E2 212: Homework - 6

1 Topics

- **LU factorization, Triangular factorizations and linear equations**
- **Hermitian matrices, Rayleigh quotient**

2 Problems

1. (Applications of LU factorization): Suggest efficient algorithms to solve the following system of equations: Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be non-singular.

   (a) $\mathbf{AX} = \mathbf{B}$, where $\mathbf{X}, \mathbf{B}$ are $n \times k$.

   (b) $\mathbf{A}^k \mathbf{x} = \mathbf{b}$. (The idea is to avoid matrix multiplications in computing $\mathbf{A}^k$ explicitly).

2. Give an algorithm for computing a non-zero $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{Ux} = 0$ where $\mathbf{U} \in \mathbb{R}^{n \times n}$ is upper triangular with $u_{nn} = 0$ and $u_{ii} \neq 0$ for all $i = 1, 2, \ldots, n-1$.

3. (Matrix forms of elementary row operations) Let $\mathbf{x}$ be such that $\mathbf{x}_k \neq 0$. Write down the matrix $\mathbf{M}$ that when multiplied with $\mathbf{x}$ produces zeros on components $[k+1 : n]$. Verify that $\mathbf{M}$ can be written as $\mathbf{M} = \mathbf{I} - \mathbf{t}\mathbf{e}_k^T$ and find the value of vector $\mathbf{t}$. In the above, $\mathbf{e}_k$ is the k^{th} standard basis vector. What would be the form of $\mathbf{M}^{-1}$?

4. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$

   $$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

   Let $\mathbf{A}_{11}$ be $r \times r$ and non-singular. Show that if $\mathbf{A}_{11}$ has an LU factorization without pivoting, then after r-steps of Gaussian elimination without pivoting on $\mathbf{A}$, $\mathbf{A}(r+1 : n, r+1 : n)$ will contain the Schur's complement of $\mathbf{A}_{11}$, defined by $\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}$.

5. Is the following statement correct: The multipliers for Gaussian elimination of matrix $\mathbf{A}^T\mathbf{A}$ are identical to the multipliers that orthogonalize the columns of $\mathbf{A}$. Why/ Why not ?

6. Show that

   (a) The inverse of an upper triangular matrix is upper triangular.

   (b) The product of two lower triangular matrices is lower triangular.

   (c) The inverse of a unit upper triangular matrix is unit upper triangular.

   (d) The product of two unit lower triangular matrices is unit lower triangular.

7. If $\mathbf{A}$ is diagonalizable and $f(\cdot)$ is a polynomial, show that $f(\mathbf{A})$ is diagonalizable.

8. Show that any 2×2 real symmetric matrix is diagonalizable.

9. If $\mathbf{A} \in \mathbb{R}^{n \times n}$ is both normal ($\mathbf{AA}^H = \mathbf{A}^H\mathbf{A}$) and nilpotent ($\mathbf{A}^k = 0$, for some k), show that $\mathbf{A} = 0$.

10. Let $\mathbf{A}, \mathbf{B}$ be $n \times n$ Hermitian matrices. Show that $\mathbf{A}$ and $\mathbf{B}$ are similar if and only if they are unitarily similar.

11. Let $\mathbf{B} \in \mathbb{C}^{n \times n}$ be skew-hermitian, i.e., $\mathbf{B}^H = -\mathbf{B}$. Then,

    (a) Show that all the eigenvalues of $\mathbf{B}$ are purely imaginary.

    (b) Show that the matrix exponential $e^{\mathbf{B}}$ is unitary.

12. Let $\mathbf{A}$ be Hermitian. Show that the rank of $\mathbf{A}$ is equal to the number of non-zero eigenvalues of $\mathbf{A}$. Give an example of non-Hermitian matrix where this is not true.

13. Let $\mathbf{A}$ be Hermitian and let $\mathbf{A}$ be non-singular. Show that:

$$\text{rank}(A) \geq \frac{[\text{tr}\mathbf{A}]^2}{\text{tr}\mathbf{A}^2}.$$

    When will the equality be achieved ?

14. Prove that if $\mathbf{A}$ is Hermitian then $\mathbf{A}^k$ is also Hermitian for all $k \geq 1$. If $\mathbf{A}$ is non-singular as well then $\mathbf{A}^{-1}$ is also Hermitian.

15. Prove that all the diagonal elements of a Hermitian matrix lie between the maximum and the minimum eigenvalue, i.e. For a Hermitian $n \times n$ matrix $\mathbf{A}$ show that: $\lambda_{min} \leq a_{ii} \leq \lambda_{max}$ for all $i = 1, 2, \ldots, n$.

16. If $\mathbf{A} = [a_{ij}]$ is a positive definite Hermitian symmetric matrix, show that

    (a) $a_{ii} > 0$ for all i,

    (b) $a_{ii}a_{jj} > |a_{ij}|^2$ for all $i \neq j$.