E9 203: Homework - 4

Assigned on: 02 Mar. 2015; does not need to be turned in.

1 Topics

- Reweighted algorithms
- Sparse Bayesian learning

2 Problems

Notation: $\mathbf{x} \in \mathbb{R}^N, \mathbf{y} \in \mathbb{R}^m, \mathbf{A} \in \mathbb{R}^{m \times N}$. x_i is the i^{th} entry of a vector $\mathbf{x}$.

1. Suppose $g(x)$ is a monotonically increasing, strictly concave function of $x \in \mathbb{R}$ for $x \geq 0$, and that $g(0)$ is bounded. Let $\mathbf{x} = \mathbf{y} - \mathbf{z}$ for some $\mathbf{y} \geq 0, \mathbf{z} \geq 0$, both in $\mathbb{R}^N$. Show that there exists a one-to-one mapping between the local minima of the following two problems, and that the objective functions coincide at the corresponding local minima:
$$\min_{\mathbf{x}} \sum_{i=1}^{N} g(|x_i|) \text{ s.t. } \mathbf{y} = \mathbf{A}\mathbf{x}$$
and
$$\min_{\mathbf{y},\mathbf{z}} \sum_{i=1}^{N} \left(g(|y_i|) + g(|z_i|) - g(0)\right) \text{ s.t. } \mathbf{y} = [\mathbf{A} \; -\mathbf{A}] \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix}, \mathbf{y} \geq 0, \mathbf{z} \geq 0.$$

2. Show that
$$\lim_{p \to 0} \frac{1}{p} \sum_{i=1}^{N} (|x_i|^p - 1) = \sum_{i=1}^{N} \log x_i.$$

3. Show that
$$\log(|x_i| + \epsilon) \leq \frac{x_i^2}{\gamma} + \log\left(\frac{(\epsilon^2 + 2\gamma_i)^{\frac{1}{2}} + \epsilon}{2}\right) - \frac{\left[(\epsilon^2 + 2\gamma_i)^{\frac{1}{2}} - \epsilon\right]^2}{4\gamma_i}$$
for all $\epsilon, \gamma_i > 0$, with equality iff $\gamma_i = x_i^2 + \epsilon |x_i|$. Use this to derive an iteratively reweighted ℓ_2 algorithm for recovery of $\mathbf{x}$ from $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$.

4. If $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$, where $\mathbf{n}$ is Gaussian with zero mean and covariance matrix $\Sigma_n \in \mathbb{R}^{m \times m}$, $\mathbf{x}$ is Gaussian with zero mean and covariance Γ, derive the conditional distribution $p(\mathbf{x}|\mathbf{y})$. What if Σ_n is rank deficient?

5. Show that
$$g_{\text{SBL}}(\mathbf{x}) \triangleq \min_{\gamma \geq 0} \mathbf{x}^T \Gamma^{-1} \mathbf{x} + \log\det\left(\sigma^2 \mathbf{I} + \mathbf{A}\Gamma\mathbf{A}^T\right)$$
(where $\Gamma = \text{diag}\gamma$) is a nondecreasing, concave function of $|\mathbf{x}| = [|x_1|, |x_2|, \ldots, |x_N|]^T$.