E1 244: Homework - 10

Assigned: 07 Apr 2011

1 Topics

- Orthonormal functions and KL expansion
- Mercer's theorem, KL theorem and consequences

2 Problems

1. When the observation interval is infinite and the processes of interest are wide-sense stationary and have band-limited power spectra, then it is sometimes convenient to use a sampled representation of the process. Let X_t be a random process with

$$\begin{aligned} \mathrm{E}\{X_t\} &= 0 \\ \mathrm{E}\{X_t X_s\} &= R_X(t-s) \\ S_X(f) &= \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau \end{aligned}$$

where $S_X(f) = 0$ for all $|f| \geq W$.

   (a) Prove

$$X_t = \text{l.i.m.}_{K\to\infty} \sum_{i=-K}^{K} X_{i/(2W)} \frac{\sin(2\pi W(t - i/(2W)))}{2\pi W(t - i/(2W))}.$$

   (b) Compute $\mathrm{E}\{X_{i/(2W)} X_{j/(2W)}\}$ for the triangular power spectral density

$$S_X(f) = \begin{cases} M\left(1 - \frac{|f|}{W}\right) & |f| \leq W \\ 0 & |f| > W \end{cases}.$$

2. Let $C_X(t,u), 0 \leq t, u \leq T$ be the covariance function of a random process $X_t, 0 \leq t \leq T$ with finite energy.

   (a) Show that the eigenvalues of C_X are all positive if and only if C_X is positive definite.

(b) Show that the eigenfunctions of C_X with positive eigenvalues form a C.O.N.S. only if C_X is positive definite.

3. Prove the following representation theorem for the Weiner process. Let $\{\phi_k(t)\}_{k=1}^{\infty}$ be any C.O.N.S. of functions on $[0, T]$. Then the Weiner process $\{W_t : 0 \le t \le T\}$ has the mean-square convergent sum representation

$$W_t = \sum_{k=1}^{\infty} \tilde{W}_k \int_0^t \phi_k(u)du, \quad 0 \le t \le T,$$

where $\tilde{W}_k = \int_0^T \phi_k(t)dW_t, \quad k = 1, 2, \ldots$.

4. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of finite variance random variables defined on a common probability space. If there exists a finite variance random variable X, defined on the same probability space, such that for every $\epsilon > 0$ there exists an integer N such that

$$n > N \Rightarrow \mathrm{E}\{|X_n - X|^2\} \le \epsilon$$

then we say that $\{X_n\}_{n=1}^{\infty}$ converges to X in mean square and write $\text{l.i.m.}_{n \to \infty} X_n = X$. An important property of the space of finite variance random variables is that a necessary and sufficient condition for there to exist a mean-square limit as above is the Cauchy criterion

$$\lim_{n,m \to \infty} \mathrm{E}\{|X_n - X_m|^2\} = 0.$$

Show that another necessary and sufficient condition for the mean-square convergence is the Loève criterion that the limit

$$\lim_{n,m \to \infty} \mathrm{E}\{X_n X_m^*\}$$

exists (finite).