E2 212: Homework - 5

1 Topics

- Schur's theorem
- Unitary equivalence
- Normal matrices
- QR decomposition and Jordan form

2 Problems

1. Prove that for every matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$, and for every $\epsilon > 0$, there exists a matrix $\mathbf{A}_\epsilon$ such that $\mathbf{A}_\epsilon$ is equivalent to a diagonal matrix, and $\|\mathbf{A} - \mathbf{A}_\epsilon\|_F < \epsilon$. *Hint:* Use Schur's theorem.

2. Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n \times n}$ be given, and suppose $\mathbf{A}$ and $\mathbf{B}$ are simultaneously similar to upper triangular matrices: that is, $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ and $\mathbf{S}^{-1}\mathbf{B}\mathbf{S}$ are both upper triangular for some nonsingular $\mathbf{S}$. Show that every eigenvalue of $\mathbf{AB} - \mathbf{BA}$ must be zero.

3. If $\mathbf{A} \in \mathbb{C}^{n \times n}$, show that rank of $\mathbf{A}$ is not less than the number of nonzero eigenvalues of $\mathbf{A}$.

4. Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ be a nonsingular matrix. Show that any matrix that commutes with A also commutes with $\mathbf{A}^{-1}$.

5. If $\mathbf{A} \in \mathbb{C}^{n \times n}$ is normal, and if $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors corresponding to distinct eigenvalues, show that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

6. Show that a normal matrix is unitary if and only if all its eigenvalues have absolute value 1.

7. Show that a given matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ is normal if and only if

$$(\mathbf{A}\mathbf{x})^H(\mathbf{A}\mathbf{y}) = (\mathbf{A}^H\mathbf{x})^H(\mathbf{A}^H\mathbf{y}).$$

8. Let $n_1, n_2, \ldots, n_k$ be given positive integers and let $A_j \in M_{n_j}$, $j = 1, 2, \ldots, k$. Show that the direct sum $\mathbf{A}_1 \oplus \ldots \oplus \mathbf{A}_k$ is normal if and only if each $\mathbf{A}_j$ is normal.

9. Show that each $\mathbf{A}_k$ produced by the QR algorithm is unitarily equivalent to $\mathbf{A}_0$, $k = 1, 2, \ldots$.

10. Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ be given. Define

$$K_\lambda \triangleq \{\mathbf{x} \in \mathbb{C}^n : (\mathbf{A} - \lambda I)^p \mathbf{x} = 0 \text{ for some integer } p > 0\}.$$

Prove that

(a) If λ is an eigenvalue of $\mathbf{A}$, then K_λ is a $\mathbf{A}-$invariant subspace of $\mathbb{C}^n$, i.e., $\mathbf{A}\mathbf{y} \subseteq K_\lambda$ for all $\mathbf{y} \in K_\lambda$.

(b) If λ is an eigenvalue of $\mathbf{A}$ with multiplicity m, then $\dim(K_\lambda) \leq m$.

11. Find an invertible matrix $\mathbf{U}$ such that $\mathbf{U}^{-1}\mathbf{A}\mathbf{U}$ is in Jordan form when

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & i \\ 0 & 2 & 0 \\ -i & 0 & 1 \end{bmatrix},$$

where $i = \sqrt{-1}$.

12. Find the Jordan form of the following matrices

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -4 & 2 \\ -i & 0 & 1 \end{bmatrix}.$$