E1 244: Homework - 3

1 Topics

1. MVUE, ML and MMSE estimation
2. Cramér-Rao bound, extensions
3. Asymptotic properties of ML estimation

Note: the problems below are from

- Jun Shao, "Mathematical statistics", 2nd Edition, Springer.
- H. Vincent Poor, "An introduction to signal detection and estimation", 2nd Edition, Springer.

2 Problems

1. Let $X_1, \ldots, X_n$ be i.i.d. binary random variables with $P(X_i = 1) = p \in (0, 1)$.
   (a) Find an MVUE of p.
   (b) Find an MVUE of $P(X_1 + X_2 + \ldots + X_n = k)$ for some $k \leq n$.
2. Let $X_1, \ldots, X_n$ be i.i.d. having the $\mathcal{N}(\mu, \sigma^2)$ with an unknown $\mu \in \mathcal{R}$ and a known $\sigma^2 > 0$.
   (a) Find an MVUE of μ and μ^3.
   (b) Find an MVUE of $P(X_1 \leq t)$ for a fixed $t \in \mathcal{R}$.
3. Let $X_1, \ldots, X_n$ be i.i.d. having the poisson distribution $P(\theta)$ with $\theta > 0$. Find the asymptotic mean square error of the MVUE of $e^{t\theta}$ with a fixed $t > 0$ and show that it meets the asymptotic Cramèr-Rao lower bound.
4. Let $X_1, \ldots, X_n$ be i.i.d. having the $\mathcal{N}(\mu, \sigma^2)$ with an unknown $\mu \in \mathcal{R}$ and a known $\sigma^2 > 0$.
   (a) Find an MVUE of $v = e^{t\mu}$ with a fixed $t \neq 0$.

(b) Determine whether the variance of the MVUE attains the Cramér-Rao lower bound.

(c) Show that it meets an asymptotic Cramér-Rao lower bound.

5. Let $X_1, \ldots, X_n$ be i.i.d. having the exponential distribution $E_{(a,\theta)}(x) = \theta^{-1} \exp(-(x - a)/\theta) 1_{x \geq a}$ with parameters $\theta > 0$ and $a \in \mathcal{R}$.

   (a) Find an MVUE of a when θ is known.

   (b) Find an MVUE of θ when a is known.

   (c) Find the MVUE's of θ and a, when both parameters are unknown.

   (d) Assuming that θ is known, find the MVUE of $P(X_1 \geq t)$ for some fixed $t > 0$.

6. Let X be a sample from P_θ. Find the Fisher information matrix in the following cases:

   (a) P_θ is the $\mathcal{N}(\mu, \sigma^2)$ distribution with $\theta = \mu \in \mathcal{R}$.

   (b) P_θ is the $\mathcal{N}(\mu, \sigma^2)$ distribution with $\theta = \sigma^2 > 0$.

   (c) P_θ is the $\mathcal{N}(\mu, \sigma^2)$ distribution with $\theta = \sigma > 0$.

   (d) P_θ is the $\mathcal{N}(\sigma, \sigma^2)$ distribution with $\theta = \sigma > 0$.

   In all cases, assume that the unspecified parameters are "known".

7. Suppose we observe a sequence $Y_1, Y_2, \ldots, Y_n$ given by

$$Y_k = N_k + \theta s_k, \quad k = 1, \ldots, n$$

   where $\underline{N} = (N_1, N_2, \ldots, N_n)^T$ is a zero mean Gaussian random vector with covariance matrix $\Sigma > 0$; $s_1, s_2, \ldots, s_n$ is a known signal sequence; and θ is a real non-random parameter.

   (a) Find the maximum-likelihood estimate of parameter θ.

   (b) Compute the bias and variance of your estimate.

   (c) Compute the Cramér-Rao lower bound for unbiased estimates of θ and compare result from (b).

   (d) What can be said about the consistency of $\hat{\theta}_{ML}$ as $n \to \infty$? Suppose, for example, that there are positive constants a and b such that

$$\frac{1}{n} \sum_{k=1}^{n} s_k^2 > a \quad \text{for all } n$$

   and

$$\lambda_{\min}(\Sigma^{-1}) > b \quad \text{for all } n.$$

8. Consider the observation model

$$Y_k = \theta^{1/2} s_k R_k + N_k \quad k = 1, \ldots, n$$

where $s_1, s_2, \ldots, s_n$ is a known signal, $N_1, N_2, \ldots, N_n, R_1, R_2, \ldots, R_n$ are i.i.d. $\mathcal{N}(0,1)$ random variables, and $\theta \geq 0$ is an unknown parameter.

   (a) Find the likelihood equation for estimating θ from $Y_1, Y_2, \ldots, Y_n$.

   (b) Find the Cramér-Rao lower bound on the variance of unbiased estimates of θ.

   (c) Suppose $s_1, s_2, \ldots, s_n$ is a sequence of $+1$'s -1's. Find the MLE of θ explicitly.

   (d) Compute the bias and variance from estimate (c), and compare the latter with the Cramér-Rao lower bound.

9. A formal definition of the consistency of an estimator is given as follows. An estimator $\hat{\theta}$ is consistent if, given any $\epsilon > 0$,

$$\lim_{N \to \infty} \Pr\{|\hat{\theta} - \theta| > \epsilon\} = 0.$$

Prove that the sample mean is consistent estimator for the problem of estimating a DC level A in white Gaussian noise of known variance. Hint: Use Chebyshev's inequality.

10. If the data set

$$x[n] = As[n] + w[n] \quad n = 0, 1, \ldots, N-1$$

is observed, where $s[n]$ is known and $w[n]$ is white Gaussian noise with known variance σ^2, find the MLE of A. Determine the PDF of the MLE and whether or not the asymptotic normality theorem holds.

11. For N i.i.d. observations from $\mathcal{N}(0, 1/\theta)$ PDF, where $\theta > 0$, find the MLE of θ and its asymptotic PDF.