E1 244: Homework - 8

Assigned: 06 Apr 2011

1 Topics

- UMP, LMP and Generalized LRTs
- The general Gaussian problem

2 Problems

1. **Srinath, Pb. 3.21** Let $X_1, X_2, \ldots, X_n$ be a random sample from the Poisson distribution

$$P_\theta(x) = \frac{e^{-\theta}\theta^x}{x!} \quad x = 0, 1, 2, \ldots$$

   Let $\theta_0 > 0$ be a specific value of θ.

   (a) Find the UMP test for deciding between $\mathcal{H}_0 : \theta = \theta_0$ vs. $\mathcal{H}_1 : \theta > \theta_0$.

   (b) Find the α-level LMP test for testing $\mathcal{H}_0 : \theta = \theta_0$ vs. $\mathcal{H}_1 : \theta \neq \theta_0$.

   (c) Find the generalized LRT for testing $\mathcal{H}_0 : \theta = \theta_0$ vs. $\mathcal{H}_1 : \theta \neq \theta_0$.

2. **Srinath, Pb. 3.27** Consider the following composite hypothesis-testing problem. The observations are

$$\begin{aligned} \mathcal{H}_1 &: Z_k = Y_k + V_k \quad k = 1, 2, \ldots, K \\ \mathcal{H}_0 &: Z_k = V_k \quad\quad\quad k = 1, 2, \ldots, K \end{aligned}$$

   with Y_k and V_k i.i.d. random variables with densities $\mathcal{N}(0, \sigma_y^2)$ and $\mathcal{N}(0, 1)$, where σ_y is unknown.

   (a) Does a UMP test exist?

   (b) If a UMP test does not exist, find a generalized LRT.

3. Suppose that the observations $y \in \mathcal{Z}$ are modeled by a family of probability densities $\{p_\theta : \theta \in \Lambda\}$ where Λ is an interval in the real line. We consider the test of a simple hypothesis $\mathcal{H}_0 : \theta = \theta_0$ against a composite two-sided alternative $\mathcal{H}_1 : \theta \neq \theta_0$. Since UMP tests for this type of problem usually do not exist we consider LMP tests. We also consider only nonrandomized tests for simplicity. Recall that the detection probability of a test δ is

$$P_D(\delta, \theta) = \int_{\mathcal{Z}_1} p_\theta(y)dy, \ \theta \in \Lambda$$

   where $\mathcal{Z}_1 \subset \mathcal{Z}$ is the critical region of δ. The LMP α-level test of this type of problem can be shown to be equivalent to the following optimization problem: find the region $\mathcal{Z}_1$ such that

$$\begin{aligned} P_D(\delta, \theta_0) &= \int_{\mathcal{Z}_1} p_{\theta_0}(y)dy = \alpha \\ P_D'(\delta, \theta_0) &= \int_{\mathcal{Z}_1} p_{\theta_0}'(y)dy = 0 \\ P_D''(\delta, \theta_0) &= \int_{\mathcal{Z}_1} p_{\theta_0}''(y)dy \text{ is maximum.} \end{aligned}$$

In the above, the derivatives are all w.r.t. θ, i.e.,

$$P'_D(\delta, \theta_0) = \left.\frac{\partial P_D(\delta, \theta_0)}{\partial \theta}\right|_{\theta=\theta_0} \quad \text{and } P'_{\theta_0}(y) = \left.\frac{\partial p_\theta(y)}{\partial \theta}\right|_{\theta=\theta_0},$$

and similarly for P''_D and $p''_\theta(y)$.

(a) Assuming whatever regularity conditions you need, give the argument leading to the optimization problem above. We require that the resulting test be locally unbiased, i.e., $P_D(\delta, \theta) \geq P_D(\delta, \theta_0)$ in the neighborhood of θ_0.

(b) Solve the optimization problem in the sense of specifying the form of the optimal critical region $\mathcal{Z}_1^*$.

(c) If p_θ is the family $\mathcal{N}(\theta, \sigma^2)$ where the variance is assumed known, show that the two-sided LMP test amounts to comparing $(y - \theta_0)^2$ to a threshold. For a given level α, compute the threshold and the resulting detection probability.

4. **Srinath, Pb. 3.16** Consider the following binary hypothesis-testing problem:

$$\mathcal{H}_0 : \mathbf{Z} = \mathbf{m}_0 + \mathbf{V} \quad \text{vs.} \quad \mathcal{H}_1 : \mathbf{Z} = \mathbf{m}_1 + \mathbf{V}$$

where $\mathbf{Z} = [Z_1\ Z_2]^T, \mathbf{m}_0 = [m_{10}\ m_{20}]^T, \mathbf{m}_1 = [m_{11}\ m_{21}]^T$, and $V = [V_1\ V_2]^T$, with V_1 and V_2 being independent Gaussian random variables with mean zero and variance σ^2.

(a) Find the minimum probability of error receiver for equally likely hypotheses if the vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ are known constants.

(b) How does the test simplify if $\mathbf{m}_1 = [1\ 0]^T$ and $\mathbf{m}_2 = [0\ 1]^T$.

(c) If $\mathbf{m}_1 = [1\ 0]^T$ and you were allowed to choose a unit-norm $\mathbf{m}_2$, which vector would you recommend?

5. **Srinath, Pb. 3.18** Consider the following binary hypothesis-testing problem:

$$\mathcal{H}_0 : \mathbf{Z} = \mathbf{m}_0 + \mathbf{V} \quad \text{vs.} \quad \mathcal{H}_1 : \mathbf{Z} = \mathbf{m}_1 + \mathbf{V}$$

where $\mathbf{Z}, \mathbf{m}_0, \mathbf{m}_1$ and $\mathbf{V}$ are vectors of dimension N and $\mathbf{V}$ is Gaussian with mean zero and covariance matrix Σ. For equally likely hypotheses and a minimum probability of error criterion, find the optimum decision rule. Find the receiver structure and the probability of error for the case $\Sigma = \sigma^2 I$.