E2 212: Matrix Theory Fall 2012 – Test 1

Rules

1. This exam counts for 30% of your final grade.
2. It is a closed book exam. There is no need for a calculator as there are no complex numerical computations involved. Please leave your cell-phones turned off, preferably in your bag.
3. Please indicate your final answer clearly (for example, by drawing a box around it) and provide justifications. In your proofs, if you want to use a result stated and proved in class, please clearly state the result you plan to use. Avoid mixing up answers to different questions, making it hard to evaluate.
4. The points assigned to a problem does not necessarily indicate its difficulty level. Generally more "basic" questions carry more points.
5. **Code of conduct:** Cheating is, needless to say, strictly forbidden and will be taken extremely seriously, so don't even think about it. I reserve the right to immediately terminate your exam at any point in time if I suspect you of cheating. So, it is your responsibility to conduct yourself in a manner that is completely above board and beyond suspicion.

Problems

1. **(Vector spaces)**
   (a) If $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ are linearly independent, then are $\mathbf{x}+\mathbf{y}$, $\mathbf{y}+\mathbf{z}$ and $\mathbf{z}+\mathbf{x}$ linearly independent? (3 points)
   (b) If $\mathcal{U}$ and $\mathcal{V}$ are finite dimensional subspaces of a vector space, then show that (3 points)
   $$\dim(\mathcal{U}) + \dim(\mathcal{V}) = \dim(\mathcal{U}+\mathcal{V}) + \dim(\mathcal{U}\cap\mathcal{V})$$
2. Let $\mathbf{X}$ be partitioned as $\mathbf{X} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$, where $\mathbf{A}$ and $\mathbf{D}$ are square matrices of possibly different dimensions.

(a) If $\mathbf{X}$ is invertible and all the relevant inverses exist, show that (3 points)

$$\mathbf{X}^{-1} = \begin{bmatrix} (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & \mathbf{A}^{-1}\mathbf{B}(\mathbf{C}\mathbf{A}^{-1}\mathbf{B} - \mathbf{D})^{-1} \\ (\mathbf{C}\mathbf{A}^{-1}\mathbf{B} - \mathbf{D})^{-1}\mathbf{C}\mathbf{A}^{-1} & (\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \end{bmatrix}$$

(b) If either $\mathbf{B} = \mathbf{0}$ or $\mathbf{C} = \mathbf{0}$, show that $\det\mathbf{X} = (\det\mathbf{A})(\det\mathbf{D})$. (3 points)

3. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ with $m \geqslant n$, $\mathbf{y} \in \mathbb{R}^n$, and define $\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A} \\ \mathbf{y}^T \end{bmatrix}$. Show that $|||\mathbf{A}|||_2 \leqslant |||\tilde{\mathbf{A}}|||_2 \leqslant \sqrt{|||\mathbf{A}|||_2^2 + \|\mathbf{y}\|_2^2}$. (6 points)

4. Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$, and suppose that at least one of $\mathbf{A}$ or $\mathbf{B}$ is nonsingular. Show that

(a) $\mathbf{AB}$ and $\mathbf{BA}$ have the same eigenvalues, counting multiplicities. (3 points)

(b) If $\mathbf{AB}$ is diagonalizable, then $\mathbf{BA}$ is diagonalizable. (3 points)

5. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\mathbf{A} \in \mathbf{C}^{n\times n}$. Then, prove the following:

(a) $\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$. (3 points)

(b) $\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$. (3 points)