E2 212: Homework - 7

1 Topics

- **Variational characterization of eigenvalues**
- **Congruent matrices**

2 Problems

1. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be a Hermitian matrix, let $\mathbf{x} \in \mathbb{C}^n$ be a given nonzero vector, and let $\alpha \triangleq \mathbf{x}^H\mathbf{A}\mathbf{x}/\mathbf{x}^H\mathbf{x}$. Show that there exists at least one eigenvalue of $\mathbf{A}$ in the interval $(-\infty, \alpha]$ and at least one in $[\alpha, \infty)$.

2. If $\mathbf{A} \in \mathbb{C}^{n\times n}$ be a Hermitian matrix. Show that the following three optimization problems all have the same solution:

   (a) $\max_{\mathbf{x}^H\mathbf{x}=1} \mathbf{x}^H\mathbf{A}\mathbf{x}$

   (b) $\max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^H\mathbf{A}\mathbf{x}}{\mathbf{x}^H\mathbf{x}}$

   (c) $\min_{\mathbf{x}^H\mathbf{A}\mathbf{x}=1} \mathbf{x}^H\mathbf{x}$

3. Show that, if λ_i is any eigenvalue of $\mathbf{A} \in \mathbb{C}^{n\times n}$ (not necessarily Hermitian), then, one has the bounds

$$\min_{\mathbf{x}\neq 0}\left|\frac{\mathbf{x}^H\mathbf{A}\mathbf{x}}{\mathbf{x}^H\mathbf{x}}\right| \leq |\lambda_i| \leq \max_{\mathbf{x}\neq 0}\left|\frac{\mathbf{x}^H\mathbf{A}\mathbf{x}}{\mathbf{x}^H\mathbf{x}}\right|, i = 1, 2, \ldots, n. \tag{1}$$

4. If $\mathbf{A} \in \mathbb{C}^{n\times n}$ is Hermitian and if $\mathbf{x}^H\mathbf{A}\mathbf{x} \geq 0$ for all $\mathbf{x}$ in a k-dimensional subspace, then prove that $\mathbf{A}$ has at least k nonnegative eigenvalues.

5. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\mathbf{B} \in \mathbb{C}^{n\times n}$ be two Hermitian matrices. Prove that

$$|\lambda_k(\mathbf{A}+\mathbf{B}) - \lambda_k(\mathbf{A})| \leq \rho(\mathbf{B})$$

   for all $k = 1, 2, \ldots, n$, where $\rho(\mathbf{B})$ is the spectral radius of $\mathbf{B}$, and where the eigenvalues of $\mathbf{A}$ and $\mathbf{A}+\mathbf{B}$ are arranged, as usual, in increasing order.

6. If $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\mathbf{B} \in \mathbb{C}^{n\times n}$ are Hermitian matrices with eigenvalues arranged in increasing order, and if $1 \leq k \leq n$, show that

$$\lambda_k(\mathbf{A}+\mathbf{B}) \leq \min\{\lambda_i(\mathbf{A}) + \lambda_j(\mathbf{B}) : i + j = k + n\}.$$

7. If $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\mathbf{B} \in \mathbb{C}^{n\times n}$ are Hermitian matrices with all the eigenvalues of $\mathbf{A}-\mathbf{B}$ being nonnegative, then prove that $\lambda_i(\mathbf{A}) \geq \lambda_i(\mathbf{B})$ for all $i = 1, 2, \ldots, n$, where the eigenvalues of both matrices are arranged in increasing order, as usual.

8. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be Hermitian. Show that $\mathbf{A}$ is congruent to the identity matrix if and only if all the eigenvalues of $\mathbf{A}$ are positive.

9. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\mathbf{B} \in \mathbb{C}^{n\times n}$ be Hermitian matrices and assume that $\mathbf{A}$ is positive definite. Then prove that

(a) There exists a non singular matrix $\mathbf{S} \in \mathbb{C}^{n \times n}$ and a real diagonal $\mathbf{D}$ such that $\mathbf{S}^H \mathbf{A}\mathbf{S} = \mathbf{I}$, and $\mathbf{S}^H \mathbf{B}\mathbf{S} = \mathbf{D}$.

(b) The diagonal entries of $\mathbf{D}$ are the eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$, and the columns of $\mathbf{S}$ are the eigenvectors of $\mathbf{A}^{-1}\mathbf{B}$ corresponding to these eigenvalues.

10. Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ have entries $a_{ij} = 1$ for all $i \neq j$, and $a_{ii} = n$ for each i. Show that $\mathbf{A}$ is invertible.