E2 312: Homework - 1

Assigned on: 10 Aug. 2015, due 24 Aug. 2015

1 Topics

- Some review problems in linear algebra
- Wishart matrices, their distribution, and properties

2 Problems

1. Let $\mathbf{v}_1, \ldots, \mathbf{v}_N$ be given. Define $\mathbf{w}_1, \ldots, \mathbf{w}_N$ as $\mathbf{w}_1 = \mathbf{v}_1$, and

$$\mathbf{w}_i = \mathbf{v}_i - \sum_{j=1}^{i-1} \mathbf{w}_j \frac{\mathbf{v}_i^T \mathbf{w}_j}{\mathbf{w}_j^T \mathbf{v}_j}, \quad i = 2, \ldots, N.$$

   Assume $\mathbf{w}_i \neq \mathbf{0}, i = 1, 2, \ldots, N$. Show that $\mathbf{w}_k$ is orthogonal to $\mathbf{w}_i$, $k < i$.

2. Given a positive definite matrix $\mathbf{R}$ and a real symmetric matrix Θ of the same size as $\mathbf{R}$, show that there exists a real nonsingular matrix $\mathbf{B}$ such that $\mathbf{B}^T \mathbf{R}^{-1} \mathbf{B} = \mathbf{I}$ and $\mathbf{B}^T \Theta \mathbf{B} = \mathbf{D}$, where $\mathbf{D}$ is a real diagonal matrix.
   (*Hint:* Theorem A.2.2 in Anderson's textbook.)

3. Show that the characteristic function of the chi-squared distribution with 1 degree of freedom, denoted by $\chi^2(1)$, is given by $1/\sqrt{1-2is}$.
   (*Hint:* We want to find $\mathbb{E}(\exp(isY^2))$ where $Y \sim \mathcal{N}(0,1)$. Expand the exponential as a power series and evaluate the expectation term-by-term.)

4. Let $\mathbf{x}_1, \ldots \mathbf{x}_{n+1} \in \mathbb{R}^N$ be i.i.d. $\sim \mathcal{N}(\mu, \mathbf{R})$. Let $\bar{\mathbf{x}} \triangleq \frac{1}{n+1} \sum_{i=1}^{n+1} \mathbf{x}_i$, and $\mathbf{A} \triangleq \sum_{i=1}^{n+1} \mathbf{x}_i \mathbf{x}_i^T$. Show that $\bar{\mathbf{x}}$ and $\mathbf{A}$ are independently distributed, and show that $\mathbf{A}$ is distributed as $\sum_{i=1}^{n} \mathbf{z}_i \mathbf{z}_i^T$ where $\mathbf{z}_1, \ldots, \mathbf{z}_n \in \mathbb{R}^N$ are i.i.d. $\sim \mathcal{N}(0, \mathbf{R})$. What is the pdf of $\bar{\mathbf{x}}$?

5. (Anderson, Pb. 7.1) A transformation from rectangular to *polar* coordinates is

$$\begin{aligned}
y_1 &= w \sin\theta_1, \\
y_2 &= w \cos\theta_1 \sin\theta_2, \\
y_3 &= w \cos\theta_1 \cos\theta_2 \sin\theta_3, \\
&\vdots \\
y_{n-1} &= w \cos\theta_1 \cos\theta_2 \cdots \cos\theta_{n-2} \sin\theta_{n-1}, \\
y_n &= w \cos\theta_1 \cos\theta_2 \cdots \cos\theta_{n-2} \cos\theta_{n-1},
\end{aligned}$$

   where $-\frac{1}{2}\pi < \theta_i \leq \frac{1}{2}\pi$, $i = 1, \ldots, n-2$, $-\pi < \theta_{n-1} \leq \pi$, and $0 \leq w < \infty$.

(a) Prove $w^2 = \sum_{i=1}^n y_i^2$.

(b) Show that the Jacobian is $w^{n-1}\cos^{n-2}\theta_1\cos^{n-3}\theta_2\cdots\cos\theta_{n-2}$

6. (Anderson, Pb. 7.2) Show that

$$\int_{-\pi/2}^{\pi/2}\cos^{h-1}\theta d\theta = \frac{\Gamma(\frac{1}{2}h)\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}(h+1))}.$$

7. (Anderson, Pb. 7.3) Use Problems 5 and 6 to prove that the surface area of a sphere of unit radius in n dimensions is

$$C(n) = \frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}.$$

8. (Anderson, Pb. 7.4) Use Problems 5, 6 and 7 to show if the density of $\mathbf{y}^T = (y_1, \ldots, y_n)$ is $f(\mathbf{y}^T\mathbf{y})$, then the density of $u = \mathbf{y}^T\mathbf{y}$ is $\frac{1}{2}C(n)f(u)u^{\frac{1}{2}n-1}$.

9. (Anderson, Pb. 7.5) Use Problem 8 to show that if $y_1, \ldots y_n$ are i.i.d. $\sim \mathcal{N}(0,1)$, then $U = \sum_{i=1}^n y_i^2$ has the density $u^{\frac{1}{2}n-1}e^{-\frac{1}{2}u}/[2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)]$.