E1 244: Homework - 6

Assigned: 30 Mar. 2011

1 Topics

- Bayesian detection
- NP Testing

2 Problems

1. Find the Bayes rule and minimum Bayes risk for a binary H.T. with uniform costs and equal priors, with

$$p_0(y) = \begin{cases} c_0 y^2 & |y| \leq 1 \\ 0 & \text{else} \end{cases}$$
$$p_1(y) = \begin{cases} c_1(3 - |y|) & |y| \leq 3 \\ 0 & \text{else} \end{cases}$$

   where c_0 and c_1 are constants.

2. A communication system uses binary signaling by sending one of the following two signals: $s_1(t)$ to send one, and $s_0(t)$ to send zero, where $s_1(t) = -As(t), s_0(t) = As(t), A > 0$, and

$$s(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

   The transmitted signal is corrupted by zero mean white Gaussian noise $n(t)$ with autocorrelation function $R_n(\tau) = \sigma^2\delta(\tau)$, so that the received signal is given by

$$y(t) = \begin{cases} -As(t) + n(t) & 1 \text{ sent} \\ As(t) + n(t) & 0 \text{ sent} \end{cases}$$

   The receiver computes the decision statistic

$$Z = \int_{-\infty}^{\infty} y(t)s(t)dt$$

The receiver would like to decide whether 0 or 1 was sent, or (if it is not confident about a 0/1 decision), erase the signal. We would like to obtain a Bayesian decision rule for doing this as follows.

Let $\Theta = \{0, 1\}$ and $\mathcal{A} = \{0, 1, e\}$. The observation is Z . Assume the cost structure

$$C(a, \theta) \quad = \quad \begin{cases} 1 & a \neq \theta, a = 0, 1 \\ 0 & a = \theta \\ c & a = e \end{cases}$$

where $\theta = 0, 1$ and where $c \in (0, 1)$ is the cost of an erasure.

(a) Find the Bayes rule for equal priors. Simplify the form of the rule as much as possible and specify its dependence on the problem parameters c, A, and σ^2.

(b) For $d = A/\sigma = 5$, find c and a corresponding Bayesian decision rule δ_A so that the probability of erasure p_1 is twice the probability of error p_2. Compare with the values of p_1 and p_2 for a Bayesian decision rule δ_B corresponding to $c = 1/2$.

3. **NP Testing.** Consider the hypothesis testing problem

$$\begin{aligned} H_0 &: \quad r_i = n_i, \quad i = 1, \ldots, N, (N \text{ odd}) \\ H_1 &: \quad r_i = m + n_i, \end{aligned}$$

and let $\mathbf{r} \triangleq [r_1, \ldots, r_N]^T$.

(a) Find the Log-Likelihood Ratio (LLR), defined as

$$\text{LLR} = \log \left(\frac{p(\mathbf{R}|H_1)}{p(\mathbf{R}|H_0)} \right)$$

(b) (LLR as a random variable.) Find $p(\text{LLR }|H_0)$ and $p(\text{LLR }|H_1)$. Plot.

(c) **Matlab.** Define $\delta(r_i)$ as

$$\delta(r_i) = \begin{cases} 1 & r_i \geq \eta \\ 0 & \text{otherwise} \end{cases}$$

And let $\delta^{(1)}(\mathbf{r})$ be

$$\delta^{(1)}(\mathbf{r}) = \begin{cases} 1 & \sum_{i=1}^{N} \delta(r_i) \geq (N-1)/2 \\ 0 & \text{otherwise} \end{cases}$$

That is, $\delta^{(1)}(\mathbf{r})$ takes a *majority vote* among $\delta(r_i)$. Find P_F, P_D and plot the ROC.

(d) Find the α-level NP test and plot its ROC. Compare with (d).