E1 244: Homework - 1

Date assigned: 13 Jan 2010

1 Topics

- Bayesian estimation
- Classical estimation

2 Problems

1. Show that, if X is a non-negative random variable, $\mathbf{E}(X) = \int_0^\infty Pr(X > t)\,dt = \int_0^\infty (1 - F(t))\,dt$.

2. Suppose that Θ is a random parameter uniformly distributed on the interval $[0, 1]$. Noisy observations

$$Y_k = X_k + \Theta, \qquad k = 1, 2, \ldots, n,$$

are made where $\{X_k\}$ are i.i.d. and independent of Θ. The common marginal pdf of the $\{X_k\}$ is

$$f_X(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

   (a) Find the minimum mean squared error estimate of Θ.
   (b) Find the maximum aposteriori probability estimate of Θ.
   (c) Find the minimum absolute error estimate of Θ.
   (d) What happens to these estimators as $n \to \infty$?

3. **Bayesian sufficiency** Under the Bayesian formulation, the parameter θ is random, $\Theta \sim p_\Theta(\theta)$ where $p_\Theta(\theta)$ is the prior distribution. The statistical model $p_\theta(x)$ is given by

$$p_\theta(x) \triangleq p_{X|\Theta}(x|\theta) = \frac{p_{X,\Theta}(x,\theta)}{p_\Theta(\theta)}$$

A statistic $t(X)$ is Bayesian sufficient, if, for whatever the prior distribution $p_\Theta(\theta)$, the posterior distribution

$$f_{\Theta|X}(\theta|x) = \frac{p_{X|\Theta}(x|\theta)p_\Theta(\theta)}{p_X(x)}$$

involves data only through $t(x)$. Show that the Bayesian sufficiency is equivalent to the classical notion of sufficiency.

4. Suppose that $Z_1, Z_2, \ldots, Z_m$ are i.i.d. samples from a uniform distribution on the interval $(0, \theta)$, $0 < \theta < \infty$. Let $Y_1, Y_2, \ldots, Y_m$ be the order statistics for the sample. Prove that Y_m is a sufficient statistic. (Hint: use the factorization theorem.)

5. Suppose that $\theta > 0$ is a parameter of interest and that given θ, $\{Y_k, 1 \leq k \leq n\}$ is a set of i.i.d. observations with marginal distribution function

$$F_\theta(y) = [F(y)]^{1/\theta}, y \in \mathbf{R}$$

where F is a known distribution function with pdf f.

(a) Show that

$$\hat{\theta}_{MV}(y) = -\frac{1}{n}\sum_{k=1}^{n} \log F(y_k)$$

is a minimum variance unbiased estimate of θ.

(b) Suppose now that θ is replaced by a random variable Θ drawn at random using the prior density

$$w(\theta) = c^m \frac{e^{-c/\theta}}{\Gamma(m)\theta^{m+1}}, \quad \theta > 0$$

where c and m are positive constants. Use the fact that $E(\Theta) = c/(m-1)$ to show that the MMSE estimator of Θ from $\mathbf{Y}$ is

$$\hat{\theta}_{MMSE}(\mathbf{y}) = \frac{c - \sum_{k=1}^{n} \log F(y_k)}{m+n-1}.$$

(c) Compare $\hat{\theta}_{MV}$ and $\hat{\theta}_{MMSE}$ with regard to the role of prior information.

6. Let $\mathbf{Y} = [Y_1, Y_2, \ldots, Y_n]^T$ be a random vector where the individual components are i.i.d Poisson random variables with parameter θ.

(a) Show that $T(\mathbf{Y}) = Y_1 + Y_2 + \cdots + Y_n$ is a complete sufficient statistic for θ. You must explain why it is sufficient and why it is complete.

(b) Show that $T(\mathbf{Y})$ is also Poisson. Start with $n = 2$ and use induction. What is the parameter for the distribution of $T(\mathbf{Y})$?

(c) For any (fixed) integer $k \geq 0$ find an MVUE of the probability

$$P_\theta\{Y_1 = k\}.$$

(d) For any (fixed) integer $k \geq 0$ find the ML estimator of the probability

$$P_\theta\{Y_1 = k\}.$$

Is this ML estimator unbiased?