E1 244: Homework - 9

Assigned: 08 Apr 2011

1 Topics

- Detection of signals in noise

2 Problems

1. Consider the M-ary decision problem in $\mathbf{R}^n$:

$$\begin{array}{ll} \mathcal{H}_0 : & \mathbf{y} = \mathbf{s}_0 + \mathbf{n} \\ \mathcal{H}_1 : & \mathbf{y} = \mathbf{s}_1 + \mathbf{n} \\ & \vdots \\ \mathcal{H}_{M-1} & \mathbf{y} = \mathbf{s}_{M-1} + \mathbf{n} \end{array}$$

   where the known signal vectors $\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_{M-1}$ have equal energies

$$\|\mathbf{s}_0\|^2 = \|\mathbf{s}_1\| = \ldots \|\mathbf{s}_{M-1}\|^2.$$

   (a) Assuming $\mathbf{n} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$, find the decision rule achieving minimum probability of error when all hypotheses are equally likely.

   (b) Assuming further that the signals are orthogonal, show that the minimum error probability is given by

$$P_e = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\Phi(x)]^{M-1} e^{-(x-d)^2/2} dx$$

   where $d = \|\mathbf{s}_0\|^2/\sigma^2$.

2. **Srinath et. al., Problem 4.3:** One of two equally likely, equal-energy ($= E$) orthogonal signals is transmitted over an interval of $[0, T]$, over an AWGN channel of spectral density $N_0/2$. A minimum probability of error receiver is to be designed for detecting the signals.

   (a) Find the probability of error in terms of E and N_0.

Due to a fault in the transmitter, when the signal to be transmitted is $y_i(t)$, $i = 1, 2$, the signal that is actually transmitted is found to be

$$y_{a_i}(t) = \epsilon y_j(t) + (1-\epsilon) y_i(t), \quad j \neq i, \quad 0 < t \leq T$$

Let us assume that it is known that the transmitter is faulty and that the receiver is designed for the actual transmitted signals $y_{a_i}(t)$.

(b) Find the energies in the actual transmitted signals $y_{a_i}(t), i = 1, 2$, and the correlation coefficient between these two signals.

(c) Draw the block diagram of the minimum probability of error receiver. ind the probability of error in terms of E, ϵ, and N_0. What happens when $\epsilon = 0.5$?

3. **Srinath et. al., Problem 4.4:** One of two equally likely, equal-energy, orthogonal signals is transmitted over an interval of $[0, T]$, over an AWGn channel of spectral density $N_0/2$. A minimum probability of error receiver is to be designed for detecting the signals. Assume that $T = 1$ and that the two orthogonal signals are:

$$y_1(t) = \begin{cases} 1 & 0 < t \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad y_2(t) = \begin{cases} -1 & 0 < t \leq 1/2 \\ 1 & 1/2 < t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the receiver structure and write down the expression for the probability of error.

(b) Due to equipment failure, the actual transmission of each symbol does not last for the entire signaling interval of 1 second, but ends abruptly τ seconds earlier than scheduled, where $\tau\,(0 < \tau < 1)$ is assumed to be known. If the receiver of part (a) is used, calculate the probability of error as a function of τ.

(c) Assume that the receiver is designed based on the knowledge that each symbol signal is nonzero for $(1-\tau)$ seconds and is zero after that. Calculate the probability of error for this receiver as a function of τ.

(d) Compare your answers in parts (b) and (c). Are the results what you might reasonably expect?