Solutions to problems in quiz-1

1. If A is $m \times k$ and B is $k \times n$, then prove that: $rank(A) + rank(B) - k \leq rank(AB) \leq min\{rank(A), rank(B)\}$.

   Solution: (a) We have showed that if $A \in R^{m\times n}$ and $B \in R^{n\times p}$ then, $rank(AB) = rank(B) - dim\{N(A) \bigcap R(B) - -(1)$. We shall use this result to prove the given inequality.

   We have from from Eq. (1), $rank(AB) \leq rank(B) - -(2)$.

   To show that, $rank(AB) \leq rank(A)$

   We know that transposing does not alter the rank of a matrix. Therefore, $rank(AB) = rank(AB)^T = rank(B^T A^T)$.

   Now, using Eqs. (1) and (2), $rank(B^T A^T) \leq rank(A^T) = rank(A)$.

   Hence, $rank(AB) {\leq} min\{rank(A), rank(B)\}$.

   (b) To show $rank(A) + rank(B) - k \leq rank(AB) - -(*)$,

   We have $N(A) \bigcap R(A) \subseteq N(A)$. Also, if P and Q are two subspaces such that $P \subseteq Q$, $dim(P) \leq dim(Q)$.

   Thus, $dim\{N(A) \bigcap R(A)\} \leq dim\{N(A)\} = n - rank(A) - -(3)$, from rank-nullity theorem.

   Using Eq. (3) in Eq. (1) gives $(*)$.

2. Show that rank of a matrix is unaltered by pre or post multiplication by a non-singular matrix of appropriate dimensions.

   Solution: Let $A \in R^{m\times n}$, $P \in R^{m\times m}$ and $Q \in R^{n\times n}$, We need to show that $rank(PA) = rank(AQ) = rank(A)$.

   Conisder $rank(PA) = rank(A) - dim(\{N(P) \bigcap R(A)\}) - -(4)$, from Eq.(1).

   Now, Since P is non-singular, $N(P) = \{0\}$and$\{N(P) \bigcap R(A)\} = \{0\}$. Thus, $dim(\{N(P) \bigcap R(A)\}) = 0$ which implies $rank(PA) = rank(A)$ from Eq. (4).

   To prove $rank(AQ) = rank(A)$,

   Conisder $rank(AQ) = rank(Q) - dim(\{N(A) \bigcap R(Q)\}) - -(5)$

   Since Q is also non-singular, $R(Q) = R^n$ and hence $rank(Q) = n$ and $dim(\{N(A) \bigcap R(Q)\}) = dim\{N(A)\} = n - rank(A)$ because, $\{N(A) \bigcap R(Q)\} = N(A)$.

   Using the above facts in Eq. (5),

   $rank(AQ) = n - (n - rank(A)) = rank(A)$.

3. Show that, for any A and b, one and only one of the following systems has a solution: (1) $Ax = b$ (2) $A^T y = 0, y^T b \neq 0$.

   Solution: Proof by contradiction.

   Suppose both the systems have solutions.

   That is, $Ax = b$ and $A^T y = 0$ for $y^T b \neq 0$.

   Since $A^T y = 0$, we have

   $y^T A = 0$

   post-multiplying the above equation by x on both sides,

   $y^T Ax = 0$

   $\Rightarrow y^T b = 0$ which is a contradiction since we have assumed $y^T b \neq 0$. Hence the proof.