+K)+1}x_{k+KQ} \qquad (5.10) \end{aligned}$$

and using linearity of the trace operator, $M(S)$ can be written as

$$\begin{aligned} M(S) &= \mathrm{tr}\left(\mathbf{V}^{\mathcal{H}}\mathbf{V}\right) - \sum_{k=0}^{K-1}\left\{\mathrm{tr}\left((T\mathbf{H})^{\mathcal{H}}\mathbf{V}\right) - \mathrm{tr}\left(\mathbf{V}^{\mathcal{H}}T\mathbf{H}\right)\right) + \mathrm{tr}\left(T^{\mathcal{H}}T\mathbf{H}\mathbf{H}^{\mathcal{H}}\right) \\ &= \sum_{k}\underbrace{\|\mathbf{V} - T\mathbf{H}\|^2}_{M_k(x_k,x_{k+K})} + M_C \end{aligned} \tag{5.11}$$

where $M_C = -(K-1)\mathrm{tr}\left(V^{\mathcal{H}}V\right)$ and $\|.\|$ denotes the Frobenius norm.

(b) Proof for the "only if part": If (5.6) is not satisfied for any $A_{k_1}, A_{l_1}, k_1 \neq l_1$ then

$$M(S) = \sum_k |\mathbf{V} - T\mathbf{H}|^2 + \mathrm{tr}\left((A_{k_1}^{\mathcal{H}} A_{l_1} + A_{l_1}^{\mathcal{H}} A_{k_1})\mathbf{H}^{\mathcal{H}}\mathbf{H}\right) y + M_C$$

$$\text{where } y = \begin{cases} x_{k_1/2I} x_{l_1/2I} & \text{if both } k_1, l_1 \text{ are even} \\ x_{(k_1-1)/2Q} x_{(l_1-1)/2Q} & \text{if both } k_1, l_1 \text{ are odd} \\ x_{(k_1-1)/2Q} x_{l_1/2I} & \text{if } k_1 \text{ odd, } l_1 \text{ even.} \end{cases}$$

Now, from the above it is clear that $M(S)$ can not be decomposed into terms involving only two variables. □

Remark 5.2.1. Observe that for a SDD $A_k^{\mathcal{H}} A_l + A_l^{\mathcal{H}} A_k = 0 \quad \forall l \neq k$ and hence (5.6) is satisfied. In other words, a SDD is also double-symbol decodable, trivially. Similarly, when $A_k = c_k A_{k+2K}, k = 0, \cdots, 2K-1$ in a DSDD, $S = \sum_{k=0}^{2K-1} x_{kI} A_{2k} + x_{kQ} A_{2k+1}$, then $S = \sum_{k=0}^{K-1} y_{kI} A_{2k} + y_{kQ} A_{2k+1}$ is a SDD where $y_{kI} = x_{kI} + c_k x_{k+KI}$ and $y_{kQ} = x_{kQ} + c_k x_{k+KQ}$. In other words S is actually a SDD.

5.2.2 Definition and examples of DSDD

Observe from remark 5.2.1 that a SDD is also trivially a DSDD and the Definition 5.1.1 of DSDD needs to be sharpened to be meaningful. In this section we formally define double-symbol decodable designs so as to avoid trivial cases discussed in remark 5.2.1. We have,

Definition 5.2.1 (Double-symbol Decodable Designs (DSDD)). A Double-symbol Decodable Design is a complex linear STBC, $S = \sum_{k=0}^{2K-1} x_{kI} A_{2k} + x_{kQ} A_{2k+1}$, whose weight matrices, $\{A_0, \cdots, A_{2K-1}\}$ satisfy

$$A_k^{\mathcal{H}} A_l + A_l^{\mathcal{H}} A_k = 0, \quad \forall l \neq k, (k+2K)_{4K} \tag{5.12}$$

$$A_k^{\mathcal{H}} A_{k+2K} + A_{k+2K}^{\mathcal{H}} A_k \neq 0, \quad 0 \leq k \leq 2K-1 \tag{5.13}$$

$$A_k - c_k A_{k+2K} \neq 0, \quad k = 0, \cdots, 2K-1 \tag{5.14}$$

where $(k+2K)_{4K} = (k+2K) \mod 4K$; and $\{c_k\}$ are constants.

Remark 5.2.2. In Definition 5.2.1 we have not only excluded trivially double-symbol decodable codes like SDD but also those STBC in which some variables are single-symbol decodable and some are double-symbol decodable. It turns out that all such codes are derived from a DSDD and hence Definition 5.2.1 is appropriate. Such codes will be considered towards the end of this chapter for completeness.

Examples of DSDD are the STBCs from QOD like,

Example 5.2.1. *consider the rate 1 QOD for $N = K = 2$*

$$S(x_0, x_1) = \begin{bmatrix} x_0 & x_1 \\ x_1 & x_0 \end{bmatrix} \tag{5.15}$$

with the corresponding weight matrices

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & \mathbf{j} \\ -\mathbf{j} & 0 \end{bmatrix};$$

and some other STBCs not covered by STBCs from QOD like the rate 1, QCIOD for $N = L = 2$

Example 5.2.2. *Consider*

$$S(x_0, x_1) = \begin{bmatrix} x_{0I} & \mathbf{j}x_{1Q} \\ x_{1I} & \mathbf{j}x_{0Q} \end{bmatrix}. \tag{5.16}$$

The corresponding weight matrices are given by

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{j} \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & \mathbf{j} \\ 0 & 0 \end{bmatrix}$$

such that

$$S^{\mathcal{H}}S = \begin{bmatrix} x_{0I}^2 + x_{1I}^2 & \mathbf{j}(x_{0I}x_{1Q} + x_{1I}x_{0Q}) \\ -\mathbf{j}(x_{0I}x_{1Q} + x_{1I}x_{0Q}) & x_{0Q}^2 + x_{1Q}^2 \end{bmatrix}$$

and det $\left(S^{\mathcal{H}}S\right) = (x_{0I}x_{0Q} - x_{1I}x_{1Q})^2$. ***Observe that S is not a real orthogonal design.*** *In fact the coding gain is linked to the CPD of x_0, x_1.*

Similarly

Example 5.2.3. *the rate 1 QCIOD for $N = 8$*

$$\mathcal{S}_8(x_0,x_1,x_2,x_3,x_4,x_5,x_6,x_7) = \begin{bmatrix} \tilde{x}_o & \tilde{x}_1 & 0 & 0 & \tilde{x}_4 & \tilde{x}_5 & 0 & 0 \\ -\tilde{x}_1^* & \tilde{x}_0^* & 0 & 0 & -\tilde{x}_5^* & \tilde{x}_4^* & 0 & 0 \\ 0 & 0 & \tilde{x}_2 & \tilde{x}_3 & 0 & 0 & \tilde{x}_6 & \tilde{x}_7 \\ 0 & 0 & -\tilde{x}_3^* & \tilde{x}_2^* & 0 & 0 & -\tilde{x}_7^* & \tilde{x}_6^* \\ \tilde{x}_4 & \tilde{x}_5 & 0 & 0 & \tilde{x}_0 & \tilde{x}_1 & 0 & 0 \\ -\tilde{x}_5^* & \tilde{x}_4^* & 0 & 0 & -\tilde{x}_1^* & \tilde{x}_0^* & 0 & 0 \\ 0 & 0 & \tilde{x}_6 & \tilde{x}_7 & 0 & 0 & \tilde{x}_2 & \tilde{x}_3 \\ 0 & 0 & -\tilde{x}_7^* & \tilde{x}_6^* & 0 & 0 & -\tilde{x}_3^* & \tilde{x}_2^* \end{bmatrix} \quad (5.17)$$

is a DSDD.

It is easily seen that these codes are not covered by QODs and satisfy the requirements of the above theorem and hence are double-symbol decodable.

5.3 Full-rank DSDD

In this section we identify all full-rank designs with in the class of DSDD, i.e., characterize the class FDSDD; subsection 5.3.1. The formal definition of FDSDD is then presented in Subsection 5.3.2 and the classification of FDSDD is presented in 5.3.3.

5.3.1 Characterization of Full-rank DSDD

In this sub-section we identify all full-rank designs with in the class of DSDD. Towards this end, we have

Proposition 5.3.1. *A square double-symbol decodable design,* $S = \sum_{k=0}^{2K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$*, exists if and only if there exists a square double-symbol decodable design,* $\hat{S} = \sum_{k=0}^{2K-1} x_{kI}\hat{A}_{2k} + x_{kQ}\hat{A}_{2k+1}$ *such that*

$$\begin{aligned}
\hat{A}_k^{\mathcal{H}}\hat{A}_l + \hat{A}_l^{\mathcal{H}}\hat{A}_k &= 0, \quad \forall l \neq k, (k+2K)_{4K}, \\
\hat{A}_k^{\mathcal{H}}\hat{A}_{k+2K} + \hat{A}_{k+2K}^{\mathcal{H}}\hat{A}_k &\neq 0, \quad 0 \leq k \leq 2K-1, \\
\hat{A}_k - c_k\hat{A}_{k+2K} &\neq 0, \quad k = 0, \cdots, 2K-1, \\
\hat{A}_k^{\mathcal{H}}\hat{A}_k &= \mathcal{D}_k, \quad \forall k
\end{aligned}$$

where $(k+2K)_{4K} = (k+2K) \mod 4K$*,* c_k *are constants* $\mathcal{D}_k, \mathcal{D}_{k,k+2K}$ *are diagonal matrices.*

Proof. Consider a DSDD, $S = \sum_{k=0}^{2K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$. Then the family of Hermitian matrices $\{A_0^{\mathcal{H}}A_0, \cdots, A_{2K-1}^{\mathcal{H}}A_{2K-1}\}$ is a family of commuting matrices by Theorem 2.4.2 and are diagonalizable by a unitary matrix U. Similarly, the family of Hermitian matrices $\{A_1^{\mathcal{H}}A_1, \cdots, A_{2K-1}^{\mathcal{H}}A_{2K-1}, A_{2K}^{\mathcal{H}}A_{2K}\}$ is a family of commuting matrices by Theorem 2.4.2 and are diagonalizable by the same unitary matrix U. It follows that $\{A_0^{\mathcal{H}}A_0, A_1^{\mathcal{H}}A_1, \cdots, A_{2K-1}^{\mathcal{H}}A_{2K-1}, A_{2K}^{\mathcal{H}}A_{2K}\}$ is a family of diagonalizable matrices. Proceeding, in the same way, $\{A_0^{\mathcal{H}}A_0, \cdots, A_{4K-1}^{\mathcal{H}}A_{4K-1}\}$ is a family of diagonalizable matrices. Define

$$\hat{S} = SU^{\mathcal{H}} = \sum_{k=0}^{2K-1} x_{kI}\hat{A}_{2k} + x_{kQ}\hat{A}_{2k+1} \text{ where } \hat{A}_k = A_kU^{\mathcal{H}}, \tag{5.18}$$

then $\hat{S}$ satisfies the requirements of the proof. For the converse, given $\hat{S}$, S can be obtained by defining $S = \hat{S}U$; completing the proof. □

Therefore, without loss of generality we can assume that for square DSDD, S is such that the weight matrices

$$
\begin{aligned}
A_k^{\mathcal{H}} A_l + A_l^{\mathcal{H}} A_k &= 0, \quad \forall l \neq k, (k+2K)_{4K} && (5.19)\\
A_k^{\mathcal{H}} A_{k+2K} + A_{k+2K}^{\mathcal{H}} A_k &\neq 0, \quad 0 \leq k \leq 2K-1 && (5.20)\\
A_k - c_k A_{k+2K} &\neq 0, \quad k = 0, \cdots, 2K-1, && (5.21)\\
A_k^{\mathcal{H}} A_k &= \mathcal{D}_k, \quad \forall k && (5.22)
\end{aligned}
$$

We conjecture (as in Chapter 2) that this is true for non-square designs also. However the characterization proceeds independent of this requirement. The following proposition gives a necessary condition for DSDDs to achieve full-diversity.

Proposition 5.3.2. *A double-symbol decodable design, $S = \sum_{k=0}^{2K-1} x_{kI} A_{2k} + x_{kQ} A_{2k+1}$, whose weight matrices A_k satisfy (5.19) achieves full-diversity only if $A_{2k}^{\mathcal{H}} A_{2k} + A_{2k+1}^{\mathcal{H}} A_{2k+1}$ is full-rank for all $k = 0, 1, \cdots, 2K-1$.*

Proof. The proof is by contradiction. First observe that $A_k^{\mathcal{H}} A_k$ is positive semidefinite. Suppose that for some l $A_{2l}^{\mathcal{H}} A_{2l} + A_{2l+1}^{\mathcal{H}} A_{2k+l}, 0 \leq l \leq 2K-1$ is not full-rank. Then for any two transmission matrices $S,\ \hat{S}$ that differ only in x_l, the difference matrix $B(S, \hat{S}) = S - \hat{S}$, will not be full-rank as $B^{\mathcal{H}}(S, \hat{S}) B(S, \hat{S}) = A_{2l}^{\mathcal{H}} A_{2l} (x_{lI} - \hat{x}_{lI})^2 + A_{2l+1}^{\mathcal{H}} A_{2k+l} (x_{lQ} - \hat{x}_{lQ})^2$ is not full-rank. □

5.3.2 Definition of FDSDD

Although, proposition 5.3.2 gives the necessary condition for full-rank STBCs, it characterizes all full-rank double-symbol decodable STBCs, as sufficient condition for full-rank is obtained by putting additional restrictions on the weight matrices or the signal set (Theorem 5.3.3). Hence all STBCs that satisfy Proposition 5.3.2 are full-rank under some restriction on the signal set. We have,

Definition 5.3.1 (Full-rank Double-symbol Decodable Designs (FDSDD)). A Full-rank Double-symbol Decodable Design is a DSDD such that

$$
A_{2k}^{\mathcal{H}} A_{2k} + A_{2k+1}^{\mathcal{H}} A_{2k+1}, \qquad 0 \leq k \leq 2K-1. \tag{5.23}
$$

is full-rank.

Examples of FDSDD are the QODs and the STBCs of examples 5.2.2,5.2.3.

Remark 5.3.1. Observe that for square FDSDD, it is sufficient to consider S such that $S^{\mathcal{H}}S = \mathcal{D}$, where $\mathcal{D}$ is diagonal.

Towards obtaining a sufficient condition for full-diversity, we classify FDSDD into classes that require different restrictions on the signal set for full-rank. We have

Proposition 5.3.3. *A linear STBC, $S = \sum_{k=0}^{2K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$ where x_k take values from a signal set $\mathcal{A}, \forall k$, satisfying the necessary condition of proposition 5.3.2 achieves full-diversity only if*

1. *either $A_k^{\mathcal{H}}A_k$ is of full-rank for all k*
2. *or the CPD of $\mathcal{A} \neq 0$*

Proof. Let $S, \hat{S}$ be such that they differ in the k-th variable only. We have $B^{\mathcal{H}}(S,\hat{S})B(S,\hat{S})$ $= \sum_{k=0}^{K-1} A_{2k}^{\mathcal{H}}A_{2k}\,(x_{kI}-\hat{x}_{kI})^2 + A_{2k+1}^{\mathcal{H}}A_{2k+1}(x_{kQ}-\hat{x}_{kQ})^2$. Since $A_k^{\mathcal{H}}A_k$ are positive semidefinite, it follows that B is full-rank either if $A_k^{\mathcal{H}}A_k, k < 2K$ are all full-rank for or $(x_{kI}-\hat{x}_{kI})^2 \neq 0$ and $(x_{kQ}-\hat{x}_{kQ})^2 \neq 0$ implying that the CPD is nonzero. □

Note that the necessary condition 1) is an additional condition on the weight matrices whereas the necessary condition 2) is a restriction on the signal set $\mathcal{A}$ and not on the weight matrices A_k. Proposition 5.3.3 essentially divides FDSDD into two classes, each of which will be treated separately in the following sections.

An important consequence of Proposition 5.3.3 is that there can exist designs that are not covered by QODs offering full-diversity and double-symbol decoding, provided the associated signal set has non-zero CPD. It is important to note that whenever we have a signal set with CPD equal to zero, by appropriately rotating it we can end with a signal set with non-zero CPD. Indeed, only for a finite set of angles of rotation we will again end up with CPD equal to zero. So, the requirement of non-zero CPD for a signal set is not at all restrictive in real sense.

5.3.3 Classification of FDSDD

In this sub-section we formally classify FSDD. When $A_k^{\mathcal{H}} A_k$ is full-rank for all k, corresponding to Proposition 5.3.3 1., we have Generalized QOD (GQOD) of which QOD is a subclass. Formally,

Definition 5.3.2 (Generalized Quasi Orthogonal Design (GQOD)). A generalized quasi orthogonal design is a FDSDD such that $A_k^{\mathcal{H}} A_k$ is full-rank for all $k = 0, \cdots, 4K-1$.

Observe that the QODs of Jafarkhani [28], Trikkonen and Hottinen [29], Sharma and Papadias [30, 31] and Su and Xia [33, 34] are all GQODs. The FDSDDs that are not GQODs are such that $A_{2k}^{\mathcal{H}} A_{2k}$ and/or $A_{2k+1}^{\mathcal{H}} A_{2k+1}$ is not full-rank for at least one k.

We call such FDSDD codes Generalized Quasi Restricted Design (**GQRD**) in symphony with the RFSDD codes that are FSDD. Formally,

Definition 5.3.3 (Generalized Quasi Restricted Design (GQRD)). A generalized quasi restricted design is a FDSDD such that $A_k^{\mathcal{H}} A_k$ is not full-rank for at least one k where $k = 0, \cdots, 4K-1$. When a GQRD satisfying the conditions for full-rank is used along with a signal set with non-zero CPD we simply refer the design as Full-rank GQRD (**FGQRD**).

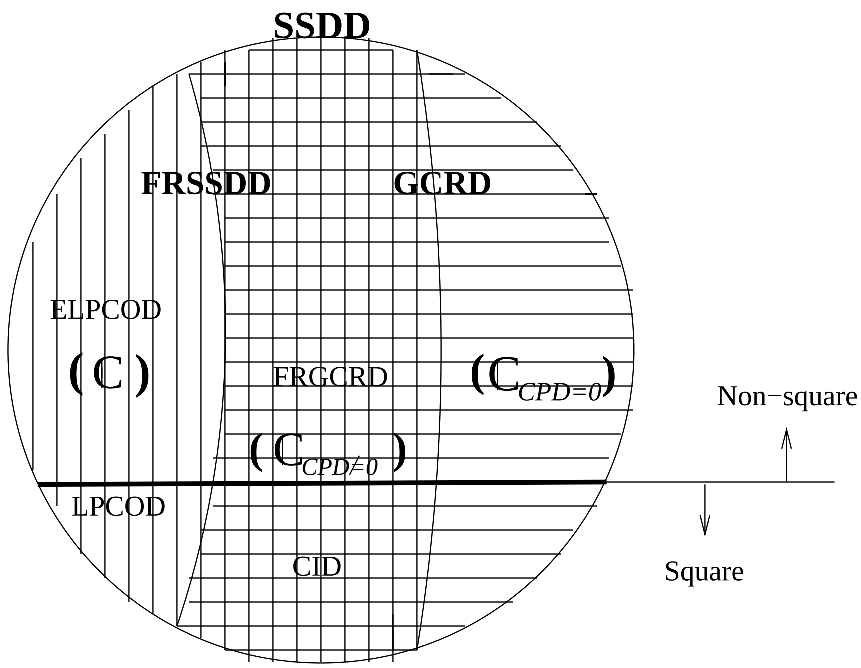
Figure 5.1 shows all the classes discussed so far, viz., DSDD, FDSDD, GQRD, GQOD, FGQRD. Observe that vertical hatches denote FDSDD, square hatches denote FGQRD and horizontal hatches denote DSDDs that are not GQODs. In Section 5.4 we analyze GQODs and in section 5.5 we focus on the square FGQRDs and show that although we gain in rate by using FGQRDs, there is a loss of performance as compared to GQODs.

5.3.4 Construction of single-symbol decodable designs from QODs

In this subsection we construct single-symbol decodable designs from Generalized Quasi-Orthogonal designs (GQODs) defined in (5.3.2).

Let $S(x_0, \cdots, x_{2K-1}) = \sum_{k=0}^{2K-1} x_{kI} A_{2k} + x_{kQ} A_{2k+1}$ be a GQOD defined in (5.3.2). Then

$$\hat{S}(x_0, \cdots, x_{2K-1}) = \sum_{k=0}^{K-1} \left\{ x_{kI} A_{2k} + x_{(k+K)I} A_{2k+1} \right\} + \sum_{k=0}^{K-1} \left\{ x_{kQ} A_{2(k+K)} + x_{(k+K)Q} A_{2(k+K)+1} \right\}$$



FRSSDD = ELPCOD ∪ FRGCRD

FRGCRD ⊂ **GCRD**

GLPCOD ⊂ **ELPCOD**

Figure 5.1: The classes of double-symbol decodable (DSDD) codes.

is single-symbol decodable. An example of such a code is given in example 2.3.1. As discussed therein, these codes have maximum coding gain when the indeterminates take value from a real constellation.

5.4 Generalized Quasi Orthogonal Designs

In this section we first derive the maximal rates of square QODs, and then present a sufficient condition for a class of GQODs to obtain full-rank. We then link the existence of GQODs to GCIODs and then use this to present various high rate GQODs. The coding gain of GQODs is then considered and the MMI of QODs analyzed. A comparison of GQODs and GCIODs is then presented.

5.4.1 Maximal rates of square GQODs

In this sub-section we find the maximal rates of square GQODs. Henceforth we assume $L = N$. Since any maximal rate square LCOD is equivalent to the LCOD of (2.9) under unitary transformation and/or change of variables [15, Theorem 2], we assume without loss of generality that the maximal rate square LCOD are of the form (2.9). We have

Theorem 5.4.1. *The maximal rate of square GQOD of size N is bounded as $\mathcal{R} \leq \frac{H(N)-2}{N} = \frac{2a}{2^a b}, N = 2^a b,\ b$ odd.*

Proof. Define $S_1 = \{A_0, A_1, \cdots, A_{2K-1}\}$ and $S_2 = \{A_{2K}, A_{2K+1}, \cdots, A_{4K-1}\}$. Following (5.28)-(5.31), both the sets S_1, S_2 independently form a set of weight matrices for ODs of rate K/N. Invoking the Hurwitz Theorem (Theorem 2.2.1) we have $K \leq \frac{H(N)}{2}$. Assume that $|S_1| = |S_2| = 2K = H(N)$.

Step 1: *Claim: The elements of S_2 are of the form UA_k or $A_kV, k = 0, \cdots, 2K-1$ where U, V are unitary.*

Observe that the sets S_1, S_2 are unitarily equivalent [15, Theorem 2]. Therefore the elements of S_2 can be written in terms of elements of S_1 as $A_k = UA_{P(k-2K)}V, k = 2K, \cdots, 4K-1$ where U, V are unitary matrices and P is a permutation over the set of indices $0, \cdots, 2K-1$. Applying (5.28) to $A_0, A_l, l > 2K, 0 \leq P(l-2K) = i \leq 2K-1$ we have

$$A_0^{\mathcal{H}}UA_iV + V^{\mathcal{H}}A_i^{\mathcal{H}}U^{\mathcal{H}}A_0 = 0$$

which implies that the set $\{U^{\mathcal{H}}A_0, A_1V, \cdots, A_{2K-1}V\}$ are the weight matrices of a LCOD and $U^{\mathcal{H}}A_0 = A_0V$. Similarly, $U^{\mathcal{H}}A_i = A_iV, i = 1, \cdots, 2K-1$. Consequently, $S_1 = S_2$ and (5.30) is not satisfied. Therefore, the elements of S_2 are of the form UA_k or $A_kV, k = 0, \cdots, 2K-1$. In what follows we assume that $S_2 = \{A_{2K}, \cdots, A_{4K-1}\} = \{UA_{P(i)}, i = 0, \cdots, 2k-1\}$. The other case can be similarly treated.

Step 2: *Claim:* $U = \pm A_pA_k^{\mathcal{H}}$*, for all* $0 \le p \ne k \le 2K-1$.

Applying (5.28) to $A_p, A_q, q > 2K, 0 \le P(q-2K) = i; p, i \le 2K-1$ we have

$$A_p^{\mathcal{H}}UA_i + A_i^{\mathcal{H}}U^{\mathcal{H}}A_p = 0. \tag{5.24}$$

But this is only possible if

$$U^{\mathcal{H}}A_p = \pm A_k, 0 \le i \ne k \ne p \le 2K-1 \tag{5.25}$$

$$\Rightarrow \quad U = \pm A_pA_k^{\mathcal{H}}, 0 \le i \ne k \ne p \le 2K-1. \tag{5.26}$$

Step 3: *Claim:* U *is anti-Hermitian.*

Since S is a square QOD $A_p^{\mathcal{H}}A_k + A_k^{\mathcal{H}}A_p = 0, p \ne k$ is equivalent to $A_pA_k^{\mathcal{H}} + A_kA_p^{\mathcal{H}} = 0, p \ne k$. Substituting in (5.26). We have $U^{\mathcal{H}} = -U$.

Step 4: *claim:* $K \le H(N) - 2$.

Applying (5.28) to $A_{2K}, A_1, P(2K) = i$ we have

$$A_1^{\mathcal{H}}UA_i + A_i^{\mathcal{H}}U^{\mathcal{H}}A_2 = 0. \tag{5.27}$$

Implying $U^{\mathcal{H}}A_1 = -UA_1 = \pm A_r \ne A_i$. This implies that $\pm A_r$ is in both S_1, S_2 which is not possible due to (5.30). Necessarily $A_r \notin S_1$ and $|S_1| = H(N) - 1$. But $|S_1|$ is even by definition 2 and $H(N)$ is even for any N, it follows that $|S_1| = |S_2| \le H(N) - 2$.

□

The following theorem shows that the upper bound of Theorem 2 is achievable.

Theorem 5.4.2. *The maximal rate, $\mathcal{R}$ of square QOD is* $\mathcal{R} = \frac{H(N)-2}{N} = \frac{2a}{2^a b}$.

Proof. By Theorem 5.4.1, $\mathcal{R} \leq \frac{H(N)-2}{N} = \frac{2a}{2^a b}$. The rate of QOD in Construction (5.1) and in (5.3) when A, B are maximal rate CODs is

$$\mathcal{R} \leq \frac{H\left(N/2\right)}{N} = \frac{2a}{2^a b}$$

which achieves the bound completing the proof. □

Remark 5.4.1. Observe that the maximal rates of square CIODs (RFSDDs), given by Theorem 2.5.2, is equal to that of square GQODs, given by Theorem 5.4.2 for the same number of transmit antennas. This suggests that there might be a possible connection between CIODs and GQODs.

5.4.2 Sufficient condition for full-rank GQOD

In this sub-section we derive the sufficient condition for a class of GQODs. However we conjecture that this class consists of all GQODs.

Theorem 5.4.3. *A $N \times N$ GQOD $S = \sum_{k=0}^{2K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$ exists iff there exists a GQOD of same size and rate, $\hat{S} = \sum_{k=0}^{2K-1} x_{kI}\hat{A}_{2k} + x_{kQ}\hat{A}_{2k+1}$ such that $\hat{A}_k^{\mathcal{H}}\hat{A}_k = I_N$.*

Proof. Consider a square GQOD, $S = \sum_{k=0}^{2K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$. The weight matrices satisfy,

$$A_k^{\mathcal{H}}A_l + A_l^{\mathcal{H}}A_k = 0, \quad \forall l \neq k, (k+2K)_{4K} \tag{5.28}$$

$$A_k^{\mathcal{H}}A_{k+2K} + A_{k+2K}^{\mathcal{H}}A_k \neq 0, \quad 0 \leq k \leq 2K-1 \tag{5.29}$$

$$A_k - c_k A_{k+2K} \neq 0, \quad k = 0, \cdots, 2K-1, \tag{5.30}$$

$$A_k^{\mathcal{H}}A_k = \mathcal{D}_k, \quad \forall k. \tag{5.31}$$

Observe that $\mathcal{D}_k$ is of full-rank for all k. Define $\hat{A}_k = A_k \hat{\mathcal{D}}_k^{-1/2}$. Then the matrices $\hat{A}_k$ satisfy

$$\begin{aligned}
\hat{A}_k^{\mathcal{H}}\hat{A}_l + \hat{A}_l^{\mathcal{H}}\hat{A}_k &= 0, \quad \forall l \neq k, (k+2K)_{4K}, && (5.32)\\
\hat{A}_k^{\mathcal{H}}\hat{A}_{k+2K} + \hat{A}_{k+2K}^{\mathcal{H}}\hat{A}_k &\neq 0, \quad 0 \leq k \leq 2K-1, && (5.33)\\
\hat{A}_k - c_k\hat{A}_{k+2K} &\neq 0, \quad k = 0, \cdots, 2K-1, && (5.34)\\
\hat{A}_k^{\mathcal{H}}\hat{A}_k &= I_N, \quad \forall k && (5.35)
\end{aligned}$$

where in proving (5.32) we have made use of the results of Subsection 2.2.1. It follows that $\hat{S} = \sum_{k=0}^{2K-1} x_{kI}\hat{A}_{2k} + x_{kQ}\hat{A}_{2k+1}$ is a GQOD having the claimed property. □

Next we present a conjecture regarding GQODs which is satisfied by all known GQODs.

Conjecture 5.4.4. *A $L \times N$ GQOD $S = \sum_{k=0}^{2K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$ exists iff there exists a GQOD of same size and rate, $\hat{S} = \sum_{k=0}^{2K-1} x_{kI}\hat{A}_{2k} + x_{kQ}\hat{A}_{2k+1}$ such that the entries of $\hat{S}^{\mathcal{H}}\hat{S}$ are linear sum of $|x_k \pm x_{k+K}|^2$ or $|x_k|^2, |x_{k+K}|^2$ with strictly positive real coefficients.*

Remark 5.4.2. From Theorem 5.4.3 it for every GQOD S there exists $\hat{S}$ whose weight matrices satisfy defined in (5.32)-5.35). In addition the entries of $\hat{\mathcal{D}}_{k,k+2K} 0 \leq k \leq 2K-1$ are $0, \pm 2$. This can be proved by observing that the columns of the matrix

$$\hat{A}_{k,K} = \begin{pmatrix} A_k & A_{k+2K} \\ A_{k+2K} & A_k \end{pmatrix}$$

are of length 2 and in addition the i-th column of $\begin{pmatrix} A_k \\ A_{k+2K} \end{pmatrix}$ is orthogonal to all except i-th column of $\begin{pmatrix} A_{k+2K} \\ A_k \end{pmatrix}$. Since $\hat{\mathcal{D}}_{k,k+2K} 0 \leq k \leq 2K-1$ need not be full-rank we have, that the entries of $\hat{S}^{\mathcal{H}}\hat{S}$ are linear sum of $|x_k \pm x_{k+K}|^2$ or $|x_k|^2, |x_{k+K}|^2$ with strictly positive real coefficients. Therefore this conjecture seems to hold for square designs. A similar argument holds for non-square designs also.

Before presenting the sufficient condition for GQODs (we assume that conjecture 5.4.4 holds) to achieve full-rank we define

Definition 5.4.1 (Minimum ζ-distance (MZD) [33, 34]). The minimum ζ-distance between two signal constellations $\mathcal{A}$ and $\mathcal{B}$ is

$$d_{min,\zeta}(\mathcal{A},\mathcal{B}) = \min_{(s_1,s_2)\neq(\hat{s}_1,\hat{s}_2)} |(s_1-\hat{s}_1)^2-(s_2-\hat{s}_2)^2|^{1/2}.$$

We have,

Theorem 5.4.5. *The GQODs achieve full-diversity iff the Minimum ζ-distances, $d_{min,\zeta}$ $(\mathcal{A}_i,\mathcal{A}_{i+K}), i=0,\cdots,K-1$ is non-zero where $\mathcal{A}_i$ is the signal constellation form which the variable x_i takes values.*

Proof. The proof is an immediate consequence of Corollary 5.4.4 and Definition 5.4.1. ☐

Su and Xia [33, 34] pointed out that such a pair of constellations can be obtained by rotating the signal constellation i.e. $\mathcal{A}_{i+K}=e^{\mathbf{j}\theta}\mathcal{A}_i$. It was also shown that for QOD the coding gain is equal to the Minimum ζ-distance. Maximization of MZD for the case when $\mathcal{A}_i=\mathcal{A}, i=0,\cdots,K-1$ and $\mathcal{A}$ is a square or triangular lattice was also done. We show in Subsection 5.4.4 that the coding gain of a class of GQOD is equal to the Minimum generalized ζ-distance.

5.4.3 GQODs from GCIODs

In this subsection we follow the intuition from Remark 5.4.1 and show that a GQOD exists if a GCIOD of same rate and same size exists. This result leads to construction of several high rate codes, which are not covered by the QODs of [28]-[34]. In particular we have a rate 1 GQOD for $N=3$, rate 6/7 GQODs for $N=5,6$ and so on as in the case of GCIODs in Chapter 3.

We have for square designs

Theorem 5.4.6. *A square GQOD of rate $2K/L$, size N exists iff exists there exits a square CIOD of same rate and size.*

Proof. Let S be a square GQOD. The set of $2K$ matrices, $S_1 = \{A_0, A_1, \cdots, A_{2K-1}$, form a Hurwitz family of matrices. By Theorem (5.4.1), $2K \leq 2a$. Accordingly S_1 can be assumed to be block diagonal as in Construction (5.1). Therefore

$$A_k = \begin{bmatrix} A_{1k} & 0 \\ 0 & A_{2k} \end{bmatrix}$$

and the sets of matrices satisfy $\{A_{1k}\}_{k=0}^{2K-1}, \{A_{2k}\}_{k=0}^{2K-1}$ are two Hurwitz families of matrices. Define

$$\hat{A}_{2k} = \begin{bmatrix} A_{1k} & 0 \\ 0 & 0 \end{bmatrix}, \hat{A}_{2k+1} = \begin{bmatrix} 0 & 0 \\ 0 & A_{2k} \end{bmatrix}, k = 0, \cdots, K-1$$

and

$$\hat{A}_{2k+1} = \begin{bmatrix} A_{1k} & 0 \\ 0 & 0 \end{bmatrix}, \hat{A}_{2k} = \begin{bmatrix} 0 & 0 \\ 0 & A_{2k} \end{bmatrix}, k = K, \cdots, 2K-1.$$

Then it follows that the design $\sum_{k=0}^{2K-1} x_{kI}\hat{A}_{2k} + x_{kQ}\hat{A}_{2k+1}$ is a CIOD of same rate and size as S.

Now given a CIOD, $\hat{S}$, then S can be obtained form it as

$$A_k = \hat{A}_{2k} + \hat{A}_{2k+1} = \begin{bmatrix} A_{1k} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & A_{2k} \end{bmatrix}, k = 0, \cdots, 2K-1$$

and

$$A_{k+2K} = \begin{bmatrix} 0 & A_{1k} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ A_{2k} & 0 \end{bmatrix}, k = 0, \cdots, 2K-1.$$

□

Next we present a weaker condition for non-square designs that is,

Theorem 5.4.7. *A non-square GQOD of rate $2K/L$, size N exists if there exits a square GCIOD of same rate and size.*

Proof. Consider a GCIOD given by Definition 3.1.1 as

$$S(x_0,\cdots,x_{K-1}) = \begin{bmatrix} \Theta_1(\tilde{x}_0,\cdots,\tilde{x}_{K/2-1}) & 0 \\ 0 & \Theta_2(\tilde{x}_{K/2},\cdots,\tilde{x}_{K-1}) \end{bmatrix} \quad (5.36)$$

where $\Theta_1(x_0,\cdots,x_{K/2-1}), \Theta_2(x_0,\cdots,x_{K/2-1})$ are GLCODs of size N_1, N_2 respectively such that$N_1+N_2 = N$, $\tilde{x}_i = \mathrm{Re}\{x_i\}+\mathbf{j}\,\mathrm{Im}\{x_{(i+K/2)_K}\}$ and where $(a)_K$ denotes $a \pmod K$. Then a simple check shows that the design

$$\begin{aligned} \hat{S}(x_0,\cdots,x_{K-1}) &= \begin{bmatrix} \Theta_1(x_0,x_1,\cdots,x_{K/2-1}) & 0 \\ 0 & \Theta_2(x_0,x_1,\cdots,x_{K/2-1}) \end{bmatrix} \\ &+ \begin{bmatrix} \Theta_1(x_{\frac{K}{2}},x_{\frac{K}{2}+1},\cdots,x_{K-1}) & 0 \\ 0 & -\Theta_2(x_{\frac{K}{2}},x_{\frac{K}{2}+1},\cdots,x_{K-1}) \end{bmatrix} \end{aligned} \quad (5.37)$$

is a GQOD of same rate and size as S. □

We can interchange the $+$ and $-$ sign between two variables x_k and $x_{k+K/2}$ in the two designs Θ_1, Θ_2 in (5.37) to obtain another GQOD.

Remark 5.4.3. Observe that Theorem 5.4.7 suggests that there might exist GQODs of rate higher than GCIODs. We conjecture that this might not be true as we could not find an example of any such GQOD and the `if` can be replaced by `iff` as and when we completely understand non-square GLCODs.

Theorem 5.4.7 leads to construction of various high rate GQODs which are not covered by known constructions of QOD.

In particular we have the rate 1 GQOD for $N = 3$ given by,

$$S = \begin{bmatrix} x_0+x_2 & x_1+x_3 & 0 \\ -(x_1+x_3)^* & (x_0+x_2)^* & 0 \\ 0 & 0 & x_0-x_2 \\ 0 & 0 & -(x_1-x_3)^* \end{bmatrix} \quad (5.38)$$

Similarly, modifying the Construction 3.1.3, we have a construction for GQOD which gives various high rate GQODs. We have

Construction 5.4.8. *Let Θ_1 be a GLCOD of size $L_1 \times N_1$, rate $r_1 = K_1/L_1$ in K_1 indeterminates $x_0, \cdots, x_{K_1-1}$ and similarly let Θ_2 be a GLCOD of size $L_2 \times N_2$, rate $r_2 = K_2/L_2$ in K_2 indeterminates $y_0, \cdots, y_{K_2-1}$. Let $K = \text{lcm}(K_1, K_2)$, $n_1 = K/K_1$ and $n_2 = K/K_2$. Construct*

$$\hat{\Theta}_1 = \begin{bmatrix} \Theta_1(x_0, x_1, \cdots, x_{K_1-1}) \\ \Theta_1(x_{K_1}, x_{K_1+1}, \cdots, x_{2K_1-1}) \\ \Theta_1(x_{2K_1}, x_{K_1+1}, \cdots, x_{2K_1-1}) \\ \vdots \\ \Theta_1(x_{(n_1-1)K_1}, x_{(n_1-1)K_1+1}, \cdots, x_{n_1K_1-1}) \end{bmatrix} \tag{5.39}$$

and

$$\hat{\Theta}_2 = \begin{bmatrix} \Theta_2(y_0, y_1, \cdots, y_{K_2-1}) \\ \Theta_2(y_{K_2}, y_{K_2+1}, \cdots, y_{2K_2-1}) \\ \Theta_2(y_{2K_2}, y_{K_2+1}, \cdots, y_{2K_2-1}) \\ \vdots \\ \Theta_2(y_{(n_2-1)K_2}, y_{(n_2-1)K_2+1}, \cdots, y_{n_2K_2-1}) \end{bmatrix}. \tag{5.40}$$

Then $\hat{\Theta}_1 \in \mathbb{C}^{n_1L_1 \times N_1}$ is a GLCOD in indeterminates $x_0, x_1, \cdots, x_{K-1}$ and $\hat{\Theta}_2 \in \mathbb{C}^{n_2L_2 \times N_2}$ is a GLCOD in indeterminates $y_0, y_1, \cdots, y_{K-1}$. The design

$$\begin{aligned} S(x_0, \cdots, x_{K-1}) &= \begin{bmatrix} \hat{\Theta}_1(x_0, x_1, \cdots, x_{K/2-1}) & 0 \\ 0 & \hat{\Theta}_2(x_0, x_1, \cdots, x_{K/2-1}) \end{bmatrix} \\ &+ \begin{bmatrix} \hat{\Theta}_1(x_{\frac{K}{2}}, x_{\frac{K}{2}+1}, \cdots, x_{K-1}) & 0 \\ 0 & -\hat{\Theta}_2(x_{\frac{K}{2}}, x_{\frac{K}{2}+1}, \cdots, x_{K-1}) \end{bmatrix} \end{aligned} \tag{5.41}$$

is a GQOD of rate

$$\mathcal{R} = \frac{2K}{n_1L_1 + n_2L_2} = \frac{2\text{lcm}(K_1, K_2)}{n_1L_1 + n_2L_2} = \frac{2\text{lcm}(K_1, K_2)}{\text{lcm}(K_1, K_2)(L_1/K_1 + L_2/K_2)} = H(r_1, r_2) \tag{5.42}$$

where $H(r_1, r_2)$ is the Harmonic mean of r_1, r_2 with $N = N_1 + N_2$ and delay, $L = n_1 L_1 + n_2 L_2$.

We illustrate the working of Construction 5.4.8 by constructing a rate 6/7 GQOD for six transmit antennas.

Example 5.4.1. *Let*

$$\Theta_1 = \begin{bmatrix} x_0 & x_1 \\ -x_1^* & x_0^* \end{bmatrix}$$

be the Alamouti code. Then $L_1 = N_1 = K_1 = 2$. Similarly let

$$\Theta_2 = \begin{bmatrix} x_0 & x_1 & x_2 & 0 \\ -x_1^* & x_0^* & 0 & x_2 \\ -x_2^* & 0 & x_0^* & -x_1 \\ 0 & -x_2^* & x_1^* & x_0 \end{bmatrix}$$

Then $L_2 = N_2 = 4$, $K_2 = 3$ and rate $3/4$. $K = \mathrm{lcm}(K_1, K_2) = 6$, $n_1 = K/K_1 = 3$ and $n_2 = K/K_2 = 2$. $\hat{Theta}_1, \hat{Theta}_2$ are defined in (3.23). Using (5.41) we have the GQOD for $N = N_1 + N_2 = 6$ given by $S =$

$$\begin{bmatrix} x_0 + x_6 & x_1 + x_7 & 0 & 0 & 0 & 0 \\ -(x_1 + x_7)^* & (x_0 + x_6)^* & 0 & 0 & 0 & 0 \\ x_2 + x_8 & x_3 + x_9 & 0 & 0 & 0 & 0 \\ -(x_3 + x_9)^* & (x_2 + x_8)^* & 0 & 0 & 0 & 0 \\ x_4 + x_{10} & x_5 + x_{11} & 0 & 0 & 0 & 0 \\ -(x_5 + x_{11})^* & (x_4 + x_{10})^* & 0 & 0 & 0 & 0 \\ 0 & 0 & x_0 - x_6 & x_1 - x_7 & x_2 - x_8 & 0 \\ 0 & 0 & -(x_1 - x_7)^* & (x_0 - x_6)^* & 0 & x_2 - x_8 \\ 0 & 0 & -(x_2 - x_8)^* & 0 & (x_0 - x_6)^* & -(x_1 - x_7) \\ 0 & 0 & 0 & -(x_2 - x_8)^* & (x_1 - x_7)^* & x_0 - x_6 \\ 0 & 0 & x_3 - x_9 & x_4 - x_{10} & x_5 - x_{11} & 0 \\ 0 & 0 & -(x_4 - x_{10})^* & (x_3 - x_9)^* & 0 & x_5 - x_{11} \\ 0 & 0 & -(x_5 - x_{11})^* & 0 & (x_3 - x_9)^* & -(x_4 - x_{10}) \\ 0 & 0 & 0 & -(x_5 - x_{11})^* & (x_4 - x_{10})^* & (x_3 - x_9) \end{bmatrix}. \quad (5.43)$$

The rate of the above design is $\frac{12}{14} = \frac{6}{7} = 0.8571 > 3/4$. *This increased rate comes at the cost of additional delay. While the rate 3/4 QOD for* $N = 6$ *has a delay of 8 symbol durations, the rate 6/7 GCIOD has a delay of 14 symbol durations. In other words, the rate* 3/4 *scheme is* ***delay-efficient****, while the rate 6/7 scheme is* ***rate-efficient****. Deleting one of the columns we have a rate 6/7 design for 5 transmit antennas.*

Similarly, taking Θ_1 to be the Alamouti code and Θ_2 to be the rate 7/11 design, given in (2.11) in Construction 5.4.8, we have a GQOD for $N = 7$ whose rate is 7/9 and delay is 36 symbol durations. For, $N = 8$ the maximum rate obtained using known GLCODs is 3/4. Significantly, **these designs are not QOD**. Next, we present the construction of rate 2/3 GQOD for all $N > 6$ in the next example.

Example 5.4.2. *For a given* N*, Let* Θ_1 *be the Alamouti code. Then* $L_1 = N_1 = K_1 = 2$ *and* $N_2 = N - 2$. *Let* Θ_2 *be the rate 1/2 GLCOD for* $N - 2$ *transmit antennas (either using the construction of [13] or [19]). Then* $r_2 = 1/2$. *The corresponding rate of the GQOD is given by*

$$\mathcal{R} = \frac{2}{2+1} = \frac{2}{3}.$$

5.4.4 Coding gain

In this sub-section we consider the coding gain for the scheme presented in the previous section, i.e. GQODs derived from GCIODs. Recollect from Subsection 5.4.2 that GQODs achieve full-diversity iff the minimum ζ-distance (MZD) of $\mathcal{A}, \mathcal{B}$ is non-zero. Here we show that the coding gain defined in (1.12) is equal to a quantity, which we call, the Generalized (MZD) as it is a generalization of MZD. The maximization of GMZD for lattice constellations is then considered by rotating the constellation. The results for MZD derived in [33, 34] are a special case of this maximization.

Towards obtaining an expression for the coding gain, we first introduce

Definition 5.4.2 (Minimum Generalized ζ-distance (MGZD)). The minimum generalized ζ-distance between two signal constellations $\mathcal{A}$ and $\mathcal{B}$ is

$$d_{min,\zeta}(\mathcal{A}, \mathcal{B}) = \min_{(s_1,s_2) \neq (\hat{s}_1,\hat{s}_2)} |(s_1 - \hat{s}_1) + (s_2 - \hat{s}_2)|^{N_1/N} |(s_1 - \hat{s}_1) - (s_2 - \hat{s}_2)|^{N_2/N}$$

where N_1, N_2 are two given integers such that $N_1 + N_2 = N$.

Remark 5.4.4. Observe that

1. when $N_1 = N_2$ then the MGZD(N_1, N_2) reduces to the MZD defined in Definition 5.4.1,
2. MGZD$(N_1, N_2) \neq$ MGZD(N_2, N_1) in general.
3. MGZD$(N_1, N_2) \neq 0$ iff GZD$\neq$ MGZD $\neq 0$

We have

Theorem 5.4.9. *The coding gain of a full-rank GQOD from GCIODs or a QOD, S, with the variables $x_0, \cdots, x_{K-1}$ taking values from a signal set $\mathcal{A}$ and the remaining from $\mathcal{B}$, is equal to the MGZD($\mathcal{A}, \mathcal{B}$).*

Proof. Consider the QODs or the GQODs from GCIODs. Since $\hat{\mathcal{D}}_{k,k+K} 0 \leq k \leq 2K-1$ is full-rank we have, that the entries of $S^{\mathcal{H}}S$ are linear sum of $|x_k \pm x_{k+K}|^2$ with strictly positive real coefficients. In addition the first N_1 columns of $S^{\mathcal{H}}S$ are of the form $|x_k + x_{k+K}|^2$ and the remaining are of the form $|x_k - x_{k+K}|^2$.

Now consider the codeword difference matrix $B(\mathbf{S}, \mathbf{S}') = \mathbf{S} - \mathbf{S}'$ which is of full-rank for two distinct codeword matrices $\mathbf{S}, \mathbf{S}'$, we have

$$B(\mathbf{S}, \mathbf{S}')^{\mathcal{H}} B(\mathbf{S}, \mathbf{S}') = \begin{bmatrix} \sum_{k=0}^{K-1} |\underbrace{(x_k - x'_k)}_{\nabla x_k} + \underbrace{(x_{k+K} - x'_{k+K})}_{\nabla x_{k+K}}|^2 I_{N_1} & 0 \\ 0 & \sum_{k=0}^{K-1} |\nabla x_k - \nabla x_{k+K}|^2 I_{N_2} \end{bmatrix} \tag{5.44}$$

where at least one x_k differs from x'_k, $k = 0, \cdots, 2K-1$. Clearly, the terms $\sum_{k=0}^{K-1} |\nabla x_k \pm \nabla x_{k+K}|^2$ are both minimum iff x_k, x_{k+K} differ from x'_k, x'_{k+K} for only one k. Therefore assume, without loss of generality, that the codeword matrices $\mathbf{S}$ and $\mathbf{S}'$ are such that they differ by only one variable, say x_0, x_K taking different values from the signal set $\mathcal{A}, e^{\mathbf{j}\theta}\mathcal{A}$.

Then, for this case, the determinant of the matrix is given by

$$\Lambda_1 = \det\left\{B^{\mathcal{H}}(\mathbf{S},\mathbf{S}')B(\mathbf{S},\mathbf{S}')\right\}^{1/N} = |\nabla x_0 + \nabla x_K|^{\frac{N_1}{N}}||\nabla x_0 \pm \nabla x_K||^{\frac{N_2}{N}} = MGZD(\mathcal{A}, e^{\mathbf{j}\theta}\mathcal{A}).$$

□

An important implication of the above result is,

Corollary 5.4.10. *The coding gain of a full-rank STBC from a QOD with the variables taking values from a signal set, is equal to the MZD of that signal set.*

Remark 5.4.5. Observe that as the MZD is independent of the constructional details of GOD i.e N_1, N_2 and is dependent only on the elements of the signal set, it becomes very amenable to maximization techniques. In contrast, for GQOD the coding gain is a function of N_1, N_2.

It is important to note that the MGZD (N_1, N_2) is non-zero iff the MZD is non-zero and consequently, **this is not at all a restrictive condition, since given any signal set $\mathcal{A}$ one can always get another signal set by rotating it $e^{\mathbf{j}\theta}\mathcal{A}$ such that the MZD is non-zero. In fact, there are infinitely many angles of rotations that will satisfy the required condition and only finitely many which will not. Moreover, appropriate rotation leads to more coding gain also.**

From the above results it follows that signal constellations with $MZD = 0$ and hence $MGZD = 0$ like $M - ary\ QAM, M - ary\ PSK$ will not achieve full-diversity. But the situation gets salvaged by simply rotating the signal set to get this condition satisfied as also indicated in [34].

Maximizing MZD for Lattice constellations

In this subsection we derive the optimal angle of rotation for QAM constellations so that MZD and hence the coding gain of QOD is maximized. Although Theorem 5.4.11 was presented in [34], our proof is easier.

Theorem 5.4.11. *Consider a square QAM constellation $\mathcal{A}$, with signal points from the square lattice $(2k-1-Q)d + \mathbf{j}(2l-1-Q)d$ where $k, l \in [1Q]$ and d is chosen so that the* *average energy of the QAM constellation is 1, and $e^{\mathbf{j}\theta}\mathcal{A}$ i.e. $\mathcal{A}$ rotated by an angle θ so as to maximize MZD. The MZD of $\mathcal{A}$,$e^{\mathbf{j}\theta}\mathcal{A}$ is maximized at $\theta = \frac{\pi}{4}$ and is given by*

$$d_{min,\zeta}(\mathcal{A}, e^{\mathbf{j}\theta}\mathcal{A}) = 4d^2. \tag{5.45}$$

Proof. First observe that

$$d_{min,\zeta}(\mathcal{A}, e^{\mathbf{j}\theta}\mathcal{A}) \leq 4d^2.$$

Now $s_1 - s_1' = p_1 2d + \mathbf{j}q_1 2d$, where p_1, q_1 are integers. Similarly $s_2 - s_2' = 2de^{\mathbf{j}\theta}(p_2 + \mathbf{j}q_2)$ where p_2, q_2 are integers such that p_1, q_1, p_2, q_2 cannot be zero at the same time. It follows that at $\theta = \pi/4$

$$\begin{aligned} |(s_1 - s_1')^2 - (s_2 - s_2')^2| &= 4d^2 \left[(p_1 + p_2\cos(\theta) - q_2\sin(\theta))^2 + (q_1 + p_2\sin(\theta) + q_2\cos(\theta))^2\right]^{1/2} \\ &\quad \left[(p_1 - p_2\cos(\theta) + q_2\sin(\theta))^2 + (q_1 - p_2\sin(\theta) - q_2\cos(\theta))^2\right]^{1/2} \\ &= 4d^2 \left[(p_1^2 - q_1^2 + 2p_2q_2)^2 + (q_2^2 - p_2^2 + 2p_1q_1)^2\right]^{1/2}. \end{aligned} \tag{5.46}$$

Now $p_1^2 - q_1^2 + 2p_2q_2$ and $q_2^2 - p_2^2 + 2p_1q_1$ cannot be simultaneously zero. Suppose they are zero then

$$\begin{aligned} p_1^2 - q_1^2 + 2p_2q_2 + q_2^2 - p_2^2 + 2p_1q_1 &= 0 \\ (p_1 + q_1)^2 + (p_2 + q_2)^2 &= 2(q_1^2 + p_2^2) \end{aligned}$$

which is not possible for $p_1, p_2, q_1, q_2 \in \mathbb{Z}$. □

A similar result is obtained for triangular constellations in [33, 34].

Maximizing MGZD for Lattice constellations

In this subsection we consider the optimal angle of rotation for QAM constellations so that MGZD and hence the coding gain of GQOD is maximized.

Proceeding as in the previous section we have, did for the case MZD we find that the optimization is difficult and varies with N_1, N_2.

5.4.5 MMI of GQOD

In this sub-section we analyze the maximum mutual information (MMI) that can be attained by the GQOD schemes presented in this section. We show that except for the Alamouti scheme all other GCOD have lower MMI than the corresponding GQOD. We also show that the MMI of GQOD is equal to that of GCIOD presented in Chapter 3.

We first consider the $N = 2, M = 1$ QOD. Equation (1.6), for the QOD code given in (5.15) with power normalization, can be written as

$$V = \sqrt{\frac{\rho}{2}}\mathcal{H}s + N \tag{5.47}$$

where

$$\mathcal{H} = \begin{bmatrix} h_0 & h_1 \\ h_1 & h_0 \end{bmatrix}$$

and $s = [s_0 \;\; s_1]^T$. If we define $C_Q(N, M, \rho)$ as the maximum mutual information of the GQOD for N transmit and M receive antennas at SNR ρ then proceeding as in [39] we have

$$\begin{aligned}
C_Q(2,1,\rho) &= \frac{1}{2}E(\log\det(I_2 + \frac{\rho}{2}\mathcal{H}^*\mathcal{H})) \\
&= \frac{1}{2}E\log(I_2 + \frac{\rho}{2}\hat{H}^*\hat{H}), \hat{H} = U\mathcal{H}U^{\mathcal{H}}; U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\
&= \frac{1}{2}E\log\left\{1 + \frac{\rho}{2}|h_0 + h_1|^2\right\} + \frac{1}{2}E\log\left\{1 + \frac{\rho}{2}|h_0 - h_1|^2\right\} \\
&= \frac{1}{2}E\log\left\{1 + \frac{\rho}{2}|h_0 + h_1|^2\right\} + \frac{1}{2}E\log\left\{1 + \frac{\rho}{2}|h_0 - h_1|^2\right\} \\
&= \frac{1}{2}E\log\left\{1 + \rho|\tilde{h}_0|^2\right\} + \frac{1}{2}E\log\left\{1 + \rho|\tilde{h}_1|^2\right\}, \left[\tilde{h}_0 \; \tilde{h}_1 \right] = [h_0 \; h_1 \,]\, U \\
&= C(1,1,\rho) = C_D(2,1,\rho) < C(2,1,\rho).
\end{aligned} \tag{5.48}$$

It is similarly seen for QOD code for $N = 4$ given by (5.1), that

$$\begin{aligned}
C_D(4,1,\rho) &= \frac{1}{4}E(\log\det(I_4 + \frac{\rho}{4}\mathcal{H}^\dagger\mathcal{H})) \\
&\qquad \text{where } \mathcal{H} = \begin{bmatrix} h_0 & h_1 & h_2 & h_3 \\ -h_1^* & h_0^* & -h_3^* & h_2^* \\ h_2 & h_3 & h_0 & h_1 \\ -h_3^* & h_2^* & -h_1^* & h_0^* \end{bmatrix} \\
&= \frac{1}{2}E\log\{(1 + \frac{\rho}{4}(|h_0 + h_2|^2 + |h_1 + h_3|^2))(1 + \frac{\rho}{4}(|h_0 - h_2|^2 + |h_1 - h_3|^2))\} \\
&= \frac{1}{2}E\log\{1 + \frac{\rho}{2}(|\tilde{h}_0|^2 + |\tilde{h}_1|^2)\} + \frac{1}{2}E\log\{1 + \frac{\rho}{2}(|\tilde{h}_2|^2 + |\tilde{h}_3|^2)\} \\
&\qquad \text{where } \begin{bmatrix} \tilde{h}_0 \ \tilde{h}_1 \ \tilde{h}_2 \ \tilde{h}_3 \end{bmatrix} = [h_0 \ h_1 \ h_2 \ h_3 \ ] \frac{1}{\sqrt{2}} \begin{bmatrix} I_2 & I_2 \\ I_2 & -I_2 \end{bmatrix} \\
&= C(2,1,\rho) = C_D(2,1,\rho) < C(4,1,\rho). \qquad (5.49)
\end{aligned}$$

Therefore even for rate 1 and $M = 1$ QODs do not achieve full channel capacity. A general expression for the MMI of GQODs of Construction 5.4.8 can be derived as follows: Recollect that it consists of two GLCODs, $\hat{\Theta}_1, \hat{\Theta}_2$ of rate $K/2L_1, K/2L_2$ as defined in (5.41). The $K/2$ variables of $\hat{\Theta}_1$, $(x_0 + x_{K/2}, \cdots, x_{K/2-1} + x_{K/2})$ and the $K/2$ variables of $\hat{\Theta}_2$, $(x_0 - x_{K/2}, \cdots, x_{K/2-1} - x_{K/2})$ are i.i.d complex Gaussian when x_k are i.i.d. complex Gaussian. Let $C_{1,O}, C_{2,O}$ be the MMI of Θ_1, Θ_2 respectively. Then the MMI of GQOD is given by

$$\begin{aligned}
C_{DQ}(N,M,\rho) &= \frac{1}{L}\{L_1C_{1,O} + L_2C_{2,O}\} \qquad (5.50) \\
&= \frac{1}{L}\left\{L_1C_O(N_1, M, \frac{2N_1\rho}{N}) + L_2C_O(N_2, M, \frac{2N_2\rho}{N})\right\} \qquad (5.51) \\
&= \frac{K}{2L}\left\{C(MN_1, 1, \frac{2N_1M\rho}{N}) + C(MN_2, 1, \frac{2N_1M\rho}{N})\right\} \\
&= C_D(N,M,\rho) \qquad (5.52)
\end{aligned}$$

where $C_D(N,M,\rho)$ is defined in (3.65). When $L_1 = L_2$ i.e. $\Theta_1 = \Theta_2$ we have

$$C_Q(N,M,\rho) = \frac{K}{L}C(\frac{MN}{2}, 1, M\rho) \qquad (5.53)$$

as we have already seen for $N = 2, 4$. It is clear that the MMI of QODs is equal to that of CIODs and hence all the arguments that hold for MMI of CIOD hold here also. That is

1. The capacity loss is negligible for one receiver as is seen from Fig. 3.6,3.7; this is because the increase in capacity is small from two to four transmitters in this case.

2. The capacity loss is substantial when the number of receivers is more than one, because these schemes achieve capacity that could be attained with half the number of transmit antennas, roughly less than half of the actual capacity at high SNR's.

3. For $N > 4$, i.e non-rate 1 codes the loss is much more.

4. The MMI of square QOD is greater than MMI of square GLCOD except when $N = 2$.

Results similar to that of GCIOD can be derived here also and hence have been omitted.

5.4.6 Comparison of GQODs and GCIODs

Here we compare GQODs and GCIODs in terms of rate, coding gain, MMI and what it implies in terms of BER performance. Recollect that

1. When ever a GCIOD exists a GQOD of same rate and size exists. Therefore in terms of rate there is parity.

2. Similarly the MMI of GCIOD and GQOD is equal for a given N.

3. In terms of coding gain, we have for lattice constellations the coding gain of CIOD is $\frac{8d^2}{\sqrt{5}}$ while the coding gain of QOD is $4d^2$ for same transmit power and a given N. Therefore QODs have an advantage of $10 * \log_1 0(\sqrt{5}/2) = 0.4$ dB over CIODs.

4. Next we compare the number of codeword matrices having minimum coding gain, i.e. the multiplicity. Assume that the rate of the designs is K/L. For lattice constellations the multiplicity of CIOD is $K * \tau_1$, where τ is the maximum number signal points of the signal set at CPD from any given signal point. For QOD the multiplicity is equal to $K * \tau_2$, where τ_2 is the maximum number signal points of the signal set at MZD from any given signal point. For square Q^2-QAM, τ_1 can be calculated by solving $|\pm nm + n^2 - m^2| = 1, n, m \in [0Q-1]$ and counting the number

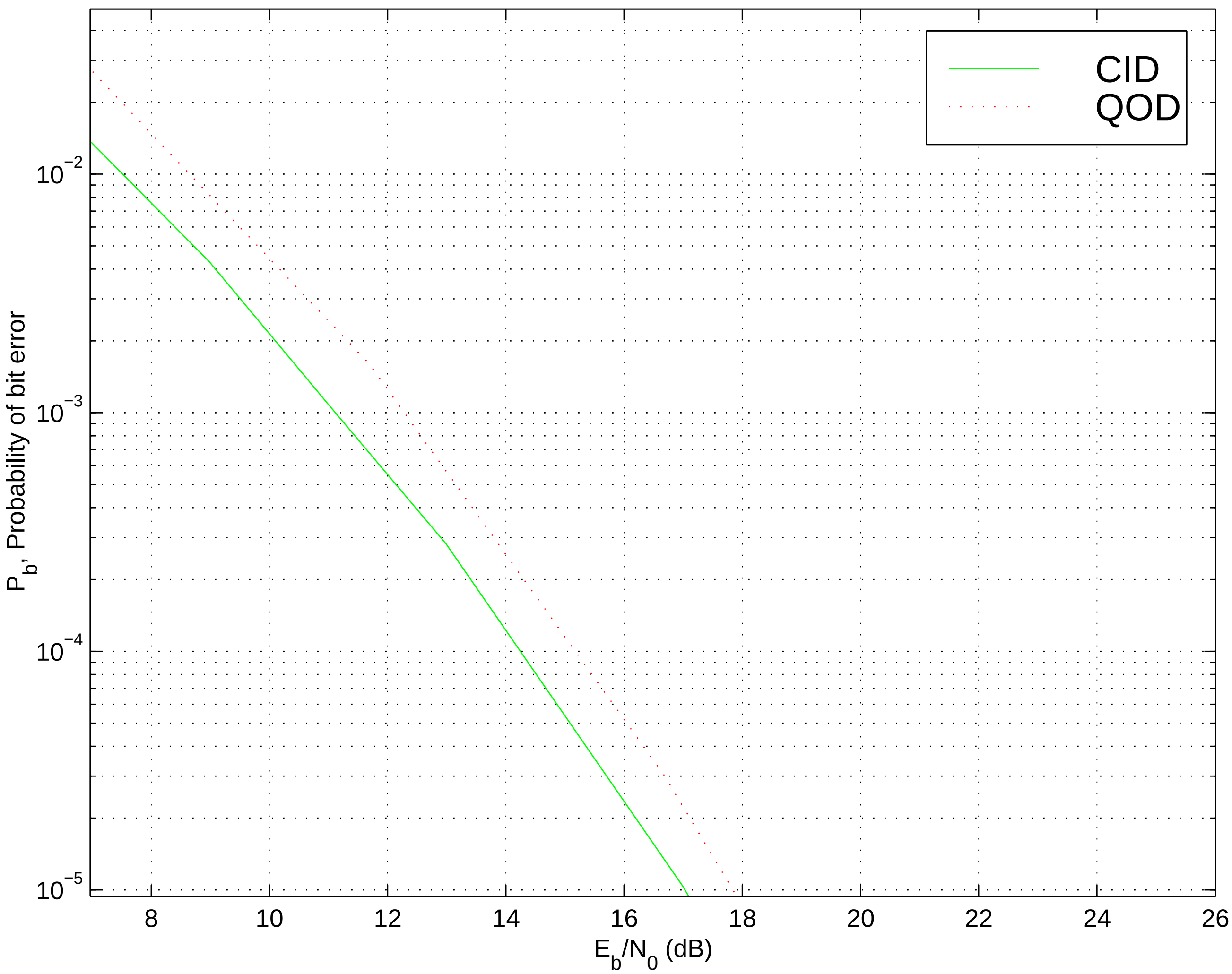


Figure 5.2: The BER performance of coherent QPSK rotated by an angle of 13.2825° (Fig. 3.2)for of the CIOD scheme for 4 transmit and 1 receive antenna compared with QOD in Rayleigh fading for the same number of transmit and receive antennas.

of solutions and τ_2 can be calculated by solving $(p_1^2 - q_1^2 + 2p_2q_2)^2 + (q_2^2 - p_2^2 + 2p_1q_1)^2 = 1; p, q \in [0Q-1]$. For QPSK we have $\tau_1 = 3, \tau_2 = 4$. Hence the multiplicity of CIOD is $3K$ and that QOD is $4K$ for QPSK.

5. Fig. 5.2 gives the simulation results QPSK for $N = 4$. Observe that the CIOD and QOD have almost similar BER performance with CIOD being better by about 0.1 dB. this is explained by the fact that CIOD has lower multiplicity as compared to QOD.

6. In terms of complexity GCIOD is single-symbol decodable while GQOD is double-symbol decodable. This implies that for a spectral efficiency of b bits/sec/Hz, the CIOD requires $K2^b$ metric computations while GQOD requires $K2^{2b}$ metric computations where K is the number of indeterminates of the designs.

5.5 Existence of Square FGQRDs

A main result in this section is the proof for the fact that **there exists square FGQRDs with rate $\frac{2(a+1)}{2^a}$ for $N = 2^a > 2$ (A rate 1 FGQRD for $N = 2$ is given in example 5.2.2), whereas only rates up to $\frac{2a}{2^a}$ is possible with square QODs with the same number of antennas**. The other results are: (i) rate-one square FGQRD of size N exist, iff $N = 2, 4, 8$ and (ii) a construction of FGQRDs with maximum rate from GCIODs. We then show by simulation results that although FGQRDs have higher rate their performance is inferior to GQODs.

Let $S = \sum_{k=0}^{K-1} x_{kI} A_{2k} + x_{kQ} A_{2k+1}$ be a FGQRD. Recollect that the weight matrices satisfy (5.19), that is

$$\begin{aligned} A_k^{\mathcal{H}} A_l + A_l^{\mathcal{H}} A_k &= 0, \quad \forall l \neq k, (k+2K)_{4K} && (5.54)\\ A_k^{\mathcal{H}} A_{k+2K} + A_{k+2K}^{\mathcal{H}} A_k &= \mathcal{D}_{k,k+2K}, \quad 0 \leq k \leq 2K-1 && (5.55)\\ A_k - c_k A_{k+2K} &\neq 0, \quad k = 0, \cdots, 2K-1, && (5.56)\\ A_k^{\mathcal{H}} A_k &= \mathcal{D}_k, \quad \forall k && (5.57) \end{aligned}$$

where $\mathcal{D}_k, k = 0, \cdots, 4K-1$ are diagonal matrices with non-negative entries such that $\mathcal{D}_{2k} + \mathcal{D}_{2k+1}$ is full-rank $\forall k$ and at least one of $\mathcal{D}_k$ is not full-rank.

We first give a construction of square FGQRDs and then show that the constructed codes are maximal rate for $N = 4, 8$.

Theorem 5.5.1. *A square FGQRD, S, of size $N = 2^a b, b$ odd, in variables $x_i, i = 0, \cdots, 2K-1$ of rate $\mathcal{R} = \frac{2+2a}{N}$, is given by*

$$S = \begin{bmatrix} \underbrace{\Theta(x_0, \cdots, x_{K-1})}_{\Theta_1} + \underbrace{\Theta(x_K, \cdots, x_{2K-1})}_{\Theta_2} & 0 \\ 0 & \Theta_1 - \Theta_2 \end{bmatrix} \qquad (5.58)$$

where $\Theta(x_0, \cdots, x_{K-1})$ *is a maximal rate square RFSDD of size* $N/2$ *(constructed in Theorem (2.5.6)).*

Proof. The proof is by direct verification. As the maximal rate of square RFSDD of size $N/2$ is $\frac{2(a-1)}{2^{a-1}b}$ (Theorem 2.5.2) the rate of S in (5.58) is $2\frac{2(a-1)}{2^a b} = \frac{4(a-1)}{2^a b}$. Next we show that S is a FGQRD. Consider

$$S^{\mathcal{H}}S = \begin{bmatrix} \Theta_1^{\mathcal{H}}\Theta_1 + \Theta_2^{\mathcal{H}}\Theta_2 + \Theta_1^{\mathcal{H}}\Theta_2 + \Theta_2^{\mathcal{H}}\Theta_1 & 0 \\ 0 & \Theta_1^{\mathcal{H}}\Theta_1 + \Theta_2^{\mathcal{H}}\Theta_2 - \Theta_1^{\mathcal{H}}\Theta_2 - \Theta_2^{\mathcal{H}}\Theta_1 \end{bmatrix},$$

by construction, the sum of weight matrices of x_{kI}^2, x_{kQ}^2 for any symbol x_k is I_N and (5.28)-(5.31) are satisfied as Θ is a RFSDD. Therefore S is a FGQRD. □

Next we show that rate 1, FGQRD exists for $N = 2, 4$ or 8 only.

Theorem 5.5.2. *A square FGQRD of size* N*, rate 1 exists iff* $N = 2, RFSDD4$ *or* 8.

Proof. Consider a square FGQRD S. Let

$$B_k = A_{2k} + A_{2k+1} + A_{2k+2K} + A_{2k+2K+1}, k = 0, \cdots, K-1$$

then

$$B_k^{\mathcal{H}}B_k = \hat{\mathcal{D}}_k = \mathcal{D}_{2k} + \mathcal{D}_{2k+1},\ k = 0, \cdots, K-1 \tag{5.59}$$

$$B_l^{\mathcal{H}}B_k + B_k^{\mathcal{H}}B_l = 0,\ 0 \le k \ne l \le K-1. \tag{5.60}$$

Observe that $\hat{\mathcal{D}}_k$ is of full-rank for all k. Define $C_k = B_k\hat{\mathcal{D}}_k^{-1/2}$. Then the matrices C_k satisfy

$$C_k^{\mathcal{H}}C_k = I_N, \quad k = 0, \cdots, K-1 \tag{5.61}$$

$$C_l^{\mathcal{H}}C_k + C_k^{\mathcal{H}}C_l = 0, \quad 0 \le k \ne l \le K-1. \tag{5.62}$$

Define

$$\hat{C}_k = C_0^{\mathcal{H}} C_k, k = 0, \cdots, K-1, \tag{5.63}$$

then $\hat{C}_0 = I_N$ and

$$\hat{C}_k^{\mathcal{H}} = -\hat{C}_k, \quad k = 1, \cdots, K-1 \tag{5.64}$$

$$\hat{C}_l^{\mathcal{H}} \hat{C}_k + \hat{C}_k^{\mathcal{H}} \hat{C}_l = 0, \quad 1 \leq k \neq l \leq K-1. \tag{5.65}$$

The normalized set of matrices $\{\hat{C}_1, \cdots, \hat{C}_{K-1}\}$ constitute a *Hurwitz family of order N [24]* and for $N = 2^a b$, b odd and $a, b > 0$ the number of such matrices $K-1$ is bounded by [24]

$$K \leq 2a + 2.$$

For rate 1, $N = K$, this equality is satisfied iff $N = 2, 4, 8$. □

Observe that other square FGQRDs can be constructed from (5.58) by applying some of the following

- permuting rows and/or columns of (2.92),
- permuting the real symbols $\{x_{kI}, x_{kQ}\}_{k=0}^{K-1}$,
- multiplying a symbol by -1 or $\pm\mathbf{j}$
- conjugating a symbol in (2.92).

It follows that the design

$$\begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_2 & \Theta_1 \end{bmatrix} \tag{5.66}$$

is also a FGQRD as it is unitary equivalent to (5.58) where Θ_1, Θ_2 are as defined in (5.58). Also observe that the square FGQRDs of Theorem 5.5.1 can be thought of as designs combining co-ordinate interleaving and GQODs.

An important question remains to be answered, before proceeding with further development of the theory of FGQRDs, that is, although FGQRDs have higher rate as compared to GQODs do we gain in terms of coding gain?

To answer this question we first present the sufficient condition for FGQRDs to achieve full-diversity and then compare the BER of rate 1 FGQRD for $N = 8$ with rate 3/4 QOD for $N = 8$. It turns out that although FGQRDs have higher rate, QODs perform better than FGQRDs. Hence the characterization of FGQRDs completes the theory of FDSDD but are not practical.

Theorem 5.5.3. *The FGQRDs of Theorem 5.5.1 achieve full-diversity iff the* $x_i \in \mathcal{A}e^{\mathbf{j}\theta_0}, x_{i+K} \in e^{\mathbf{j}\theta_1}\mathcal{A}$, $\theta_0 \neq \theta_1$ *and CPD(*$\mathcal{A}e^{\mathbf{j}\theta_0}$*), CPD*$(\mathcal{A}e^{\mathbf{j}\theta_1}) \neq 0$.

Proof. It is sufficient to consider the difference matrix $S-\hat{S}$ that differ only in the variables x_0, x_K. Observing that Θ_1, Θ_2 are RFSDDs, we have CPD$(\mathcal{A}e^{\mathbf{j}\theta_0})$, CPD$(\mathcal{A}e^{\mathbf{j}\theta_1}) \neq 0$. The other requirement is straight forward. □

5.5.1 Coordinate-Interleaved Design for Eight Tx Antennas

Let $x_0, x_1, x_2, x_3, x_4, x_5, x_6$ and x_7 denote eight complex indeterminates, where $x_i = x_{iI} + \mathbf{j}x_{iQ}$. We define,

$$\begin{aligned} &\tilde{x}_0 = x_{0I} + \mathbf{j}x_{2Q}, \quad \tilde{x}_1 = x_{1I} + \mathbf{j}x_{3Q}, \quad \tilde{x}_2 = x_{2I} + \mathbf{j}x_{0Q} \\ &\tilde{x}_3 = x_{3I} + \mathbf{j}x_{1Q}, \quad \tilde{x}_4 = x_{4I} + \mathbf{j}x_{6Q} \quad \tilde{x}_5 = x_{5I} + \mathbf{j}x_{7Q} \\ &\tilde{x}_6 = x_{6I} + \mathbf{j}x_{4Q} \quad \tilde{x}_7 = x_{7I} + \mathbf{j}x_{5Q} \end{aligned} \tag{5.67}$$

to be the new complex indeterminates, called interleaved indeterminates henceforth, obtained by coordinate interleaving the in-phase and quadrature-phase components of the original indeterminates. Then using (5.66) the FGQRD for eight transmit antennas is given by

$$\mathcal{S}_8(x_0,x_1,x_2,x_3,x_4,x_5,x_6,x_7)=\begin{bmatrix} \tilde{x}_o & \tilde{x}_1 & 0 & 0 & \tilde{x}_4 & \tilde{x}_5 & 0 & 0\\ -\tilde{x}_1^* & \tilde{x}_0^* & 0 & 0 & -\tilde{x}_5^* & \tilde{x}_4^* & 0 & 0\\ 0 & 0 & \tilde{x}_2 & \tilde{x}_3 & 0 & 0 & \tilde{x}_6 & \tilde{x}_7\\ 0 & 0 & -\tilde{x}_3^* & \tilde{x}_2^* & 0 & 0 & -\tilde{x}_7^* & \tilde{x}_6^*\\ \tilde{x}_4 & \tilde{x}_5 & 0 & 0 & \tilde{x}_0 & \tilde{x}_1 & 0 & 0\\ -\tilde{x}_5^* & \tilde{x}_4^* & 0 & 0 & -\tilde{x}_1^* & \tilde{x}_0^* & 0 & 0\\ 0 & 0 & \tilde{x}_6 & \tilde{x}_7 & 0 & 0 & \tilde{x}_2 & \tilde{x}_3\\ 0 & 0 & -\tilde{x}_7^* & \tilde{x}_6^* & 0 & 0 & -\tilde{x}_3^* & \tilde{x}_2^* \end{bmatrix} \quad (5.68)$$

We will use only S_8 for the matrix above, dropping the arguments. It is easily verified that

$$\mathcal{S}_8^{\mathcal{H}}\mathcal{S}_8=\begin{bmatrix} a & 0 & 0 & 0 & b & 0 & 0 & 0\\ 0 & a & 0 & 0 & 0 & b & 0 & 0\\ 0 & 0 & c & 0 & 0 & 0 & d & 0\\ 0 & 0 & 0 & c & 0 & 0 & 0 & d\\ b & 0 & 0 & 0 & a & 0 & 0 & 0\\ 0 & b & 0 & 0 & 0 & a & 0 & 0\\ 0 & 0 & d & 0 & 0 & 0 & c & 0\\ 0 & 0 & 0 & d & 0 & 0 & 0 & c \end{bmatrix} \quad (5.69)$$

where

$$\begin{aligned} a &= \tilde{x}_0\tilde{x}_0^*+\tilde{x}_1\tilde{x}_1^*+\tilde{x}_4\tilde{x}_4^*+\tilde{x}_5\tilde{x}_5^*=|\tilde{x}_0|^2+|\tilde{x}_1|^2+|\tilde{x}_4|^2+|\tilde{x}_5|^2\\ b &= \tilde{x}_0\tilde{x}_4^*+\tilde{x}_4\tilde{x}_0^*+\tilde{x}_1\tilde{x}_5^*+\tilde{x}_5\tilde{x}_1^*=2(\tilde{x}_{0I}\tilde{x}_{4I}+\tilde{x}_{0Q}\tilde{x}_{4Q}+\tilde{x}_{1I}\tilde{x}_{5I}+\tilde{x}_{1I}\tilde{x}_{5I})\\ c &= \tilde{x}_2\tilde{x}_2^*+\tilde{x}_3\tilde{x}_3^*+\tilde{x}_6\tilde{x}_6^*+\tilde{x}_7\tilde{x}_7^*=|\tilde{x}_2|^2+|\tilde{x}_3|^2+|\tilde{x}_6|^2+|\tilde{x}_7|^2\\ d &= \tilde{x}_2\tilde{x}_6^*+\tilde{x}_6\tilde{x}_2^*+\tilde{x}_3\tilde{x}_7^*+\tilde{x}_7\tilde{x}_3^*=2(\tilde{x}_{2I}\tilde{x}_{6I}+\tilde{x}_{2Q}\tilde{x}_{6Q}+\tilde{x}_{3I}\tilde{x}_{7I}+\tilde{x}_{3I}\tilde{x}_{7I}). \end{aligned} \quad (5.70)$$

The STBC based on the design $\mathcal{S}_8$ transmits signal matrices obtained by replacing the indeterminates x_i by s_i in S_8 and allowing $s_i,\ i=0,1,2,3$ to take values from a signal set $\mathcal{A}$ and the remaining four variables $s_i,\ i=4,5,6,7$ to take values from a rotated version $\mathcal{A}e^{\mathbf{j}\theta}$ of $\mathcal{A}$. This rotation is needed to achieve full-diversity as elaborated in Theorem 5.5.3. It is important to note that when the variables $s_i,\ i=0,1,2,3,$

take values from $\mathcal{A}$ the transmission matrix have entries which are coordinate interleaved versions of the variables and hence the actual signal points transmitted are not from $\mathcal{A}$ but from an *expanded version of* $\mathcal{A}$ which we denote by $\tilde{\mathcal{A}}$. Figure 5.3(a) shows $\tilde{\mathcal{A}}$ when $\mathcal{A} = \{1, -1, \mathbf{j}, -\mathbf{j}\}$ which is shown in Figure 5.3(c). Notice that $\tilde{\mathcal{A}}$ has 8 signal points whereas $\mathcal{A}$ has 4. Figure 5.3(b) shows $\tilde{\mathcal{A}}'$ where $\mathcal{A}'$ is the four point signal set obtained by rotating $\mathcal{A}$ by 13.2825 degrees counter clockwise i.e., $\mathcal{A}' = \{e^{\mathbf{j}\theta}, -e^{\mathbf{j}\theta}, \mathbf{j}e^{\mathbf{j}\theta}, -\mathbf{j}e^{\mathbf{j}\theta}\}$ where $\theta = 13.2825$ degrees as shown in Figure 5.3(d). Notice that now the expanded signal set has 16 signal points (The choice of the value $\theta = 13.2825$ has no particular significance). Another important aspect to notice in the signal transmission matrix (5.68) is that at

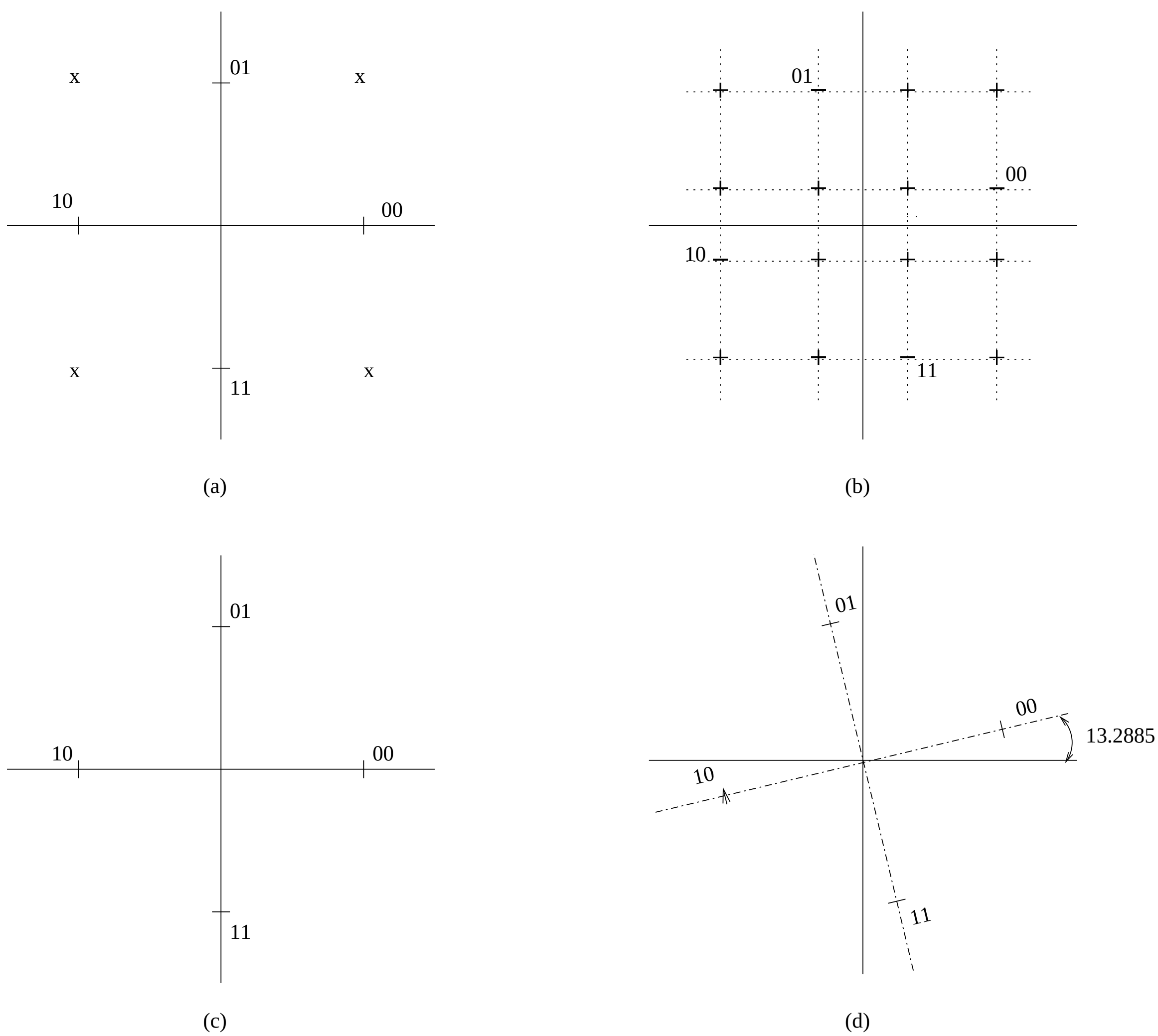


Figure 5.3: Expanded signal sets $\tilde{\mathcal{A}}$ for $\mathcal{A} = \{1, -1, \mathbf{j}, -\mathbf{j}\}$ and a rotated version of it.

any time only four antennas transmit signals. However, at any time the symbols that are transmitted by four antennas carry a component (in- or quadrature-) of all the eight indeterminates $s_i, \;\; i = 0, 1, ..., 7$. Moreover the decoding is for the variables s_i's and not for the interleaved versions of these as is clear from Proposition 5.2.1.

Simulation Results

Figure 5.4 shows simulation results for data rate of 1.5 bits/sec/Hz over a quasi-static Rayleigh fading channel. The fading is assumed to be constant over a fade length of 120 symbol durations.

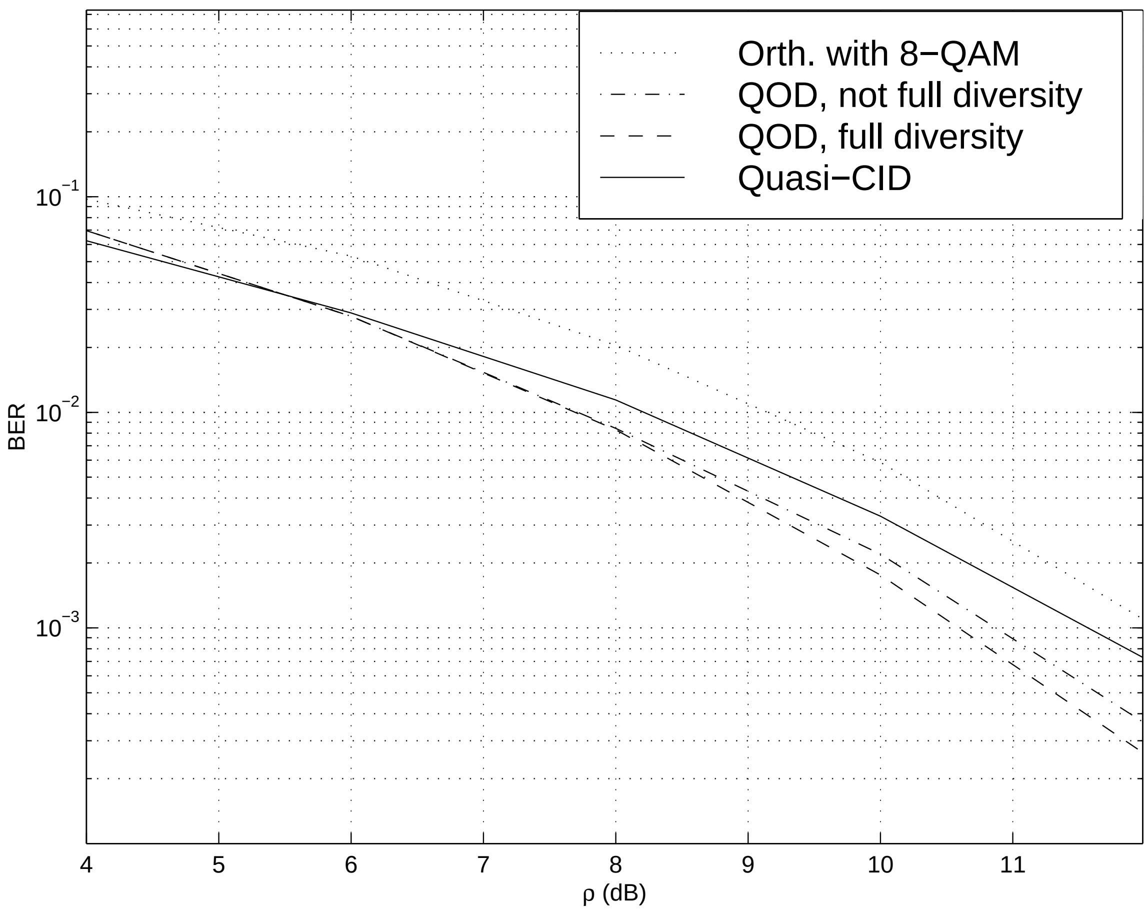


Figure 5.4: The BER performance of STBCs from OD, QODs and the Quasi-CIOD for $N = 8$ at 1.5 bits/sec/Hz in quasi-static Rayleigh fading channel.

We compare our scheme for 8 transmitters with rate 3/4 QODs and rate 1/2 real OD at same energy per bit (E_b/N_0). For the proposed scheme we choose signals from 3-PSK rotated by angles 15° for the first two variable and −15° for the last two. This results in a rate of 1.58 bits/sec/Hz. For QODs we select symbols from 4-QAM and for OD from 8-QAM. Two curves for QODs are shown: one with the symbols rotated to achieve full-diversity and other where the symbols are not rotated. Note that the scheme of this correspondence out-performs QODs at low ρ (SNR), but is inferior at $\rho > 5.5dB$. This is due to different coding gains of these schemes. The coding gain of the proposed scheme is 0.09 while that of quasi-orthogonal design (full-diversity) is 0.27 and that of orthogonal designs is 0.24. It is well known that the rank and determinant criteria apply at high SNR's and at low SNR's and/or when the number of antennas is large the performance is determined by the trace criterion [65]. This explains the behavior of the curves shown in the figure.

5.6 Discussion

In this chapter, we characterized all double-symbol decodable STBCs called DSDD. Among DSDD we have characterized all STBCs that can achieve full-diversity called FDSDD. The FDSDD is then classified into Generalized Quasi-orthogonal Designs (GQODs) and FGQRDs.

The maximal rates of square GQODs is then derived and the existence of GQODs is linked to that of GCIODs. In particular rate 1 GQODs exist for $N = 2, 4$ only. Based on this link a construction of GQODs is presented that results in high rate GQODs. The coding gain of GQODs is also analyzed and a comparison of GQODs and GCIODs is presented that brings out the difference between the two designs.

For sake of completeness square FGQRDs are also analyzed. It is shown that rate 1 FGQRDs exist for $N = 2, 4, 8$. The rate 1 FGQRD for $N = 8$ is then compared with rate 3/4 QOD for $N = 8$ and it is shown that although FGQRDs have higher rate their performance is inferior to that of GQODs and hence does not merit a detailed analysis.

Important directions for future research are

1. Maximal rates of non-square GQODs and

2. Maximization of the coding gain of GQODs when $N_1 \neq N_2$.

Chapter 6

Space-Time Block Codes from Designs for Fast-Fading Channels

Space-Time block codes (STBC) obtained from OD (Orthogonal Designs), QOD (Quasi-Orthogonal Designs) and their variations [13]-[34] are attractive due to their fast ML decoding (single/ double-symbol decoding) when used over quasi-static fading channels. However, these STBCs from Designs have not been studied well for use in fast-fading channels. In this chapter, we study these codes for use in fast-fading channels by giving a matrix representation of the multi-antenna fast-fading channels. We first characterize all linear STBCs that allow single-symbol ML decoding when used in fast-fading channels. Then, among these we identify those with full-diversity, i.e., those with diversity L when the STBC is of size $L \times N, (L \geq N)$, where N is the number of transmit antennas and L is the time interval. The maximum rate for such a full-diversity, single-symbol decodable code is shown to be $2/L$ from which it follows that rate 1 is possible only for 2 Tx. antennas. The co-ordinate interleaved orthogonal design (CIOD) for 2 Tx (introduced in Chapter 2) is shown to be one such full-rate, full-diversity and single-symbol decodable code. (It turns out that Alamouti code is not single-symbol decodable for fast-fading channels.) This code performs well even when the channel is varying in the sense that sometimes it is quasi-static and other times it is fast-fading. We then present simulation results for this code in such a scenario. For sake of completeness, we also consider double-symbol decodable STBCs for fast-fading channels.

6.1 Introduction

A design criterion for designing codes for multiple transmit and receive antennas suitable for fading channels was first proposed in [8] and the implementation of multi-antenna systems in the context of transmit diversity was extensively investigated in [12] with space-time trellis codes (STTC). However, most of the subsequent research was directed toward STBCs over quasi-static fading channels due to their amenability for fast decoding: Following Alamouti [16], lot of work has gone into fast decoding of STBC over quasi-static fading channels [13]-[35],[28]-[34]. These are based on Orthogonal Designs (OD) and Quasi-Orthogonal Designs (QOD). In Chapters 2,3,5 a complete characterization of single-symbol and double-symbol ML decodable, full-diversity linear STBC has been presented. All these are applicable for quasi-static fading channels. The use of ODs, QODs and their variations like CIODs for fast-fading channels have not been studied so far to the best of our knowledge. In this chapter, we study linear STBCs for fast-fading channels with emphasis on fast decoding (single/double-symbol decoding).

6.2 Channel Model

In this section we present the channel model for fast-fading channels. Let the number of transmit antennas be N and the number of receive antennas be M. At each time slot t, complex signal points, $s_{it},\ i = 0, 1, \cdots, N-1$ are transmitted from the N transmit antennas simultaneously. Let $h_{ijt} = \alpha_{ijt} e^{\mathbf{j}\theta_{ijt}}$ denote the path gain from the transmit antenna i to the receive antenna j at time t, where $\mathbf{j} = \sqrt{-1}$. The received signal v_{jt} at the antenna j at time t, is given by

$$v_{jt} = \sum_{i=0}^{N-1} h_{ijt} s_{it} + n_{jt}, \tag{6.1}$$

$j = 0, \cdots, M-1;\quad t = 0, \cdots, L-1$. Assuming that perfect channel state information (CSI) is available at the receiver, the decision rule for ML decoding is to minimize the metric

$$\sum_{t=0}^{L-1} \sum_{j=0}^{M-1} \left| v_{jt} - \sum_{i=0}^{N-1} h_{ijt} s_{it} \right|^2 \tag{6.2}$$

over all codewords. This results in exponential decoding complexity, because of the joint decision on all the symbols s_{it} in the matrix $\mathbf{S}$. If the throughput rate of such a scheme is R in bits/sec/Hz, then 2^{RL} metric calculations are required; one for each possible transmission matrix $\mathbf{S}$. Even for modest antenna configurations and rates this could be very large resulting in search for codes that admit a simple decoding while providing full-diversity gain.

6.2.1 Quasi-Static Fading Channels

For quasi-static fading channels $h_{ijt} = h_{ij}$ and (6.1) can be written in matrix notation as,

$$\mathbf{V} = \mathbf{S}\mathbf{H} + \mathbf{N} \tag{6.3}$$

where $\mathbf{V} \in \mathbb{C}^{L\times M}$ ($\mathbb{C}$ denotes the complex field) is the received signal matrix, $\mathbf{S} \in \mathbb{C}^{L\times N}$ is the transmission matrix (codeword matrix), $\mathbf{H} \in \mathbb{C}^{N\times M}$ denotes the channel matrix and $\mathbf{W} \in \mathbb{C}^{L\times M}$ has entries that are Gaussian distributed with zero mean and unit variance and also are temporally and spatially white. In $\mathbf{V}, \mathbf{S}$ and $\mathbf{W}$ time runs vertically and space runs horizontally. The channel matrix $\mathbf{H}$ and the transmitted codeword $\mathbf{S}$ are assumed to have unit variance entries. The ML metric can then be written as

$$M(\mathbf{S}) = \mathrm{tr}\left((\mathbf{V} - \mathbf{S}\mathbf{H})^{\mathcal{H}}(\mathbf{V} - \mathbf{S}\mathbf{H})\right). \tag{6.4}$$

Recollect that all linear STBCs (not necessarily of full-rank) that admit single-symbol decoding for quasi-static fading channels have been characterized as follows in Chapter 2, Theorem 2.3.1:

Theorem 6.2.1. *For a linear STBC,* $\mathbf{S} = \sum_{k=0}^{K-1} \mathbf{A}_{2k}x_{kI} + \mathbf{A}_{2k+1}x_{kQ}$ *in* K *complex variables, the ML metric,* $M(\mathbf{S})$ *defined in (6.4) decomposes as* $M(\mathbf{S}) = \sum_k M_k(x_k) + M_C$ *where* $M_C = -(K-1)\mathrm{tr}\left(\mathbf{V}^{\mathcal{H}}\mathbf{V}\right)$ *and* $M_k(x_k)$ *is a function of one variable* x_k*, iff*

$$\mathbf{A}_k^{\mathcal{H}}\mathbf{A}_l + \mathbf{A}_l^{\mathcal{H}}\mathbf{A}_k = 0, 0 \le k \ne l \le 2K-1. \tag{6.5}$$

We exploit this result for fast-fading channels in the sequel.

6.2.2 Fast-Fading Channels:

We recall that the design criteria for fast-fading channels are [12]:

- The Distance Criterion : In order to achieve the diversity rM in fast-fading channels, for any two distinct codeword matrices $\mathbf{S}$ and $\hat{\mathbf{S}}$, the strings $s_{0t}, s_{1t}, \cdots, s_{(N-1)t}$ and $\hat{s}_{0t}, \hat{s}_{1t}, \cdots, \hat{s}_{(N-1)t}$ must differ at least for r values of $0 \leq t \leq L-1$. (Essentially, the distance criterion implies that if a codeword is viewed as a L length vector with each row of the transmission matrix viewed as a single element of $\mathbb{C}^N$, then the diversity gain is equal to the Hamming distance of this L length codeword over $\mathbb{C}^N$.
- The Product Criterion : Let $\mathcal{V}(\mathbf{S}, \hat{\mathbf{S}})$ be the indices of the non-zero rows of $\mathbf{S} - \hat{\mathbf{S}}$ and let$|\mathbf{s}_t - \hat{\mathbf{s}}_t|^2 = \sum_{i=0}^{N-1} |s_{it} - \hat{s}_{it}|^2$, where $\mathbf{s}_t$ is the t-th row of $\mathbf{S}$, $0 \leq t \leq L-1$. Then the coding gain is
$$\min_{\mathbf{s} \neq \hat{\mathbf{s}}} \prod_{t \in \mathcal{V}(\mathbf{s}, \hat{\mathbf{s}})} |\mathbf{s_t} - \hat{\mathbf{s}}_\mathbf{t}|^2.$$

6.3 Extended Codeword Matrix and the Equivalent Matrix Channel

The inability to write (6.1) in the matrix form as in (6.3) for fast-fading channels seems to be the reason for scarce study of STBCs for use in fast-fading channels. In this section we solve this problem by introducing proper matrix representations for the codeword matrix and the channel. In what follows we assume that $M = 1$, for simplicity. For a fast-fading channel (6.1) can be written as

$$\mathbf{V} = SH + \mathbf{N} \tag{6.6}$$

where $\mathbf{V} \in \mathbb{C}^{L \times 1}$ ($\mathbb{C}$ denotes the complex field) is the received signal vector, $S \in \mathbb{C}^{L \times NL}$ is the **Extended codeword matrix (ExCM)** (as opposed to codeword matrix $\mathbf{S}$) given by

$$S = \begin{bmatrix} S_0 & 0 & 0 & 0 \\ 0 & S_1 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & S_{L-1} \end{bmatrix} \tag{6.7}$$

where $S_t = \begin{bmatrix} s_{0t} & s_{1t} & \cdots & s_{(N-1)t} \end{bmatrix}$, $H \in \mathbb{C}^{NL \times 1}$ denotes the **equivalent channel matrix (EChM)** formed by stacking the channel vectors for different t i.e.

$$H = \begin{bmatrix} H_0 \\ H_1 \\ \vdots \\ H_{L-1} \end{bmatrix} \text{ where } H_t = \begin{bmatrix} h_{0t} \\ h_{1t} \\ \cdots \\ h_{(N-1)t} \end{bmatrix},$$

and $\mathbf{W} \in \mathbb{C}^{L \times 1}$ has entries that are Gaussian distributed with zero mean and unit variance and also are temporally and spatially white. We denote the codeword matrices by boldface letters and the ExCMs by normal letters. For example, the ExCM S for the Alamouti code, $\mathbf{S} = \begin{bmatrix} x_0 & x_1 \\ -x_1^* & x_0^* \end{bmatrix}$, is given by

$$S = \begin{bmatrix} x_0 & x_1 & 0 & 0 \\ 0 & 0 & -x_1^* & x_0^* \end{bmatrix}. \tag{6.8}$$

Observe that for a linear space-time code, its ExCM S is also linear in the indeterminates $x_k, k = 0, \cdots, K-1$ and can be written as $S = \sum_{k=0}^{K-1} x_{kI} A_{2k} + x_{kQ} A_{2k+1}$, where A_k are referred to as **extended weight matrices** to differentiate from weight matrices corresponding to the codeword matrix $\mathbf{S}$.

Diversity and Coding gain criteria for fast-fading channels

With the notions of ExCM and EChM developed above and the similarity between (6.3) and (6.6) we observe that,

1. The **distance criterion** on the difference of two distinct codeword matrices is equivalent to the **rank criterion** for the difference of two distinct ExCM.

2. The **product criterion** on the difference of two distinct codeword matrices is equivalent to the **determinant criterion** for the difference of two distinct ExCM.

3. The trace criterion on the difference of two distinct codeword matrices derived for quasi-static fading in [65] applies to fast-fading channels also-following the observation that $\text{tr}\left(\mathbf{S}^{\mathcal{H}}\mathbf{S}\right) = \text{tr}\left(S^{\mathcal{H}}S\right)$.

4. The ML metric (6.2) can again be represented as (6.4) with the code word $\mathbf{S}$ replaced by the ExCM, S i.e.

$$M(S) = \operatorname{tr}\left((\mathbf{V} - SH)^{\mathcal{H}}(\mathbf{V} - SH)\right). \tag{6.9}$$

This amenability to write the ML decoding metric in matrix form for fast-fading channels (6.9) allows the results on single/double-symbol decodable designs of Chapter 2/5 to be applied to fast-fading channels.

6.4 Single-symbol decodable codes

Substitution of the codeword matrix $\mathbf{S}$ by the ExCM, S in Theorem 6.2.1 leads to characterization of single-symbol decodable STBCs for fast-fading channels. We have,

Theorem 6.4.1. *For a linear STBC in K complex variables, whose ExCM is given by, $S = \sum_{k=0}^{K-1} x_{kI} A_{2k} + x_{kQ} A_{2k+1}$, the ML metric, $M(S)$ defined in (6.9) decomposes as $M(S) = \sum_k M_k(x_k) + M_C$ where $M_C = -(K-1)\operatorname{tr}\left(V^{\mathcal{H}} V\right)$, iff*

$$A_k^{\mathcal{H}} A_l + A_l^{\mathcal{H}} A_k = 0, 0 \leq k \neq l \leq 2K - 1. \tag{6.10}$$

Theorem 6.4.1 characterizes all linear designs which admit single-symbol decoding over fast-fading channels in terms of the extended weight matrices.

Example 6.4.1. *The Alamouti code is not single-symbol decodable for fast-fading channels. The extended weight matrices are*

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} \mathbf{j} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{j} \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & \mathbf{j} & 0 & 0 \\ 0 & 0 & \mathbf{j} & 0 \end{bmatrix}.$$

It is easily checked that the pair A_0, A_2 does not satisfy equation (6.10).

6.5 Full-diversity, Single-symbol decodable codes

In this section we proceed to identify all full-diversity codes among single-symbol decodable codes. Recall that for single-symbol decodability in quasi-static fading the weight matrices have to satisfy (6.5) while for fast-fading the extended weight matrices, have to satisfy (6.10).

In contrast to quasi-static fading (6.10) is not easily satisfied for fast-fading due to the structure of the equivalent weight matrices imposed by the structure of S given in (6.7). The weight matrices A_k are block diagonal of the form (6.7)

$$A_k = \begin{bmatrix} A_k^{(0)} & 0 & 0 & 0 \\ 0 & A_k^{(1)} & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & A_k^{(L-1)} \end{bmatrix}. \tag{6.11}$$

where $A_k^{(t)} \in \mathbb{C}^{1\times N}$. In other words even for square codeword matrix the equivalent transmission matrix is rectangular. For example consider the Alamouti code, $A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ etc., (6.10) is not satisfied as a result we have

$$S^{\mathcal{H}}S = \begin{bmatrix} |x_0|^2 & x_0^*x_1 & 0 & 0 \\ x_1^*x_0 & |x_1|^2 & 0 & 0 \\ 0 & 0 & |x_1|^2 & -x_1x_0^* \\ 0 & 0 & -x_1^*x_0 & |x_0|^2 \end{bmatrix}, \tag{6.12}$$

and hence single-symbol decoding is not possible for the Alamouti code over fast-fading channels.

The structure of equivalent weight matrices that satisfy (6.10) is given in Proposition 6.5.1.

Proposition 6.5.1. *All the matrices A_l that satisfy (6.10), with a specified non-zero* *matrix A_k in (6.11) are of the form*

$$\begin{bmatrix} a_0 A_k^{(0)} & 0 & 0 & 0 \\ 0 & a_1 A_k^{(1)} & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & a_{L-1} A_k^{(L-1)} \end{bmatrix}. \tag{6.13}$$

where $a_i = 0, \mathbf{j}\ \forall i$.

Proof. The the matrix A_k can satisfy the condition of Theorem 6.4.1 iff $A_k^{(t)H} A_l^{(t)} = -A_l^{(t)H} A_k^{(t)}, \forall t$. For a given t, $A_k^{(t)H} A_l^{(t)}$ is skew-Hermitian and rank one, it follows that $A_k^{(t)H} A_l^{(t)} = UDU^{\mathcal{H}}$ where U is unitary and D is diagonal with one imaginary entry only. Therefore $A_k^{(t)} = \pm \mathbf{j} c A_l^{(t)}$ where c is a real constant-in fact only the values $c = 0, 1$ are of interest as other values can be normalized to 1, completing the proof. □

We give a necessary condition, derived from the rank criterion for ExCM, in terms of the extended weight matrices A_k for the code to achieve diversity $r \leq L$. This necessary condition results in ease of characterization.

Lemma 6.5.2. *If a linear STBC in K variables, whose ExCM is given by, $S = \sum_{k=0}^{K-1} x_{kI}$ $A_{2k} + x_{kQ} A_{2k+1}$, achieves diversity r then the matrices A_{2k}, A_{2k+1} together have at least r different non-zero rows for every k, $0 \leq k \leq K-1$.*

Proof. This follows from the rank criterion of ExCM interpretation of the distance criterion. If, for a given k, A_{2k}, A_{2k+1} together have at less than r different non-zero rows then the difference of ExCMs, $S - \hat{S}$ which differ in x_k only, has rank less than r. □

The conditions of Lemma 6.5.2 is only a necessary condition since either $(x_{kI} - \hat{x}_{kI})$ or $(x_{kQ} - \hat{x}_{kI})$ may be zero for $x_k \neq \hat{x}_k$. The sufficient condition is obtained by a slight modification of Theorem 2.4.6 and is given by

Corollary 6.5.3. *A linear STBC, $S = \sum_{k=0}^{K-1} x_{kI} A_{2k} + x_{kQ} A_{2k+1}$ where x_k take values from a signal set $\mathcal{A}, \forall k$, satisfying the necessary condition of Lemma 6.5.2 achieves diversity $r \geq N$ iff*

1. *either* $A_k^{\mathcal{H}} A_k$ *is of rank* r *(*r *different non-zero rows) for all* k

2. *or the* CPD *of* $\mathcal{A} \neq 0$.

Using Lemma 6.5.2 with $r = L$, we have

Theorem 6.5.4. *For fast-fading channel, the maximum rates possible for a full-diversity single-symbol decodable STBC using* N *transmit antennas is* $2/L$.

Proof. We have two cases corresponding to the two cases of Corollary 6.5.3 and we consider them separately.

Case 1: A_k *has* L *non-zero rows* $\forall k$. The number of matrices that satisfy Proposition 6.5.1 are 2, and the maximal rate is $R = 1/L$. The corresponding STBC is given by its equivalent transmission matrix $S = x_0 A_0$, where A_0 is of the form given in (6.11).

Case 2: A_k *has less than* L *non-zero rows for some* k. As Lemma 1 requires A_{2k}, A_{2k+1} to have L non-zero rows, we can assume that A_{2k} has r_1 non-zero rows and A_{2k+1} has non-overlapping $L - r_1$ non-zero rows. The number of such matrices that satisfy Proposition 6.5.1 are 4, and hence the maximal rate is $R = 2/L$. □

From Theorem 6.5.4 it follows that the maximal rate full-diversity single-symbol decodable code is given by its ExCM

$$S = x_{0I} A_0 + x_{0Q} A_1 + x_{1I} A_2 + x_{1Q} A_3, \tag{6.14}$$

where $A_{2k}, A_{2k+1}, k = 0, 1$ are of the form

$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \mathbf{j}A & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{j}B \end{bmatrix} \tag{6.15}$$

where A, B are of the form given in (6.11) with $L = r_1$ and $L = L - r_1$ respectively. Observe that other STBC's can be obtained from the above, by change of variables, multiplication by unitary matrices etc. Of interest is the code for $L = 2$ due to its full-rate. Setting

$A = [1\ 0], B = [0\ 1]$ we have the ExCM,

$$S = \begin{bmatrix} x_{0I} + \mathbf{j}x_{1Q} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{1I} + \mathbf{j}x_{0Q} \end{bmatrix} \quad (6.16)$$

and the corresponding codeword matrix is

$$\mathbf{S} = \begin{bmatrix} x_{0I} + \mathbf{j}x_{1Q} & 0 \\ 0 & x_{1I} + \mathbf{j}x_{0Q} \end{bmatrix}. \quad (6.17)$$

Observe that this is the CIOD of size 2 presented in Chapter 2,3. Also observe that other full-rate STBC's that achieve full-diversity can be achieved from S by performing linear operations (not necessarily unitary) on S and/or permutation of the real symbols(for each complex symbol there are two real symbols). Consequently the most general full-diversity single-symbol decodable code for $N = 2$ is given by the codeword matrix

$$\mathbf{S} = \begin{bmatrix} x_{0I} + \mathbf{j}x_{1Q} & b(x_{0I} + \mathbf{j}x_{1Q}) \\ c(x_{1I} + \mathbf{j}x_{0Q}) & x_{1I} + \mathbf{j}x_{0Q} \end{bmatrix}, b, c \in \mathbb{C}. \quad (6.18)$$

An immediate consequence is

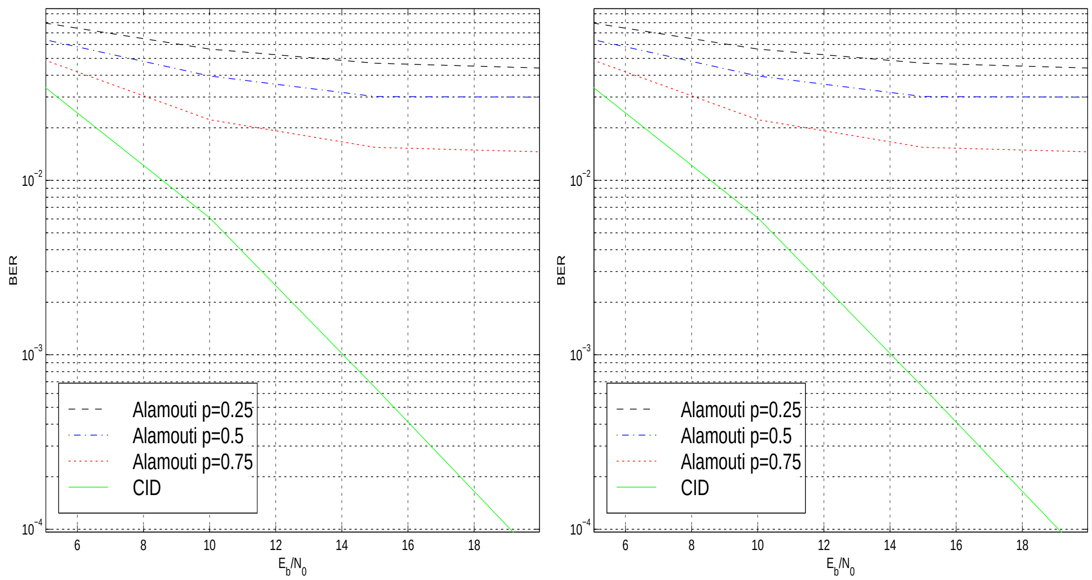
Theorem 6.5.5. *A rate 1 full-diversity single-symbol decodable design for fast-fading channel exists iff $L = N = 2$.*

Following the results of Chapter 2,

Theorem 6.5.6. *The CIOD of size 2 is the only STBC that achieves full-diversity over both quasi-static and fast-fading channels and provides single-symbol decoding.*

Other STBC that achieves full-diversity over both quasi-static fading channels and provides single-symbol decoding are unitarily equivalent to the CIOD.

Remark 6.5.1. Contrast the rates of single-symbol decodable codes for quasi-static and fast-fading channels. From Theorem 6.5.4 we have the maximal rate is $2/L$ for fast-fading channels, while that of square matrix OD [15] is given by $\frac{\lceil \log_2 N \rceil + 1}{2^{\lceil \log_2 N \rceil}}$ and that of square FSDD is given by $\frac{\lceil \log_2 N/2 \rceil + 1}{2^{\lceil \log_2 N \rceil - 1}}$ respectively. The maximal rate is independent of the number of transmit antennas for fast-fading channels.



(a) (b)

Figure 6.1: BER curves for Alamouti and CIOD schemes for a) QPSK and b) BPSK over varying fading channels where the probability that the channel is quasi-static is p and single-symbol decoding.

6.6 Robustness of CIOD to channel variations

In this section we intend to bring out the contrast in BER performance of CIOD for $N = 2$ and the Alamouti scheme when the channel is quasi-static with probability p and fast-fading with probability $1 - p$.

The CIOD for $N = 2$ is suitable for both quasi-static fading and fast-fading channels, in the sense it gives full-diversity and amenable for single-symbol decoding in both the cases, whereas Alamouti code is not. This makes the CIOD suitable for varying-fading scenario, i.e., the channel sometimes quasi-static and other times fast-fading. Here we provide simulation results for the case when the channel is either fast-fading or quasi-static with different probabilities to show that the CIOD is robust to channel variations. The corresponding curves for Alamouti are also presented for comparison. The quasi-static symbol duration is assumed to be 12 symbols. The probability that the channel is quasi-static is denoted by p and perfect CSI is assumed to be available at the receiver when the channel is quasi-static and fast-fading. At the decoder, single-symbol decoding is performed for both Alamouti and the CIOD schemes-observe that this results in loss of diversity in the case of Alamouti scheme. It is observed from Fig. 1 that the BER of CIOD is invariant for all p while the BER curves for Alamouti scheme shows considerable loss of performance.

6.7 Double-symbol decodable codes

In this section we characterize all double-symbol decodable STBCs for fast-fading channels. Basically, we will tend to repeat all that was done for single-symbol decodable STBCs in the previous two sections for double-symbol decodable STBCs in this and the following section.

Substitution of the codeword matrix $\mathbf{S}$ by the ExCM, S in Theorem 5.2.1 leads to characterization of double-symbol decodable STBCs for fast-fading channels. We have,

Theorem 6.7.1. *For a linear STBC in $2K$ variables, whose ExCM is given by $S = \sum_{k=0}^{2K-1} x_{kI}A_{2k} + x_{kQ}A_{2k+1}$, the ML metric, $M(S)$ defined in (6.9) decomposes as $M(S) = \sum_{k=0}^{K-1} M_k(x_k, x_{k+K}) + M_C$ where $M_C = -(K-1)\mathrm{tr}\left(V^{\mathcal{H}}V\right)$, iff*

$$A_k^{\mathcal{H}}A_l + A_l^{\mathcal{H}}A_k = 0 \quad \forall l \neq k, (k+2K)_{4K} \tag{6.19}$$

where $(k+2K)_{4K} = (k+2K) \mod 4K$.

Theorem 6.7.1 characterizes all linear designs which admit double-symbol decoding over fast-fading channels in terms of the extended weight matrices.

Examples of double-symbol decodable codes are

Example 6.7.1. *The Alamouti code whose ExCM is given in example 6.4.1.*

Example 6.7.2. *Another code that is not double-symbol decodable for fast-fading channels is the CIOD for 4 Tx. antennas, given by its codeword matrix (2.54). Unlike the quasi-static fading channel, the block Alamouti structure does not help and does not lead to any decoding simplicity. This can be seen by writing the extended weight matrices, few of* *which are given below for clarity.*

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} \mathbf{j} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathbf{j} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_7 = \begin{bmatrix} 0 & \mathbf{j} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{j} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easily verified that any pair of the above do not satisfy (6.19).

6.8 Full-diversity, Double-symbol decodable codes

In this section we proceed to identify all full-diversity codes among single-symbol decodable codes. Recall that for double-symbol decodability for fast-fading the extended weight matrices, have to satisfy (6.19).

Again in contrast to quasi-static fading (6.19) is not easily satisfied for fast-fading due to the structure of the block diagonal structure of equivalent weight matrices imposed by the structure of S given in (6.7).

The structure of equivalent weight matrices that satisfy (6.10) is given in Proposition 6.5.1.

Also observe that the necessary and sufficient condition for diversity $r \geq N$ for single-symbol decodable codes given in Lemma 6.5.2 and Corollary 6.5.3 are necessary conditions for full-diversity but not sufficient. This claim is proved by setting all those variables in **S** to zero due to which **S** becomes single-symbol decodable. There are two possible realizations for this and in each realization the number of such variables can be at-most half of the total number of variables of **S**. It is immediate then that both Lemma 6.5.2 and Corollary 6.5.3 are necessary. Also observe that these conditions are not sufficient observe that any two variables whose extended weight matrices do not satisfy the orthogonality condition, it is still possible that the resultant be zero.

Using Theorem 6.5.4 with we therefore have that

Theorem 6.8.1. *For fast-fading channel, the maximum rates possible for a full-diversity single-symbol decodable STBC using N transmit antennas is $4/L$.*

Proof. The proof is straight forward as we can only have twice the number of matrices as for single-symbol decodable codes; given in Theorem 6.5.4. □

Towards characterizing such codes we consider the two cases of Corollary 6.5.3 separately.

Case 1: A_k has L non-zero rows $\forall k$. The number of matrices that satisfy Proposition 6.5.1 are 2, and the maximal rate is $R = 2/L$. A possible realization of such STBCs is given by its equivalent transmission matrix $S = x_0 A_0 + x_1 A_1$, where A_0 is of the form given in (6.11) and $A_1 = A_0 U$ where U is a unitary matrix. Observe that we have not presented the sufficient condition for such codes to achieve diversity N. Towards this end we present a construction of maximal rate double-symbol decodable codes whose coding gain is equal to that of ζ-distance defined in Definition 5.4.1 for QODs.

Construction 6.8.2. *Let $L = N$ for N even and $L = N + 1$ N odd. Let the i-th row of identity matrix of size L be denoted as $I^{(i)}$. Let A_0, A_1 be as defined in (6.11) such that $A_0^{(i)} = I^{(i)}$ and $A_i^{(i)} = (-1)^i I^{(i)}$. Then $S = x_0 A_0 + x_1 A_1$ defines a double-symbol decodable STBC for fast-fading channels.*

Observe that for the STBC of construction 6.8.2 $\det\{S^{\mathcal{H}}S\} = (x_0^2 - x_1^2)^{L/2}$. It is therefore it is clear that the performance of the constructed STBC is decided by the minimum-ζ distance. Hence these codes can be thought of as an analogous of GQODs.

Case 2: A_k has less than L non-zero rows for some k. As Lemma 6.5.2 requires A_{2k}, A_{2k+1} to have L non-zero rows, we can assume that A_{2k} has r_1 non-zero rows and A_{2k+1} has non-overlapping $L - r_1$ non-zero rows. The number of such matrices that satisfy Proposition 6.5.1 are 4, and hence the maximal rate is $R = 4/L$. But in analogy to the Case 1) codes we can observe that although these codes higher rate but lower performance due to lower coding gain.

6.9 Discussion

In this chapter, we considered the use of designs for use in fast-fading channels by giving a matrix representation of the multi-antenna fast-fading channels. We first characterized all linear STBCs that allow single-symbol ML decoding when used in fast-fading channels. Then, among these we identify those with full-diversity, i.e., those with diversity L when the STBC is of size $L \times N, (L \geq N)$, where N is the number of transmit antennas and L is the time interval. The maximum rate for such a full-diversity, single-symbol decodable code is shown to be $2/L$ from which it follows that rate 1 is possible only for 2 Tx. antennas. The co-ordinate interleaved orthogonal design (CIOD) for 2 Tx (introduced in Chapter 2) is shown to be one such full-rate, full-diversity and single-symbol decodable code. (It turns out that Alamouti code is not single-symbol decodable for fast-fading channels.) This code performs well even when the channel is varying in the sense that sometimes it is quasi-static and other times it is fast-fading. We then present simulation results for this code in such a scenario. For sake of completeness, we also consider double-symbol decodable STBCs for fast-fading channels.

Chapter 7

Conclusions and Perspectives

In this thesis we have characterized all single and double symbol decodable space-time block codes (STBCs), both for quasi-static and fast-fading channels. Further, among the classes of single and double symbol decodable designs, we have characterized those that can achieve full-diversity.

As a result of this characterization of single-symbol decodable codes for quasi-static fading channels, we observe that when there is no restriction on the the signal set then STBCs from orthogonal design (OD) are the only STBCs that are single-symbol decodable and achieve full-diversity. But when there is a restriction on the signal set, that the co-ordinate product distance is non-zero (CPD $\neq 0$), then there exists a separate class of codes, which we call Full-rank generalized restricted designs (RFSDD), that allows single-symbol decoding and can achieve full-diversity. This restriction on the signal set allows for increase in rate (symbols/channel use), coding gain and maximum mutual information over STBCs from ODs except for two transmit antennas. Significantly, rate 1, STBCs from RFSDDs are shown to exist for 2,3,4 transmit antennas while rate 1 STBCs from while rate 1 STBCs from ODs exist only for 2 transmit antennas. The maximal rates of square RFSDDs were derived and a sub-class of RFSDDs called generalized co-ordinate interleaved orthogonal designs (GCIOD) were presented and their performance analyzed.

A similar characterization of double-symbol decoding STBCs was carried out of which the Quasi-orthogonal designs are a proper sub-class. Significantly, maximal rates of square QODs were derived and various high rate ($>1/2$) double-symbol decodable codes were presented.

An important contribution of this thesis is the novel application of designs to fast-fading channels, as a result of which we find that the CIOD for two transmit antennas is the only design that allows single-symbol decoding over both fast-fading and quasi-static channel.

Other contributions are the unified perspective for quasi-static and fast-fading channels (Chapter 6) and also the development of differential schemes for Full-rank single-symbol decodable designs that are themselves single-symbol decodable.

It is also worth mentioning that most of the gains obtained in this thesis can be thought of as being obtained by applying co-ordinate interleaving across the space-dimension and not time only as is clear seen from the intuition given in Chapter 3. The starting point of this intuition was the application of co-ordinate interleaving to bit and co-ordinate interleaved coded modulation [55].

Although we have rigorously pursued square STBCs, much is left to be desired in non-square STBCs. Although non-square STBCs are shown to be useless for fast-fading channels there Su and Xia [22] have shown for STBCs from ODs that higher rates can be obtained. A complete characterization of such codes in terms of achievable rates and their constructions is an open problem for all single and double symbol decodable designs.

Appendix A

A construction of non-square RFSDDs

In this appendix, we present a method of obtaining non-square RFSDDs from square CIODs whose coding gain is greater than CPD. We also present a construction derived from construction 3.1.3 whose coding gain is still given by GCPD but the difference between N_1, N_2 is small. These codes do not belong to the class of GCIODs.

A.1 Non-square RFSDDs from CIODs

Recall from Chapter 3 that we can obtain non-square GCIODs from CIODs by dropping columns. But the coding gain of such GCIODs is a power of GCPD, whose value is not known for any class of signal sets. Here we present a construction that gives non-square RFSDDs whose coding gain is greater than CPD and has higher MMI than GCIOD codes.

Construction A.1.1. *Let*

$$S(x_0, \cdots, x_{K-1}) = \begin{bmatrix} \Theta(\tilde{x}_0, \cdots, \tilde{x}_{K/2-1}) & 0_{L,N} \\ 0_{L,N} & \Theta(\tilde{x}_{K/2}, \cdots, \tilde{x}_{K-1}) \end{bmatrix} \tag{A.1}$$

be a $2L \times 2N$ CIOD where $0_{L,N}$ is the $L \times N$ zero matrix, then

$$\hat{S}(x_0, \cdots, x_{K-1}) = \begin{bmatrix} \Theta(\tilde{x}_0, \cdots, \tilde{x}_{K/2-1}) & 0_{L,N-n} \\ 0_{L,N-n} & \Theta(\tilde{x}_{K/2}, \cdots, \tilde{x}_{K-1}) \end{bmatrix} \tag{A.2}$$

is a $2L \times 2N - n$ non-square RFSDD for $2N - n$ antennas, $n < N$.

Now we present some examples of the above construction.

Example A.1.1. *Let Θ be the Alamouti scheme, then we have*

$$S(x0, \cdots, x3) = \begin{bmatrix} x_{0I} + \mathbf{j}x_{2Q} & x_{1I} + \mathbf{j}x_{3Q} & 0 \\ -x_{1I} + \mathbf{j}x_{3Q} & x_{0I} - \mathbf{j}x_{2Q} & 0 \\ 0 & x_{2I} + \mathbf{j}x_{0Q} & x_{3I} + \mathbf{j}x_{1Q} \\ 0 & -x_{3I} + \mathbf{j}x_{1Q} & x_{2I} - \mathbf{j}x_{0Q} \end{bmatrix} \tag{A.3}$$

then S is a rate 1, RFSDD for three transmit antennas, as

$$S^{\mathcal{H}}S = \begin{bmatrix} x_{0I}^2 + x_{1I}^2 + x_{2Q}^2 + x_{3Q}^2 & 0 & 0 \\ 0 & \sum_{k=0}^{3} |x_k|^2 & 0 \\ 0 & 0 & x_{0Q}^2 + x_{1Q}^2 + x_{2I}^2 + x_{3I}^2 \end{bmatrix}. \tag{A.4}$$

Observe that this code is quite different from that of CIODs in that both the in-phase and quadrature component of the variables see the second transmit antenna. Towards finding *the coding gain, let* $S, \hat{S}$ *be two codeword matrices that differ in only* x_0. *Then*

$$\begin{aligned} \Lambda = det\left((S-\hat{S})^{\mathcal{H}}(S-\hat{S})\right)^{1/3} &= [(x_{0I}-\hat{x}_{0I})^2(x_{0Q}-\hat{x}_{0Q})^2(|x_0-\hat{x}_0|^2)]^{1/3} \quad (A.5)\\ &= CPD^{2/3}|\nabla x_0|^{2/3} \quad (A.6) \end{aligned}$$

using $|x_0-\hat{x}_0|^2 \geq 2|x_{0I}-\hat{x}_{0I}||x_{0Q}-\hat{x}_{0Q}|$, *we have*

$$\begin{aligned} &\geq 2^{1/3}[(x_{0I}-\hat{x}_{0I})(x_{0Q}-\hat{x}_{0Q})] \quad (A.7)\\ &\geq 2^{1/3}CPD \quad (A.8) \end{aligned}$$

As another example consider the construction for $N=5$. Using the rate 3/4 design (2.10), we have rate 3/4 non-square RFSDD for 5 transmit antennas which is given by

$$S = \begin{bmatrix} \Theta_4(x_{0I}+\mathbf{j}x_{3Q}, x_{1I}+\mathbf{j}x_{4Q}, x_{2I}+\mathbf{j}x_{5Q}) & 0_{4,1} \\ 0_{4,1} & \Theta_4(x_{3I}+\mathbf{j}x_{0Q}, x_{4I}+\mathbf{j}x_{1Q}, x_{5I}+\mathbf{j}x_{2Q}) \end{bmatrix}, \quad (A.9)$$

such that

$$S^{\mathcal{H}}S = \begin{bmatrix} \sum_{k=0}^{3}(x_{kI}^2+x_{k+3Q}^2) & 0 & 0 & 0 & 0 \\ 0 & \sum_{k=0}^{6}|x_k|^2 & 0 & 0 & 0 \\ 0 & & \sum_{k=0}^{6}|x_k|^2 & 0 & 0 \\ 0 & 0 & 0 & \sum_{k=0}^{6}|x_k|^2 & 0 \\ 0 & 0 & 0 & 0 & \sum_{k=0}^{3}(x_{kQ}^2+x_{k+3I}^2) \end{bmatrix}.$$

The improvement in coding gain is apparent for these cases. Next we show that the coding gain is greater than CPD for the general case.

Theorem A.1.2. *The coding gain of non-square RFSDDs of construction A.1.1 with the variables taking values from a signal set, is greater than CPD of the signal set.*

Proof. Consider $\hat{S}$ defined in (A.2), then

$$\hat{S}^{\mathcal{H}}\hat{S} = \begin{bmatrix} I_{N-n}(\sum_{k=0}^{K/2-1}|\tilde{x}_k|^2) & 0 & 0 \\ 0 & I_n(\sum_{k=0}^{K}|x_k|^2) & 0 \\ 0 & 0 & I_{N-n}(\sum_{k=K/2}^{K-1}|\tilde{x}_k|^2) \end{bmatrix} \quad \text{(A.10)}$$

where $\tilde{x}_i = \text{Re}\{x_i\} + \mathbf{j}\,\text{Im}\{x_{(i+K/2)_K}\}$ and where $(a)_K$ denotes $a \pmod K$. Observe that the total number of transmit antennas is $2N - n$.

Now consider the codeword difference matrix $B(\mathbf{S}, \mathbf{S}') = \mathbf{S} - \mathbf{S}'$ which is of full-rank for two distinct codeword matrices $\mathbf{S}, \mathbf{S}'$, we have

$$B(\mathbf{S},\mathbf{S}')^{\mathcal{H}}B(\mathbf{S},\mathbf{S}') = \begin{bmatrix} I_{N-n}(\sum_{k=0}^{K/2-1}|\tilde{x}_k - \tilde{x}'_k|^2) & 0 & 0 \\ 0 & I_n(\sum_{k=0}^{K}|x_k - x'_k|^2) & 0 \\ 0 & 0 & I_{N-n}(\sum_{k=K/2}^{K-1}|\tilde{x}_k - \tilde{x}'_k|^2) \end{bmatrix} \quad \text{(A.11)}$$

where at least one x_k differs from x'_k, $k = 0, \cdots, K-1$. Clearly, all the three terms in the determinant of the above matrix are minimum iff x_k differs from x'_k for only one k. Therefore assume, without loss of generality, that the codeword matrices $\mathbf{S}$ and $\mathbf{S}'$ are such that they differ by only one variable, say x_0 taking different values from the signal set $\mathcal{A}$.

Then, the coding gain is given by

$$\begin{aligned} \Lambda &= \min_{x_0 \neq x'_0} \det\left\{B^{\mathcal{H}}(\mathbf{S},\mathbf{S}')B(\mathbf{S},\mathbf{S}')\right\}^{\frac{1}{2N-n}} = |x_{0I} - {x'}_{0I}|^{\frac{2(N-n)}{2N-n}}|x_{0Q} - {x'}_{0Q}|^{\frac{2(N-n)}{2N-n}}|\nabla x_0|^{\frac{2n}{2N-n}} \\ &\geq 2^{\frac{n}{2N-n}} \min_{x_0 \neq x'_0} |x_{0I} - {x'}_{0I}||x_{0Q} - {x'}_{0Q}| \\ &= 2^{\frac{n}{2N-n}} CPD \end{aligned} \quad \text{(A.12)}$$

where we have used $|x_0 - \hat{x}_0|^2 \geq 2|x_{0I} - \hat{x}_{0I}||x_{0Q} - \hat{x}_{0Q}|$. The additional factor $2^{\frac{n}{2N-n}}$ is due to the additional power transmitted on n antennas as compared to GCIOD and on normalizing the transmission matrices, vanishes. □

Another important property of these non-square RFSDDs is that they have higher MMI as compared to the corresponding GCIOD.

Towards this end observe that the RFSDDs of construction A.1.1 consists of two ODs of size N that are separated in time. Proceeding as in Section 3.5 we have

$$C_F(2N-n, M, \rho) = \frac{K}{L} C(NM, 1, M\rho) \tag{A.13}$$

where C_F is the MMI of RFSDD and the number of transmit antennas is $2N - n$ and the number of receive antennas is M. Comparing with C_D in (3.68) it is easily seen that these codes have higher MMI as compared to the corresponding GCIODs.

A.2 Non-square RFSDDs from GCIODs

Here we present a construction of non-square RFSDDs from GCIODs whose coding gain depends on $GCPD_{N_1,N_2}(x, y)$ but the difference between x, y is smaller as compared to the corresponding GCIOD.

Construction A.2.1. *Let*

$$S(x_0, \cdots, x_{K-1}) = \begin{bmatrix} \hat{\Theta}_1(\tilde{x}_0, \cdots, \tilde{x}_{K/2-1}) & 0_{L_1,N_2} \\ 0_{L_2,N_1} & \hat{\Theta}_2(\tilde{x}_{K/2}, \cdots, \tilde{x}_{K-1}) \end{bmatrix} \tag{A.14}$$

be a $(L_1 + L_2) \times (N_1 + N_2)$ *GCIOD constructed in construction 3.1.3 where* $0_{L,N}$ *is the*

$L \times N$ zero matrix, then

$$\hat{S}(x_0, \cdots, x_{K-1}) = \begin{bmatrix} \hat{\Theta}_1(\tilde{x}_0, \cdots, \tilde{x}_{K/2-1}) & 0_{L_1, N_2 - N_1} \\ \hat{\Theta}_2(\tilde{x}_{K/2}, \cdots, \tilde{x}_{K-1}) & \end{bmatrix} \tag{A.15}$$

is a $L_1 + L_2 \times N_2$ non-square RFSDD for N_2 antennas, $N_2 > N_1$.

We illustrate the working of Construction A.2.1 by constructing another rate 2/3 GCIOD in the next example which has better coding gain than the GCIOD of example 3.1.4.

Example A.2.1. *For a given N, Let Θ_1 be the Alamouti code. Then $N_1 = 2$ and $N_2 = N$. Let Θ_2 be the rate 1/2 GLCODN transmit antennas (either using the construction of [13] or [19]). Then $r_2 = 1/2$. Then $\hat{\Theta}_1, \hat{\Theta}_2$ are constructed as in construction 3.1.3. Using the construction A.2.1, the rate of the RFSDD constructed is given by*

$$\mathcal{R} = \frac{2}{2+1} = \frac{2}{3}.$$

That the coding gain of the constructed RFSDD is greater than the coding gain of the GCIOD can be seen by the fact that first two entries of $\hat{S}^{\mathcal{H}}\hat{S}$ are $\sum_k |x_k|^2$.

A.2.1 Comparison of coding gains of ACIOD and OD

We first define a normalized the ACIOD, S, such that,

$$\hat{S}(x_0, \cdots, x_{K-1}) = a \begin{bmatrix} \Theta(\tilde{x}_0, \cdots, \tilde{x}_{K/2-1}) & 0_{L,N-n} \\ 0_{L,N-n} & \Theta(\tilde{x}_{K/2}, \cdots, \tilde{x}_{K-1}) \end{bmatrix} \begin{bmatrix} I_{N-n} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} I_n & 0 \\ 0 & 0 & I_{N-n} \end{bmatrix}, \tag{A.16}$$

where a is a constant for normalization. $\hat{S} \in \mathbb{C}^{2L \times 2N-n}$ has the property that $\mathrm{E}\left[\{\right] \hat{S}^{\mathcal{H}}\hat{S}\} = \frac{K}{2}I_{2N-n}$, where the K variables of $\hat{S}$ take values from a unit average energy constellation. Now the total average transmit power constraint is given by $\mathbb{E}\left\{\mathrm{tr}\left(\hat{S}^{\mathcal{H}}\hat{S}\right)\right\} = 2L$. Therefore normalizing the ACIOD, to satisfy the the total average transmit power constraint, we have $a = \sqrt{\frac{4L}{K(2N-n)}} = \sqrt{\frac{2}{R(2N-n)}}$, where R is the symbol rate of ACIOD (recall that $2N-n$ is the number of transmit antennas and $2L$ is the code length). For a bit rate of r bits/sec/Hz, the variables of ACIOD choose symbols from a $2^{r/R}$-QAM signal set, rotated to maximize the CPD. Following, Theorem A.1.2 the coding gain after normalization for a spectral efficiency of r bits/sec/Hz is given by,

$$\Lambda_{CIOD} = \frac{2}{R(2N-n)}\frac{d^2\left(2^{r/R}\right)}{\sqrt{5}} \tag{A.17}$$

where $d^2(M)$ is the minimum Euclidean distance of M-QAM of unit average energy. Note that for square QAM

$$d^2(M) = \frac{6}{M-1}.$$

For a OD, Θ in variables $x_0, \cdots, x_{K-1}$ of size N we have $\mathrm{E}\left[\{\right] \Theta^{\mathcal{H}}\Theta\} = KI_N$. Proceeding as for ACIOD, the coding gain of OD after normalization for a spectral efficiency of r bits/sec/Hz is given by,

$$\Lambda_{OD} = \frac{1}{RN}d^2\left(2^{r/R}\right) \tag{A.18}$$

where N is the number of transmit antennas and R is the rate of the OD.

Example A.2.2. *Consider for example three transmit antennas. For CIOD $N = 2, n = 1, R = 1$, using (A.17), the coding gain using rotated QAM for a spectral efficiency of r bits/sec/Hz is*

$$\Lambda_{CIOD} = \frac{2}{3}\frac{d^2\left(2^r\right)}{\sqrt{5}}. \tag{A.19}$$

For OD $N = 3, R = 3/4$, using (A.18), the coding gain using QAM for a spectral efficiency of r bits/sec/Hz is

$$\Lambda_{OD} = \frac{4}{9}d^2\left(2^{4r/3}\right). \tag{A.20}$$

Table A.1: Comparison of coding gains of ACIOD and OD for $N = 3$

spectral efficiency (bits/sec/Hz)	Λ_{CIOD}	Λ_{OD}
6	*0.0284*	*0.0105*
12	4.36×10^{-4}	4.06×10^{-5}
18	6.28×10^{-6}	1.59×10^{-7}

Table A.1 gives comparison of coding gains for spectral efficiency of $6i$ *bits/sec/Hz such that both OD and CIOD use square QAM.*

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