

E2 212: Homework - 4

1 Topics

- Eigenvalues and Eigenvectors

2 Problems

1. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $\mathbf{A} \in \mathbb{C}^{n \times n}$. Then, prove the following:
 - (a) $\text{tr } \mathbf{A} = \sum_{i=1}^n \lambda_i$.
 - (b) $\det(\mathbf{A}) = \prod_{i=1}^n \lambda_i$.
2. Let the eigenvectors and eigenvalues of the matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ be \mathbf{v}_i and λ_i such that $\lambda_i \neq \lambda_j$ for all $j \neq i, i = 1, 2, \dots, r$. Then, prove that $\mathbf{v}_1, \dots, \mathbf{v}_r$ is a linearly independent set.
3. Let the subspace $U \subseteq \mathbb{C}^{n \times n}$ be an invariant subspace under the matrix transformation $\mathbf{A} \in \mathbb{C}^{n \times n}$, i.e., $\mathbf{A}\mathbf{u} \subseteq U$ for all $\mathbf{u} \in U$. Then, prove the following
 - (a) There exists a vector $\mathbf{u} \in U$ and $\lambda \in \mathbb{C}$ such that $\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$.
 - (b) If $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_k$, then $\dim(U) \geq k$.
 - (c) The subspace $U^\perp \triangleq \{\mathbf{v} \in V : \langle \mathbf{u}, \mathbf{v} \rangle = 0, \text{ for all } \mathbf{u} \in U\}$ is also an invariant subspace with respect \mathbf{A} .
4. Let $\sigma(\mathbf{A})$ denote the spectrum of the matrix \mathbf{A} . Show that for a triangular matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ with entries a_{ij} , $\sigma(\mathbf{A}) = \bigcup_{i=1}^n \{a_{ii}\}$.
5. Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ has n distinct eigenvalues in \mathbb{C} . Then, prove that there exists an invertible matrix $U \in \mathbb{C}^{n \times n}$ and a diagonal matrix D such that

$$\exp\{\mathbf{A}\} = UDU^{-1}.$$

6. Let $\mathbf{A} \in \mathbb{C}^{n \times n}$, $\lambda \in \mathbb{C}$. Prove that if $(\mathbf{A} - \lambda I_n)^j \mathbf{u} = 0$ for some $j \geq 1$, and $\mathbf{u} \in \mathbb{C}^n$, then $(\mathbf{A} - \lambda I_n)^n \mathbf{u} = 0$.
7. Let $\lambda_1, \dots, \lambda_n$ (not necessarily distinct) be the eigenvalues of $\mathbf{A} \in \mathbb{C}^{n \times n}$. Then,

$$\sum_{i=1}^n |\lambda_i|^2 \leq \sum_{i,j=1}^n |a_{ij}|^2.$$

8. Let $\mathbf{A} \in \mathbb{C}^{n \times n}$, and $\beta \triangleq \max\{|a_{ij}| : i, j = 1, 2, \dots, n\}$. Then,

$$|\det \mathbf{A}| \leq \beta^n n^{n/2}.$$