Modulo Compressed Sensing

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Agenda

Introduction

Outline

Compressed Sensing

- Sub-Nyquist sampling approach for sparse signals
- Sparse vector $\mathbf{x} \in \mathbb{R}^N$:

$$\|\mathbf{x}\|_0 = |\text{supp}(\mathbf{x})| = |\mathcal{S}| \leq K \ll N$$

- Recovery of a sparse vector from underdetermined linear system

$$\mathbf{y} = \mathbf{A}\mathbf{x} \in \mathbb{R}^M (M < N)$$

- Plethora of algorithms
- Assumption: Infinite precision

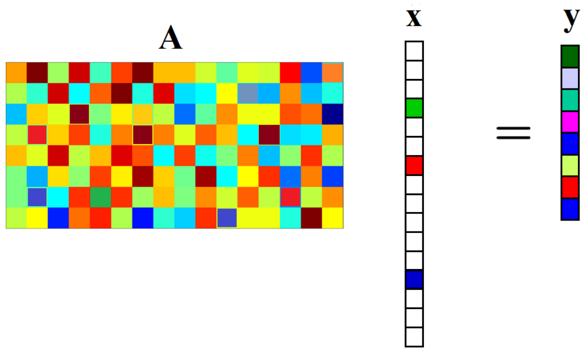
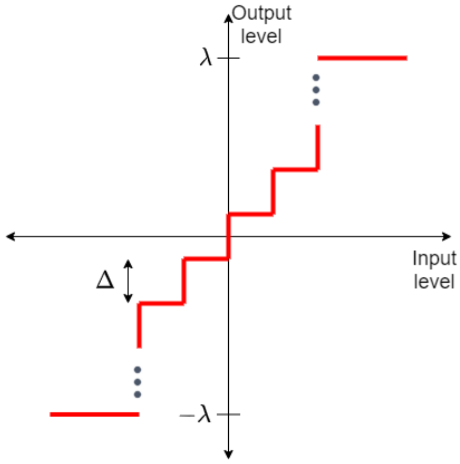


Figure: Underdetermined linear measurement[a]

[a] Figure Source: http://informationtransfereconomics.blogspot.com/2017/10/compressed-sensing-and-information.html

Analog-to-digital-converters



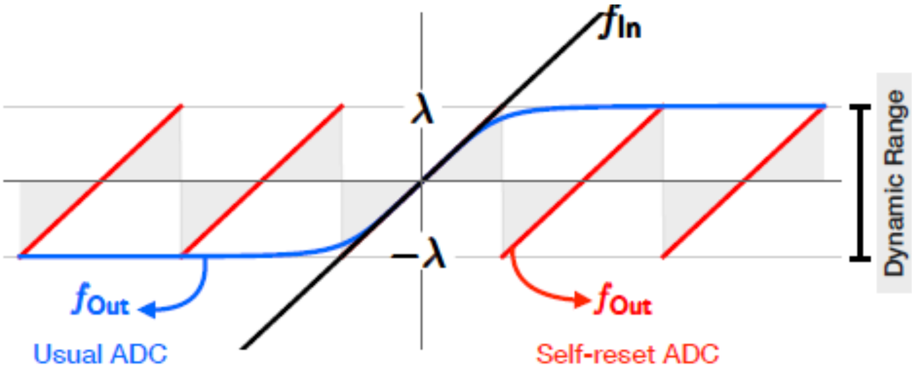
Dynamic Range (DR):

$$DR = 20 \log \left(\frac{\text{ADC range}}{\text{Step size}} \right) = 20 \log \left(\frac{2\lambda}{\Delta} \right) \propto L$$

- Finite dynamic range (precision): Number of bits L is finite
- Quantization error dependent on resolution Δ and clipping rate

Self-reset ADCs

What if the measurement setup can record amplitude only upto $\pm\lambda$?



[1]

Loss of information due to clipping!

[1] A. Bhandari, F. Krahmer, and R. Raskar, "Unlimited sampling of sparse signals," in Proc. ICASSP, 2018

Introduction Motivation

Self-reset ADCs

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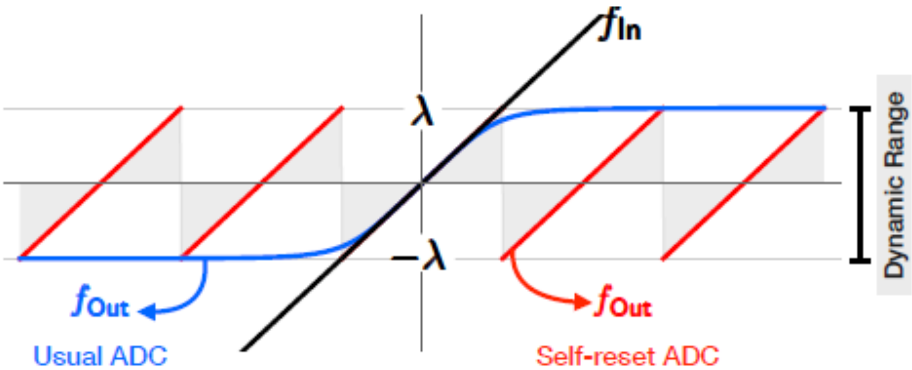

[1]

Loss of information due to clipping!
Wrap around the signal!

[1] A. Bhandari, F. Krahmer, and R. Raskar, "Unlimited sampling of sparse signals," in Proc. ICASSP, 2018

Self-reset ADCs

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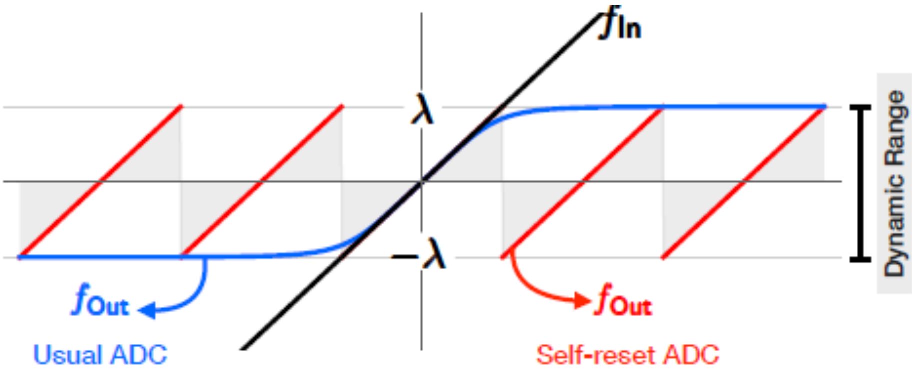

[1]

Loss of information due to clipping!
Wrap around the signal!

- Self-reset ADC measurements:
$$z_i = \mathscr{M}_\lambda(y_i) = \text{mod}\,(y_i + \lambda, 2\lambda) - \lambda$$
- Similar transfer function (modulo-1 operation):
$$z_i = \text{mod}\,(y_i, 1)$$

[1] A. Bhandari, F. Krahmer, and R. Raskar, "Unlimited sampling of sparse signals," in Proc. ICASSP, 2018

Why not scale the signal?

Scale signal down to ADC range
↓
Quantize signal
↓
Scale the values up by same amount

- Signal resolution remains same
- Quantization error $\propto$ Maximum value of input signal

SR-ADCs can be used to reduce quantization error

Issues:

- Information loss due to modulo operation
- Key question: Is recovery of sparse vectors possible from modulo measurements?

Connection to Unlimited Sampling[2]

- Unlimited sampling: Recovery of bandlimited signals from modulo samples

Theorem (Unlimited sampling of sparse signals)

Let $g = s_K * \psi$ for a known low-pass filter $\psi \in \mathcal{B}_\pi$ and s_K be the unknown K -sparse signal to be recovered, and assume one has access to an a priori bound $\beta_g \geq \|\psi\|_\infty \|s_K\|_{\mathrm{TV}}$.
Let

$$y_n = \mathscr{M}_\lambda(g(t))|_{t=nT}, n = 0, \ldots, N-1$$

be the modulo samples with sampling period T.
Then a sufficient condition for recovery of s_K from the y_n (up to additive multiples of 2λ) is

$$T \leq \frac{1}{2\pi e} \text{ and } N \geq 2K + 1 + 7\frac{\beta_g}{\lambda}$$

Low pass filtered measurements $\rightarrow$ CS measurements

[2] A. Bhandari, F. Krahmer, and R. Raskar, "Unlimited sampling of sparse signals," in Proc. ICASSP, 2018

Outline

Modulo-CS

Modulo Compressed Sensing

$$\mathbf{z} = [\mathbf{y}]^* = [\mathbf{A}\mathbf{x}]^* \quad (1)$$

- $\mathbf{A} \in \mathbb{R}^{M \times N}$ $(M < N)$ is the measurement matrix
- $\mathbf{x} \in \mathbb{R}^N$ is an s-sparse vector $(s \ll N)$
- $[\,.\,]^*$ denotes the element-wise modulo-1 operation (fractional part).

Questions:

1. What are the conditions on $\mathbf{A}$ such that each s-sparse vector $\mathbf{x}$ results in an unique $\mathbf{z}$?
2. What is the minimum M such that the conditions hold?
3. How to construct $\mathbf{A}$?
4. How to solve the problem efficiently?

Theory

Modulo decomposition property

Key idea: Any real number y can be written as an integer part $v \in \mathbb{Z}$ and a fractional part $z \in [0, 1)$.

Simple extension to vectors

Modular decomposition property

Any vector $\mathbf{y} \in \mathbb{R}^M$ can be decomposed as

$$\mathbf{y} = \mathbf{z} + \mathbf{v} \tag{2}$$

where $\mathbf{z} \in [0, 1)^M$ and $\mathbf{v} \in \mathbb{Z}^M$.

Remarks:

- There is no one-to-one correspondence between all possible vectors $\mathbf{y}$ and $\mathbf{z}$.
- Does $\mathbf{y} = \mathbf{A}\mathbf{x}$ with $\mathbf{x} \in \mathbb{R}^N$ help in possibility of one-to-one correspondence?

Theory Formulation

Sparse recovery problem

Proposition

For a given matrix $\mathbf{A} \in \mathbb{R}^{M \times N}$ and s-sparse $\mathbf{x} \in \mathbb{R}^N$, the following properties are equivalent.

1. *The vector $\mathbf{x}$ is the unique s-sparse solution to $\mathbf{A}\mathbf{w} = \mathbf{y}$ with $\mathbf{z} = \mathbf{y} + \mathbf{v}$ where $\mathbf{z} = [\mathbf{A}\mathbf{x}]^*$ and $\mathbf{v} \in \mathbb{Z}^M$.*
2. *The vector $\mathbf{x}$ can be reconstructed as the unique solution of*

$$\arg\min_{\mathbf{w},\mathbf{v}} \|\mathbf{w}\|_0$$

$$\text{subject to } \mathbf{A}\mathbf{w} + \mathbf{v} = [\mathbf{A}\mathbf{x}]^* \; ; \; \mathbf{v} \in \mathbb{Z}^M. \tag{P$_0$}$$

Sparse recovery problem

Proposition

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1. *The vector $\mathbf{x}$ is the unique s-sparse solution to $\mathbf{Aw} = \mathbf{y}$ with $\mathbf{z} = \mathbf{y} + \mathbf{v}$ where $\mathbf{z} = [\mathbf{Ax}]^*$ and $\mathbf{v} \in \mathbb{Z}^M$.*
2. *The vector $\mathbf{x}$ can be reconstructed as the unique solution of*

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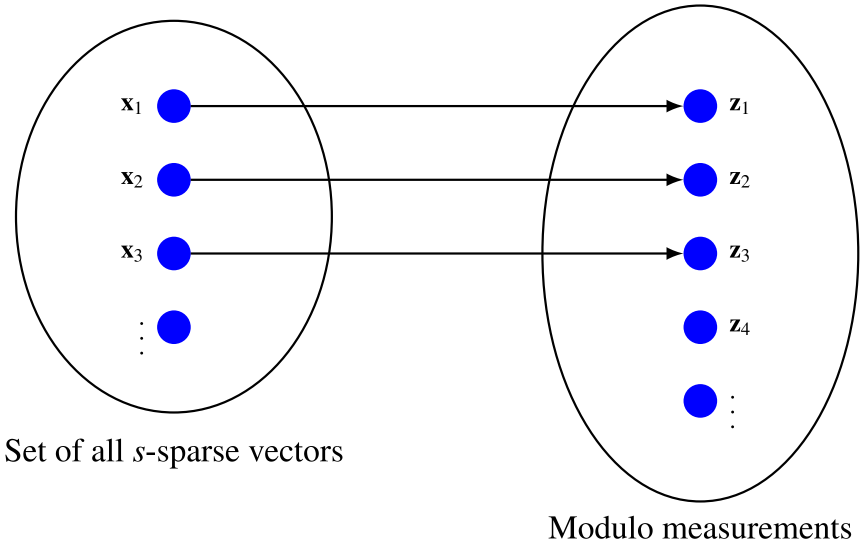
Proof.

$(1) \Rightarrow (2)$: Let $\mathbf{x}$ be the unique s-sparse solution of $\mathbf{Aw} = \mathbf{y}$ with $\mathbf{z} = \mathbf{y} + \mathbf{v}$ where $\mathbf{z} = [\mathbf{Ax}]^*$ and $\mathbf{v} \in \mathbb{Z}^M$. Then a solution $\mathbf{x}^\#$ of (P_0) is s-sparse and satisfies $\mathbf{Ax}^\# + \mathbf{v} = \mathbf{z}$, so that $\mathbf{x} = \mathbf{x}^\#$.

The implication $(2) \Rightarrow (1)$ is direct. □

Outline

Identifiability from modulo measurements



When does this not hold?

- Let $\mathbf{z}_1 = \mathbf{z}_2 \Rightarrow \mathbf{A}\mathbf{x}_1 + \mathbf{v}_1 = \mathbf{A}\mathbf{x}_2 + \mathbf{v}_2$

$$\boxed{\mathbf{A}\left(\mathbf{x}_1 - \mathbf{x}_2\right) = \mathbf{v}_2 - \mathbf{v}_1 \triangleq \mathbf{v} \in \mathbb{Z}^M}$$

Necessary and sufficient conditions

Lemma (Necessary and sufficient condition)

Any vector $\mathbf{x}$ *satisfying* $\|\mathbf{x}\|_0 \leq s$ *is a unique solution to the optimization problem* (P_0)

$$\Updownarrow$$

"Any $2s$ *columns of matrix* $\mathbf{A}$ *are linearly independent of all* $\mathbf{v} \in \mathbb{Z}^M \setminus \{\mathbf{0}\}$*".*

Necessary and sufficient conditions

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Comparison to Compressed sensing:

Corollary (Other necessary conditions)

The following two conditions are necessary for recovering any vector $\mathbf{x}$ *satisfying* $\|\mathbf{x}\|_0 \leq s$ *as a unique solution of the optimization problem* (P_0)*:*

1. $M \geq 2s + 1$ *(Compared to* $M \geq 2s$ *in CS)*
2. *Any* $2s$ *columns of matrix* $\mathbf{A}$ *are linearly independent.*

Outline

Sufficiency: Is there such a matrix?

Theorem (Construction of $\mathbf{A}$)

For any $N \geq 2s+1$, there exists a measurement matrix $\mathbf{A} \in \mathcal{R}^{M \times N}$ with $M = 2s+1$ rows such that every s-sparse vector $\mathbf{x} \in \mathcal{R}^N$ can be reconstructed from its modulo measurement vector $\mathbf{z} = [\mathbf{Ax}]^*$ as a solution of P_0-optimization problem.

Theory | Sufficiency

Sufficiency: Is there such a matrix?

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$$\boxed{M = 2s + 1 \text{ will suffice.}}$$

Proof idea:

- For $\mathbf{A} \in \mathcal{R}^{(2s+1) \times N}$, $\mathbf{u} \in \mathbb{Z}^M$, and $(|\mathcal{S}| \leq 2s)$, construct $\mathbf{B}(\mathbf{u}, S) = \begin{bmatrix} \mathbf{u} & \mathbf{A}_S \end{bmatrix}$
- Condition not satisfied $\Rightarrow \det\left(\mathbf{B}(\mathbf{u}, S)\right) = 0$
- Consider $\cup_{|\mathcal{S}| \leq 2s} \cup_{\mathbf{u} \in \mathbb{Z}^M} \{\mathbf{A} | \det\left(\mathbf{B}(\mathbf{u}, S)\right) = 0\} \Rightarrow$ Lebesgue measure 0
- Choose $\mathbf{A}$ outside the Lebesgue measure 0 set

Measurement matrix examples

Example 1: Gaussian random matrices
Claim: Will work!

- It is outside the Lebesgue measure 0 set
- Any continuous distribution based random matrices

Example 2: Integer matrices
Claim: Will not work[3]

Proposition

For any integer vector $\mathbf{a} \in \mathbb{Z}^K$ *and* $\mathbf{x} \in \mathbb{R}^K$*, it holds that*

$$[\mathbf{a}^T \mathbf{x}^*]^* = [\mathbf{a}^T \mathbf{x}]^*$$

[3] E. Romanov and O. Ordentlich, "Blind unwrapping of modulo reduced Gaussian vectors: Recovering MSBs from LSBs", 2019.

Algorithms

Related works

Phase unwrapping:[4]

- Comparison to phase retrieval problem
- Limited to Gaussian measurement matrix
- First stage: Initial estimate of bin index
- Second stage: Alternating minimization framework
- Application considered: Modulo Camera

Generalized Approximate Message Passing (GAMP):[5]

- Assume Bernoulli-Gaussian distribution for $\mathbf{x}$
- GAMP algorithm

[4] V. Shah and C. Hegde, "Sparse signal recovery from modulo observations", EURASIP Journal on Advances in Signal Processing, 2021.

[5] O. Musa, P. Jung and N. Goertz, "Generalized approximate message passing for unlimited sampling of sparse signals", Proc. IEEE Global Conf. Signal Inf. Process., pp. 336-340, 2018.

Outline

Convex Relaxation

Optimization problem:

$$\arg\min_{\mathbf{x},\mathbf{v}} \|\mathbf{x}\|_0$$
$$\text{subject to } \mathbf{A}\mathbf{x} = \mathbf{z} + \mathbf{v};\ \mathbf{v} \in \mathbb{Z}^M. \qquad (\mathrm{P}_0)$$

- NP-hard problem

Convex relaxation:

Modulo $\ell 1$ recovery problem

$$\arg\min_{\mathbf{x},\mathbf{v}} \|\mathbf{x}\|_1$$
$$\text{subject to } \mathbf{A}\mathbf{x} = \mathbf{z} + \mathbf{v};\ \mathbf{v} \in \mathbb{Z}^M. \qquad (\mathrm{P}_1)$$

- Combinatorial optimization problem

IRSP

Definition (Integer range space property (IRSP))

A matrix $\mathbf{A}$ is said to satisfy the integer range space property of order s if for all sets $\mathcal{S} \subset [N]$ with $|\mathcal{S}| \leq s$,

$$\|\mathbf{u}_{\mathcal{S}}\|_1 < \|\mathbf{u}_{\mathcal{S}^C}\|_1,$$

holds for every $\mathbf{u} \in \{\mathbf{u}|\mathbf{A}\mathbf{u} = \mathbf{v} \in \mathbb{Z}^M\}$.

- If $\mathbf{v}$ is restricted to be equal to $\mathbf{0}$ $\Rightarrow$ Null space property

Theorem (ℓ_1 recovery from modulo-CS)

Every s-sparse $\mathbf{x}$ is the unique solution of (P_1) if and only if the matrix $\mathbf{A}$ satisfies the IRSP of order s.

- Design of matrices that satisfy the above property is an open problem

Measurements restricted to $2k$ modulo periods

$$\arg\min_{\mathbf{x},\mathbf{v}} \|\mathbf{x}\|_1$$
$$\text{subject to } \mathbf{A}\mathbf{x} = \mathbf{z} + \mathbf{v},\ \|\mathbf{A}\mathbf{x}\|_\infty < k;\ \mathbf{v} \in \mathbb{Z}^M. \quad (\mathrm{P}_{1k})$$

Definition ($\mathcal{L}$-restricted integer range space property ($\mathcal{L}$-restricted IRSP))

A matrix $\mathbf{A}$ is said to satisfy the $\mathcal{L}$-restricted integer range space property of order s if for all sets $\mathcal{S} \subset [N]$ with $|\mathcal{S}| \leq s$,

$$\|\mathbf{u}_\mathcal{S}\|_1 < \|\mathbf{u}_{\mathcal{S}^C}\|_1,$$

holds for every $\mathbf{u} \in \mathcal{L} \subseteq \{\mathbf{u} | \mathbf{A}\mathbf{u} = \mathbf{v} \in \mathbb{Z}^M\}$.

- If $\mathcal{L} = \{\mathbf{u} | \mathbf{A}\mathbf{u} = \mathbf{0}\} \Rightarrow$ Null space property
- If $\mathcal{L} = \{\mathbf{u} | \mathbf{A}\mathbf{u} = \mathbf{v} \in \mathbb{Z}^M\} \Rightarrow$ Integer range space property

ℓ_1 Recovery Performance

Define three sets:

- $\mathcal{L}_l = \{\mathbf{u} | \mathbf{A}\mathbf{u} = \mathbf{v} \in \mathbb{Z}^M, \|\mathbf{v}\|_\infty \leq l\}$
- $\mathcal{K}_{l,\mathcal{S}} = \{\mathbf{u} | \mathbf{A}\mathbf{u} = \mathbf{v} \in \mathbb{Z}^M, \|\mathbf{A}\mathbf{u}_\mathcal{S}\|_\infty < l, \|\mathbf{A}\mathbf{u}_{\mathcal{S}^C}\|_\infty < l\}$
- $\mathcal{K}_l = \bigcup_{\mathcal{S}:|\mathcal{S}|\leq s} (\mathcal{K}_{l,\mathcal{S}})$

Note: $\mathcal{K}_{l,\mathcal{S}} \subset \mathcal{L}_{2l-1}$ and $\mathcal{K}_l \subseteq \mathcal{L}_{2l-1}$

Theorem

Given a matrix $\mathbf{A} \in \mathbb{R}^{M\times N}$, the guarantees for unique recovery of every s-sparse vector $\mathbf{x}$ as a solution to (P_{1k}) with the additional constraint $\|\mathbf{A}\mathbf{x}\|_\infty < k$ are:

- *Necessary condition: $\mathbf{A}$ satisfies $\mathcal{K}_k$-restricted IRSP.*
- *Sufficient condition: $\mathbf{A}$ satisfies $\mathcal{L}_{2k-1}$-restricted IRSP.*

Remarks:

- Gap between both conditions: $\mathcal{L}_{2k-1} \setminus \mathcal{K}_k$-restricted IRSP
- For (P_1) problem ($k \to \infty$): IRSP is both necessary and sufficient

Outline

Mixed integer linear program (MILP)

$\ell 1$ norm:

- $\|\mathbf{x}\|_1 = \sum_{i=1}^{N} |x_i| = \sum_{i:x_i \geq 0} x_i + \sum_{i:x_i < 0} (-x_i)$
- First set: $\mathbf{x}^+$ and second set: $\mathbf{x}^-$ with $\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^-$

Bound constraint:

- $\|\mathbf{A}\mathbf{x}\|_\infty < k \Rightarrow v_i \in [-k, k-1]$

Modulo MILP

$$\begin{aligned} &\arg\min_{\mathbf{x}^+, \mathbf{x}^-, \mathbf{v}} \mathbf{1}^T \left(\mathbf{x}^+ + \mathbf{x}^-\right) \\ &\text{subject to } \begin{bmatrix} \mathbf{A} & -\mathbf{A} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \\ \mathbf{v} \end{bmatrix} = \mathbf{z} \\ &\mathbf{v} \in [-k, k-1]^M \subseteq \mathbb{Z}^M, \quad \mathbf{x}^+, \mathbf{x}^- \geq 0 \end{aligned} \tag{P_{MILP}}$$

- Matlab optimization toolbox: `intlinprog` function

Success recovery percentage

- $N = 50$
- $\delta = \frac{M}{N}$ and $\rho = \frac{s}{N}$
- $\mathbf{A}_{i,j} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1/m)$
- Non-zero entries of $\mathbf{x} \overset{i.i.d}{\sim} Unif[-1, 1]$

Key observation: Transition for MILP close to the theoretical result

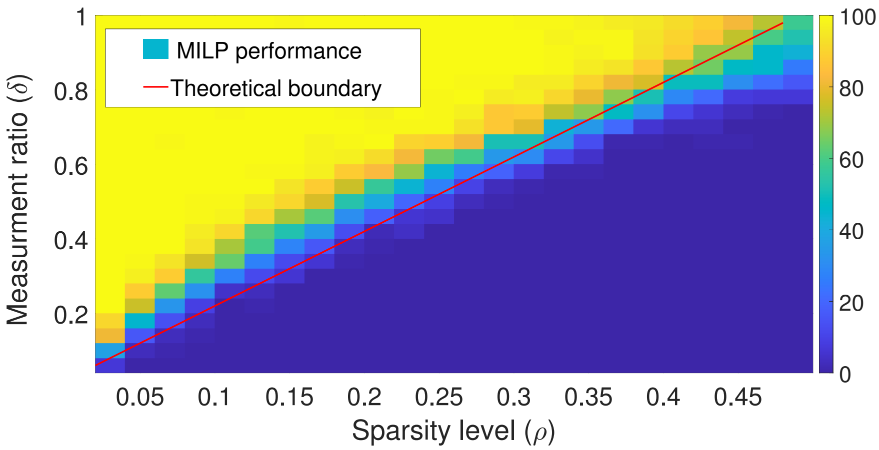


Figure: Phase transition for MILP

Phase transition curves

- N = 50
- $\mathbf{A}_{i,j} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1/M)$

Key observation: Good performance for low variance signals

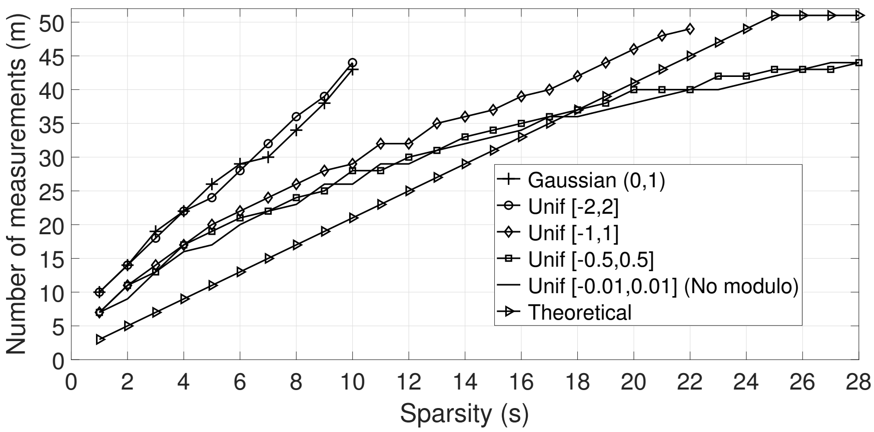


Figure: Phase transition with different distributions for 80% accuracy

Modulo-ADC for Compressed Sensing

Quantization

Quantized measurements

$$w_i = Q_{\lambda,L}\left(f_\lambda\left(y_i\right)\right); \quad i = 1, 2, \ldots, M$$

- $Q_{\lambda,L}$: Uniform mid-rise quantizer in $[-\lambda, \lambda]$ using L bits.
- $y_i = [\mathbf{Ax}]_i$

f_λ **function:**

- **Scaled measurements:** $f_\lambda(y_i) = \frac{1}{\alpha} y_i \in [-\lambda, \lambda]$ where $\alpha = \left\lceil \frac{1}{\lambda} max_i |y_i| \right\rceil$
- **Clipped measurements:** $f_\lambda(y_i) = \begin{cases} \lambda & \text{if } y_i \geq \lambda \\ -\lambda & \text{if } y_i \leq -\lambda \\ y_i & \text{otherwise} \end{cases}$
- **Modulo measurements:** $f_\lambda(y_i) = \mathscr{M}_\lambda(y_i)$ (Termed Modulo-ADC)

Recovery techniques

- **Scaled measurements:**

$$\arg\min_{\mathbf{x}^+,\mathbf{x}^-} \mathbf{1}^T\left(\mathbf{x}^+ + \mathbf{x}^-\right)$$
$$\text{subject to } \begin{bmatrix}\mathbf{A} & -\mathbf{A}\end{bmatrix}\begin{bmatrix}\mathbf{x}^+\\ \mathbf{x}^-\end{bmatrix} = \mathbf{z}, \quad \mathbf{x}^+, \mathbf{x}^- \geq 0 \qquad (\mathrm{P}_{LP})$$

- **Clipped measurements:** 2 approaches presented[6]
  - **Rejection:** Discard saturated measurements and run P_{LP}
  - **Consistency constraints:** Rejection approach with additional constraint for the saturated measurements

$$\begin{bmatrix}\Phi^{S^+}\\ -\Phi^{S^-}\end{bmatrix}\mathbf{x} \geq \lambda\mathbf{1}$$

- **Modulo measurements:** MILP algorithm

[6] Laska et. al., Democracy in action: Quantization, saturation, and compressive sensing, Applied and Computational Harmonic Analysis, 2011

Analysis setup

Default Parameters:

- $N = 50$, $s = 4$, $M = 30$
- $\lambda = 0.5$, $L = 6$ bits

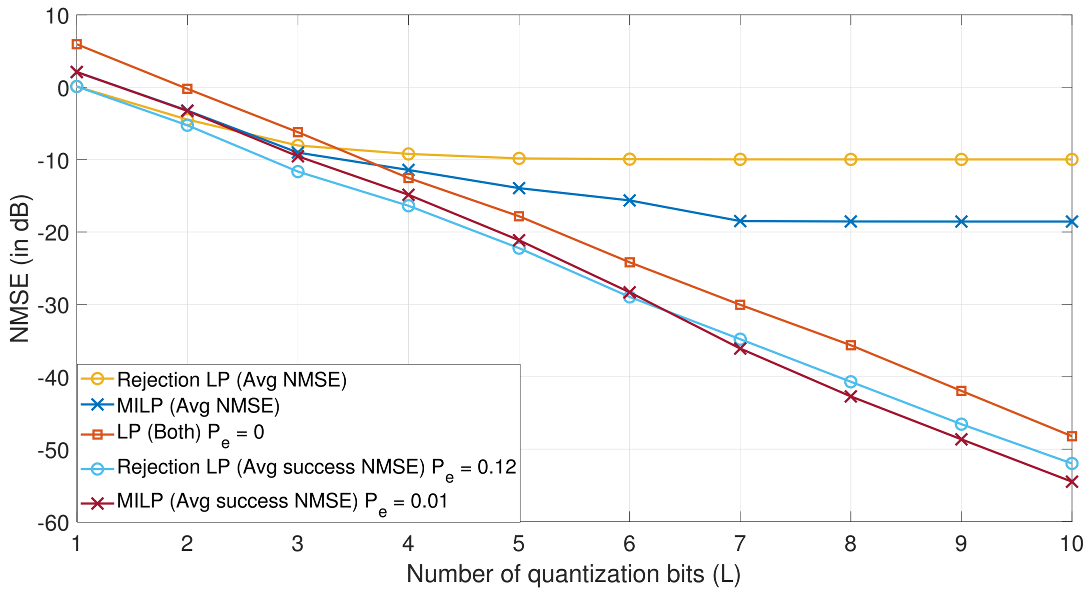
Signal generation:

- $\mathbf{A}_{i,j} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1/M)$
- Support of $\mathbf{x}$: s index drawn uniformly.
- Nonzero entries of $\mathbf{x}$: $\mathcal{N}(0, 1)$

Metrics:

- Instantaneous NMSE (for each Monte Carlo (MC) simulation): $\frac{||\mathbf{x}-\mathbf{x}_{\text{out}}||^2}{||\mathbf{x}||^2}$
- Successful recovery: If Instantaneous NMSE < 0.1 for unquantized case
- Probability of error: $\frac{\text{Number of MC sims with unsuccessful recovery}}{\text{Total number of MC sims}}$
- Average success NMSE calculated as average of Instantaneous NMSE for the MC sims with successful recovery

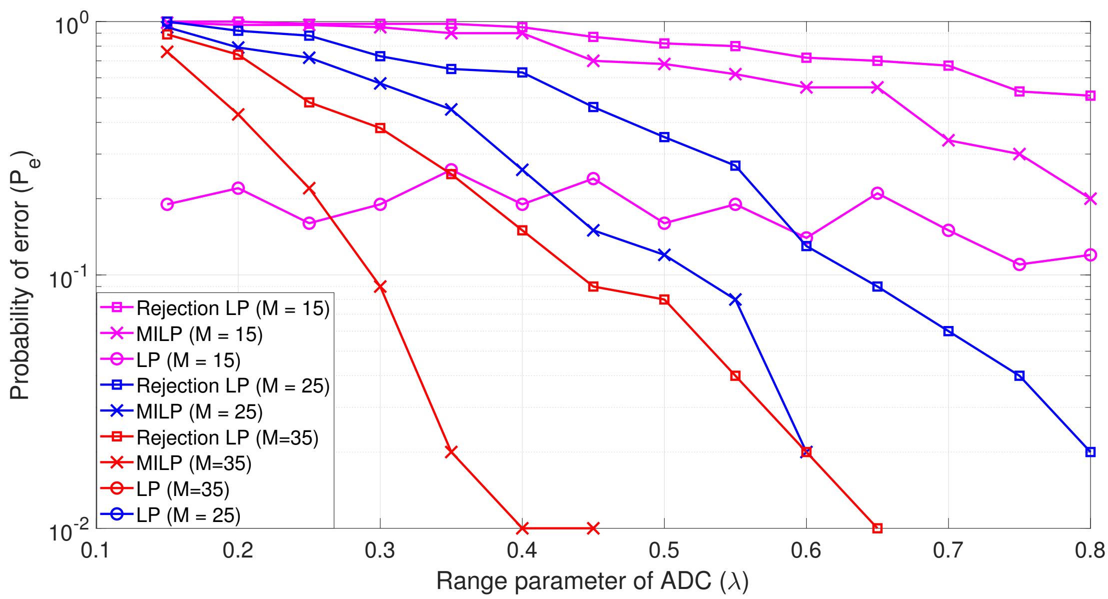
Varying quantization levels



Observations:

- NMSE floor for MILP and Rejection LP
- MILP has lower probability of error when compared to Rejection LP

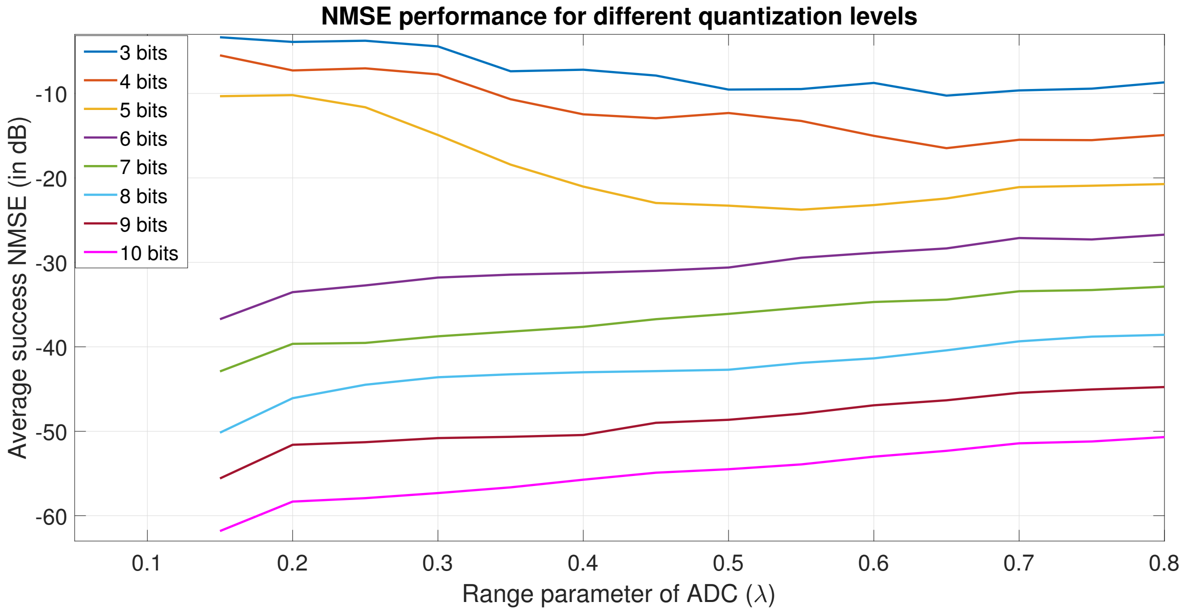
Varying ADC range- Probability of error



Observations:

- Probability of error decreases with range due to lesser folding
- **Note:** Average success NMSE increases with λ for all algorithms and all values of M

Varying ADC range- NMSE



Observations:

- NMSE increases with increase in range when sufficient resolution is present
- Increase in NMSE is 6-7dB for increase in range from 0.2 to 0.8

Summary

Summary

Key takeaways:

- Modulo-CS is identifiable
- Penalty for modulo operation is a single measurement
- Gaussian random matrices are candidate measurement matrices
- MILP algorithm can be used for modulo recovery
- Modulo-ADCs can lead to lower quantization errors under certain constraints

Future work:

- Extension to noisy case
- Alternative algorithms: e.g. SBL based algorithms
- Modulo-ADCs: Characterize tradeoff between number of folds and quantization levels

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