E1 244: Homework - 4

Assigned: 03 Mar. 2011

1 Topics

- Kalman Filtering

2 Problems

1. Consider a sequence $\{X_k\}_{k=0}^{\infty}$ of binary random variables, each taking on the values 0 or 1. Suppose that this sequence has the Markov property

   $$P\{X_k = x_k | X_0 = x_0, \ldots, X_{k-1} = x_{k-1}\} = P\{X_k = x_k | X_{k-1} = x_{k-1}\} \triangleq p_{x_k, x_{k-1}}$$

   for all integers $k \geq 1$, and for all binary sequences $\{x_k\}_{k=0}^{\infty}$. Consider the observation model

   $$Y_k = X_k + N_k, \quad k = 0, 1, 2, \ldots$$

   where $\{N_k\}_{k=0}^{\infty}$ is an i.i.d. sequence, independent of $\{X_k\}_{k=0}^{\infty}$, and having the common marginal probability density function f. For each integer $k \geq 0$, let $\hat{X}_{k/k}$ denote the MMSE estimate of X_k given measurements $\{Y_0, Y_1, \ldots, Y_k\}$, and let $\hat{X}_{k/k-1}$ denote the MMSE estimate of X_k given the measurements $\{Y_0, Y_1, \ldots, Y_{k-1}\}$. Define

   $$\hat{X}_{0/-1} = E\{X_0\} = P\{X_0 = 1\}.$$

   Show that $\hat{X}_{k/k}$ and $\hat{X}_{k/k-1}$ satisfy the joint recursions

   $$\begin{aligned} \hat{X}_{k/k} &= \frac{\hat{X}_{k/k-1} f(y_k - 1)}{\hat{X}_{k/k-1} f(y_k - 1) + (1 - \hat{X}_{k/k-1}) f(y_k)} \\ \hat{X}_{k+1/k} &= p_{1,1} \hat{X}_{k/k} + p_{1,0}(1 - \hat{X}_{k/k}) \end{aligned}$$

   for $k \geq 0$.

2. Suppose that the steady state equation in the Kalman filter model is modified as follows:

   $$X_{k+1} = F_k X_k + G_k W_k + \Gamma_k s_k \qquad k \geq 0$$

   where s_k is a known sequence of vectors (a control input) and Γ_k is a known sequence of matrices of appropriate dimension.

   (a) Find the appropriate modification of the Kalman recursions.

   (b) Repeat where each s_k is allowed to be a function of the past measurement, i.e., of $Z(k)$ (feedback control).

3. Let $L(\cdot)$ be a scalar function with $L(0) = 0, L(y) \geq L(z)$ for $\|y\| \geq \|z\|$, $L(y) = L(-y)$, and with $L(\cdot)$ convex. Let $P_{X|Y}(x|y)$ be symmetric about $\hat{x} = E\{X|Y = y\}$. Prove that for all z

   $$E\{L(X - \hat{x}|Y = y)\} \leq E\{L(X - z)|Y = y\}.$$

4. With $P_k = E\{X_k X_k^T\}$ and assuming the standard Kalman filter setup from class show that

$$P_{k+1} - \Sigma_{k+1|k} \geq 0$$

and interpret the result.

5. Let P_k be a sequence of nonnegative definite matrices such that for some nonnegative definite matrix P and for all k

$$P \geq P_{k+1} \geq P_k,$$

Show that the limit $\lim_{k\to\infty} P_k$ exists.

6. Consider the following Bayesian estimation problem. Observations of the following form are made of a random variable X

$$Y_k = X + (1 + \alpha D)V_k, \quad 1 \leq k \leq N,$$

where $\alpha > 0$, and the statistical assumptions are that:

- $X \sim \mathcal{N}(\mu_x, \sigma_x^2)$ and the measurement noise is i.i.d. with $V_k \sim \mathcal{N}(0, \sigma_v^2)$.
- D is a zero-one random variable with $P\{D = 1\} = q$.
- The random variables X, D, and $\{V_k\}_{k=1}^{\infty}$ are statistically independent.

The model is for a measurement scenario where two sensors are available to make measurements of a quantity. One sensor is "good" (variance σ_v^2) and the other sensor is "bad" (variance $(1+\alpha)^2\sigma_v^2$). We do not know apriori which sensor has been used.

(a) The minimum mean-squared error estimator of X given $\underline{Y} = [Y_1, Y_2, \ldots, Y_N]$ is of the form

$$E\{X|\underline{Y} = \underline{y}\} = C_0(\underline{y})L_0(\underline{y}) + C_1(\underline{y})L_1(\underline{y})$$

where the $L_i(\underline{y})$ are linear functions and the coefficients $C_i(\underline{y})$ are nonlinear functions.

i. Explain how the formula above comes about. (Hint: condition on the value of D.)
ii. Find explicit formulas for $L_0(\underline{y})$ and $L_1(\underline{y})$.
iii. Demonstrate how to get the coefficients $C_0(\underline{y})$ and $C_1(\underline{y})$ but don't carry out the calculations.

(b) Derive the complete formula for the minimum mean-squared error *linear* estimator of X given $\underline{Y}$.