E9 203: Homework - 2

Assigned on: 28 Jan. 2015, due 09 Feb. 2015

1 Topics

- Review of norms and related concepts
- Uniqueness and uncertainty
- Basic algorithms

2 Problems

1. Show that the solution to
$$\min_{\mathbf{x}} J(\mathbf{x}) \triangleq \|B\mathbf{x}\|_2^2 \text{ s. t. } A\mathbf{x} = \mathbf{b},$$
where $A \in \mathbb{R}^{m \times N}, B \in \mathbb{R}^{N \times N}, \mathbf{x} \in \mathbb{R}^N, \mathbf{b} \in \mathbb{R}^m$ and when $B^T B$ is invertible, is given by
$$\hat{\mathbf{x}} = (B^T B)^{-1} A^T (A (B^T B)^{-1} A^T)^{-1} \mathbf{b}.$$

2. Given $A \in \mathbb{C}^{m \times N}$ and $\tau > 0$, show that the solution of
$$\min_{\mathbf{z} \in \mathbb{C}^N} \|A\mathbf{z} - \mathbf{y}\|_2^2 + \tau \|\mathbf{z}\|_2^2$$
is given by
$$z^{\#} = \left(A^H A + \tau \mathbf{I}\right)^{-1} A^H \mathbf{y}.$$

3. Assume $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $A = I + \mathbf{u}\mathbf{v}^T$. Show that if A is nonsingular, then $A^{-1} = I + \alpha \mathbf{u}\mathbf{v}^T$, for some scalar α. Find the corresponding α.

4. (Golub and Van Loan, P2.2.2) Prove the Cauchy-Schwartz inequality
$$\left|\mathbf{x}^T \mathbf{y}\right| \leq \|\mathbf{x}\|_2 \|\mathbf{y}\|_2. \tag{1}$$
Hint: Use the inequality $(a\mathbf{x} + b\mathbf{y})^T (a\mathbf{x} + b\mathbf{y}) \geq 0$ for suitable scalars a and b.

5. (Golub and Van Loan, P2.2.4) Show that, if $\mathbf{x} \in \mathbb{R}^n$,
$$\begin{aligned} \|\mathbf{x}\|_2 &\leq \|\mathbf{x}\|_1 \leq \sqrt{n} \|\mathbf{x}\|_2 \\ \|\mathbf{x}\|_\infty &\leq \|\mathbf{x}\|_2 \leq \sqrt{n} \|\mathbf{x}\|_\infty \\ \|\mathbf{x}\|_\infty &\leq \|\mathbf{x}\|_1 \leq n \|\mathbf{x}\|_\infty \end{aligned}$$
When is the equality attained?

6. (Golub and Van Loan, P2.2.8) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and define $\psi : \mathbb{R} \to \mathbb{R}$ by $\psi(\alpha) \triangleq \|\mathbf{x} - \alpha\mathbf{y}\|_2$. Show that ψ is minimized when $\alpha = \mathbf{x}^T\mathbf{y}/\mathbf{y}^T\mathbf{y}$.

7. Given $\mathbf{x} \in \mathbb{R}_+^N$ with $\mathbf{x}_1 \geq \mathbf{x}_2 \geq \cdots \geq \mathbf{x}_N \geq 0$, show that for each $1 \leq s \leq N$ and $r > 1$

$$f(\mathbf{x}) \triangleq \sum_{j=s}^{N} \mathbf{x}_j^r$$

is a convex function of $\mathbf{x}$.

8. (Determinant of a Vandermonde Matrix) The *Vandermonde matrix* associated with $x_0, x_1, \ldots, x_N \in \mathbb{C}$ is defined as

$$\mathbf{V} \triangleq \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^N \\ 1 & x_1 & x_1^2 & \cdots & x_1^N \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^N \end{bmatrix}.$$

Show that the determinant of $\mathbf{V}$ equals

$$\det\mathbf{V} = \prod_{0 \leq k < l \leq N} (x_l - x_k).$$

9. (Foucart and Rauhut, Thm. 2.13) Given $A \in \mathbb{C}^{m\times N}$, the following statements are equivalent:

   (a) Every s-sparse vector $\mathbf{x} \in \mathbb{C}^N$ is the unique s-sparse solution of $A\mathbf{z} = A\mathbf{x}$, that is, if $A\mathbf{x} = A\mathbf{z}$ and both $\mathbf{x}$ and $\mathbf{z}$ are s-sparse, then $\mathbf{x} = \mathbf{z}$.

   (b) The null space $\mathcal{N}(A)$ does not contain any $2s$-sparse vector other than the zero vector, that is,

   $$\mathcal{N}(A) \Cup \{\mathbf{z} \in \mathcal{C}^N : \|\mathbf{z}\|_0 \leq 2s\} = \{\mathbf{0}\}.$$

   (c) For every $S \in [N]$ with $\mathrm{card}(S) \leq 2s$, the submatrix A_S is injective as a map from $\mathbb{C}^S$ to $\mathbb{C}^m$.

   (d) Every set of $2s$ columns of A is linearly independent.

10. (Foucart and Rauhut, Ex. 3.1) Let $q > 1$ and let $A \in \mathbb{C}^{m\times N}$ with $m < N$. Prove that there exists a 1-sparse vector that is *not* a minimizer of

$$\min_{\mathbf{z}\in\mathbb{C}^N} \|\mathbf{z}\|_q \text{ subject to } A\mathbf{z} = \mathbf{y}.$$

11. (Foucart and Rauhut, Ex. 3.2) Given $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$, show that the vector $\mathbf{x} = \left[1, e^{i2\pi/3}, e^{i4\pi/3}\right]^T$ is the unique solution to

$$\min_{\mathbf{z}\in\mathbb{C}^N} \|\mathbf{z}\|_1 \text{ subject to } Az = A\mathbf{x}.$$

This shows that, in the complex setting, a unique ℓ_1 minimizer is not necessarily m-sparse, where m is the number or rows of A.