E1 244: Homework - 2

Assigned: 2 Feb 2011

1 Topics

- MVUE, ML and MMSE estimation

2 Problems

1. Consider the observation model

$$Z = \frac{1}{\theta} + V$$

where V is a Gaussian random variable with zero mean and unit variance. Let $\psi = g(\theta) = 1/\theta$. In this problem we consider the estimation of the unknown parameters θ and ψ.

   (a) Assume that θ is a nonrandom parameter.

      i. Find the ML estimator $\hat{\psi}_{ML}$ of ψ.

      ii. Find the ML estimator $\hat{\theta}_{ML}$ os θ.

   (b) Assume that θ is a realization of a random parameter Θ with probability density

$$p_\Theta(\theta) = \frac{1}{\theta^2\sqrt{2\pi}} \exp\{-\frac{1}{2\theta^2}\} \quad \theta \neq 0.$$

   Assume also that Θ and V are independent.

      i. Find the maximum a posteriori estimator $\hat{\Phi}_{MAP}$ of $\Phi = g(\Theta)$.

      ii. Find the maximum a posteriori estimator $\hat{\Theta}_{MAP}$ of Θ.

2. This problem refers back to Problem 6 of HW1. Recall that $\mathbf{Y} = [Y_1 Y_2 \cdots Y_n]^T$ was a random vector where the individual components were i.i.d. Poisson random variables with parameter θ. Previously, you showed that $T(\mathbf{Y}) = Y_1 + Y_2 + \cdots + Y_n$ is a complete sufficient statistic for θ, that $T(\mathbf{Y})$ is Poisson with parameter $n\theta$, and you found the MVUE and ML estimators for

$$\phi = g(\theta) = P_\theta\{Y_1 = 0\} = e^{-\theta}.$$

(Actually, a slightly more general result was found before.) These estimators were (for $n > 1$)

$$\begin{aligned} \hat{\phi}_{MVUE} &= \left(\frac{n-1}{n}\right)^{T(\mathbf{y})} \\ \hat{\phi}_{ML} &= e^{-T(\mathbf{y})/n}. \end{aligned} \qquad (1)$$

   (a) Solve part (d) of HW1, Problem 6, if you have not yet done so.

   (b) Find the CRLB for estimating θ based upon the n observations.

   (c) The ML estimate for θ is $\hat{\theta}_{ML}(\mathbf{y}) = T(\mathbf{y})/n$. Is $\hat{\theta}_{ML}(\mathbf{y})$ unbiased? Is it efficient? (Note: an estimator is *efficient* if it achieves the CRB.) Is it an MVUE?

(d) Now consider estimation of $\phi = e^{-\theta}$. Why can we say that $\hat{\phi}_{ML}$ is biased without calculation? Calculate the bias directly. From your result show that the ML estimator is asymptotically ($n \to \infty$) unbiased.

(e) Directly calculate the variance of $\hat{\phi}_{ML}$ as a function of θ and n and argue that the estimator is consistent.

(f) Find the CRLB for estimating $\phi = e^{-\theta}$ based upon the n observations and express it as a function of θ.

(g) Does the unbiased estimator $\hat{\phi}_{MVUE}$ meet the CRLB for finite n? Answer the question without calculation.

(h) Now calculate the variance of $\hat{\phi}_{MVUE}$ expressed as a function of θ and n. Form the ratio

$$\frac{\mathrm{Var}_\theta\{\hat{\phi}_{MVUE}(\mathbf{Y})\}}{\text{the CRLB for } \phi}$$

and directly explore the efficiency question of (f).

3. Prove the factorization criterion for a sufficient statistic in the case of discrete distributions. Attempt to generalize your proof to the case of continuous distributions assuming whatever regularity conditions you need.

4. Srinath et al., Problem 5.5. We have N independent observations, $z_i, i = 1, \ldots, N$ of a Gaussian variable with mean m (known) and variance σ^2, which is unknown. Obtain $\hat{\sigma}^2_{\mathrm{ML}}$. Is this estimate unbiased? Is this efficient? What is the variance of this estimate?

5. Srinath et al., Problem 5.13. Let $X_1, \ldots, X_n$ be a random sample from a density that is uniform on $(\theta - \frac{1}{2}, \theta + \frac{1}{2})$. Note that θ is a location parameter and that $X_i - \theta$ is uniformly distributed on $(-\frac{1}{2}, \frac{1}{2})$. Let $Y_1, \ldots, Y_n$ denote the corresponding interval for θ. Find the confidence coefficient for this interval.

6. Srinath et al., Problem 5.15. Let X_1, X_2 denote a random sample of size 2 from a $\mathcal{N}(\theta, 1)$ distribution. let Y_1, Y_2 be the corresponding ordered sample. Determine γ where $P(Y_1 < \theta < Y_2) = \gamma$. Find the expected length of the interval (Y_1, Y_2).