On the Relationship Between Mean Absolute Error and Age of Incorrect Information in the Estimation of a Piecewise Linear Signal over Noisy Channels

Subham Saha, Harkirat Singh Makkar, Vineeth Bala Sukumaran, and Chandra R. Murthy

***Abstract*—We consider the remote estimation of a stochastic piecewise linear signal, observed by a sensor, at a monitor. The sensor transmits a packet whenever the observed signal's slope changes. The packets are transmitted from the sensor to the monitor through an unreliable channel which randomly loses packets. The monitor sequentially estimates the signal using the information obtained from successfully received packets. The sensor does not have any feedback from the monitor. We derive an analytical expression for the average age of incorrect information, a recently proposed information freshness metric. The average age of incorrect information is shown to be a function of success probability of transmission and signal parameters representing the rate and clustering of slope changes. We obtain an upper bound on the mean absolute error of the remote estimate using the slope-weighted age of incorrect information. The age of incorrect information is also studied for a homogeneous multisensor scenario, where sensors use slotted ALOHA and the links between the sensors and the monitor are unreliable due to contention.**

I. Introduction

The remote estimation of a physical process observed by a sensor at a remote monitor is a problem of current interest [1], [2]. We consider a scenario in which the physical process is modelled as a stochastic piecewise linear (PL) signal, with randomly chosen slopes and slope-change epochs. Such models are relevant in wireless sensor networks with energy, memory, and computation constrained nodes which use low-complexity lossy compression techniques such as lightweight temporal compression (LTC) [3]–[5] to obtain PL signals from monitored processes. The sensor monitors the PL signal and uses an event-triggered transmission policy that transmits a packet containing the current value and slope whenever the slope changes. The packets are transmitted through a channel which randomly loses packets due to noise or contention. An estimate of the PL signal is obtained by the remote monitor using successfully received packets. There is no feedback assumed from the monitor to the sensor. We analyze the age of incorrect information (AoII) [6], which is a recently proposed information freshness metric, and explore its connection with mean absolute error (MAE) of the remote estimate. We propose a slope-weighted AoII and use it to obtain an upper bound on the MAE. The results are also applied to a homogeneous multisensor scenario with slotted ALOHA.

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The analysis and design of sampling and transmission policies for remote estimation problems has been considered in [1], [2], [6]–[12]. Sun et al. [1] consider the remote estimation of a Wiener process over a channel with random delay and with a mean-squared error (MSE) performance metric. A threshold-based policy is shown to be optimal for minimizing the MSE. The authors also show the equivalence of MSE with age of information for *signal oblivious* sampling schemes. Design of sampling policies for remote estimation of Ornstein-Uhlenbeck processes is studied in [8]. Under the assumption that the signal model is a Markov process, the best sampling scheme for an energy harvesting sensor has been considered by Nayyar et al. [7]. The sampling policy is shown to be a threshold policy when the performance metric is the MSE. Markov sources have also been considered in [2] and [11], although these works do not analyze remote estimation error. Joshi et al. [12] consider the remote estimation of a autoregressive Markov process and obtain optimal threshold policies that minimize a combination of estimation error and communication cost. In many practical sensor networks applications (e.g. [13]), the sensors use lossy data compression techniques in order to reduce memory requirements for storage as well as reduce energy requirements for transmission of sensed data [4], [5]. Compression techniques such as LTC [3] or PLAMLiS [13] result in PL signals which are then communicated and remotely estimated at the receiver. A comparative study in [4] shows that LTC is energy efficient in wireless networks with a good compression ratio. Therefore, we consider the remote estimation of PL signal models in this work. Although PL signal models (more precisely, the slope models) can be related to the Markov signal models considered in [2], [11], and [12], we note that analytical characterizations of its age metrics and connections with remote estimation error are not available in prior work for PL signal models.

Information freshness metrics such as age of information and AoII have been used to design sampling and transmission policies for remote estimation. Minimization of age of information has been shown to be equivalent to minimization of MSE in [1], [8]. However, many other information freshness metrics, e.g, AoII [6] do not have a direct relationship with remote estimation performance metrics. We explore the connection between an error metric (MAE) for the remote estimation of PL signals and AoII in this paper.

Contributions: (a) We obtain analytical expressions for AoII as well as the MAE of the slope estimate for the PL signal model, (b) we propose a weighted AoII to obtain an upper bound on MAE, where the weight at a time is determined by the absolute

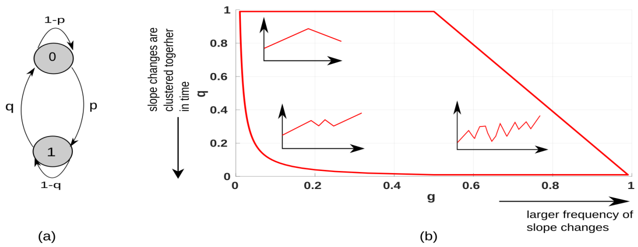

Fig. 1: (a) The two state Markov chain modelling the slope transitions for the PL signal model with state 1 denoting a slope change, p and $q \in (0, 1)$. (b) The set of allowed (g, q) values is the convex region bounded by the red perimeter. Examples of $X(t)$ corresponding to (g, q) in different regions are also illustrated.

error in the slope estimate of the PL process at that time, (c) we apply the above results to a homogeneous N-sensor scenario with a remote monitor which estimates the independent PL signals monitored by each sensor separately. The sensors use slotted ALOHA for transmission. We obtain insights into the behaviour of AoII as well as the source statistics which meet AoII constraints.

Notation: The set of non-negative integers is $\mathbb{Z}_{+}$. Capital letters (e.g., X) are used to denote random variables. The distribution of random variables is denoted using $\sim$, (e.g. $X \sim f_X$), and $\mathbb{E}$ denotes expectation. The uniform distribution on $[a, b]$ is denoted as $U[a, b]$. Independent and identically distributed random variables are denoted as IID.

II. System model and problem statement

We consider a discrete time system with slots indexed by $t \in \mathbb{Z}_{+}$. The PL signal model $X(t)$ is defined as follows. The slope of $X(t)$ in slot t is denoted by $M(t)$. We define the times at which the slope $M(t)$ changes using a time homogeneous Markov chain $S(t)$ with transition probabilities as shown in Figure 1(a). The process $M(t)$ evolves as:

1) if $S(t) = 0$, then $M(t) = M(t-1)$,
2) if $S(t) = 1$, then a new slope $M(t)$ is sampled.

The new slope is a sample of a continuous random variable with zero mean, finite variance, and a distribution $f_M(m)$ symmetric about 0. In this paper, as a specific example, we consider $f_M(m)$ to be $U[-m_0, m_0]$ where $0 < m_0 < \infty$. The process $X(t)$ then evolves as

$$X(t+1) = X(t) + M(t), \forall t.$$

We assume that $S(0)$ is sampled from the stationary distribution of the Markov chain $(S(t))$, $M(0) = m_0$, and $X(0) = x_0$. The initial values m_0 and x_0 are also assumed to be known by the receiver. We define $g = p/(p+q)$ which is the stationary probability of $S(t) = 1$. We can interpret g as the rate at which slope changes and q as deciding whether the slope changes occur close together (or *clustered*) or not (see Figure 1(b)). The values of g and q can be used to qualitatively classify the PL signal as follows: (a) $g \approx 0$ and $q \approx 0$: rare changes and changes are clustered, (b) $g \approx 1$ and $q \approx 0$: frequent changes and changes are clustered, (c) $g \approx 0$ and $q \approx 1$: rare changes and changes are not clustered, (d) $g \approx 1$ and $q \approx 1$: frequent changes and changes are not clustered.

We note that PL signals could have rare/frequent and clustered/non-clustered changes either naturally or as a result of the choice of parameters in a sampling/compression algorithm that produces a PL signal as its output. For example, suppose the monitored signal is the velocity of a vehicle. When the vehicle is moving on a highway with low traffic, its velocity would have rare and non-clustered changes. If the vehicle is moving in the downtown area of a city with low traffic, there would be frequent, but non-clustered changes. On the other hand, traffic may cause a number of velocity changes to cluster together. As another example, the LTC algorithm [13] uses a parameter, e, to control the maximum absolute error between the actual signal and the PL signal. If e is large, the PL signal would have rare changes, and the frequency of slope changes would increase as e is decreased. Furthermore, if e is varied over time, the changes become clustered.

We assume that there is no feedback from the monitor to the sensor. Without any additional information regarding the remote estimate the sensor transmits a packet to the monitor whenever the *slope* of $X(t)$ changes (i.e., at t such that $S(t) = 1$).[1] The transmission is not always reliable. If there is a transmission in slot t, it is successful (denoted by $D(t) = 1$) with probability p_s and unsuccessful ($D(t) = 0$) with probability $1 - p_s$, where $p_s \in (0, 1)$. We assume that $D(t)$ is independent across transmission slots. At a transmission slot t, the generated packet[2] $P(t)$ consists of $(X(t), M(t))$. Our objective is to remotely estimate $M(t)$ and $X(t)$ at the monitor[3]. We denote the estimate of $X(t)$ obtained remotely at the monitor by $\hat{X}(t)$ and that of $M(t)$ by $\hat{M}(t)$.[4] Let $N(t)$ be the number of successfully received packets till slot t (including any packet at slot t). The time index at which the ith packet is successfully received is denoted by T_i. Then,

$$\hat{X}(t) = \begin{cases} X(t), & \text{if } D(t) = 1 \\ \hat{X}(t-1) + \hat{M}(t-1), & \text{otherwise.} \end{cases}$$

Here, $\hat{M}(t) = M(t)$ if $D(t) = 1$ and $\hat{M}(t) = \hat{M}(t-1) = M(T_{N(t)})$ otherwise. Note that, at time slot t, $T_{N(t)} \leq t$ denotes the time of reception of the latest successfully received packet. Thus, the remote estimator sets the estimates $\hat{X}(t) = X(t)$ and $\hat{M}(t) = M(t)$ when a packet is successfully received at slot t. Otherwise, it uses the last successfully received slope value to update $\hat{X}(t)$ and $\hat{M}(t)$. We also let $\hat{X}(0) = x_0$ and $\hat{M}(0) = m_0$. The quality of the estimates for PL signal and slope are measured using

$$\begin{aligned} \text{MAE} &= \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left(|\hat{X}(t) - X(t)|\right) \text{ and} \\ \text{MAE}_S &= \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left(|\hat{M}(t) - M(t)|\right), \end{aligned}$$

[1] In Section III, we also consider a case where the sensor transmits periodically every T slots after a slope-change transmission or when a slope change occurs, whichever is earlier.

[2] Note that both $D(t)$ and $P(t)$ are used to define events only at slots where *transmissions* occur.

[3] The estimation of $M(t)$ is similar to that of remotely estimating a piecewise constant function.

[4] The remote estimates $\hat{X}(t)$ and $\hat{M}(t)$ are computed at the end of the slot t; but we consider the time index to be t itself.

respectively, conditioned on the initial values.

III. Age of Incorrect Information

The error metrics $|\hat{X}(t) - X(t)|$ or $|\hat{M}(t) - M(t)|$ can be viewed as cost functions that penalize the sensor-remote estimator system when there is change in the process, but that change is not successfully communicated to the remote estimator.[5] This is similar to AoII [6] which penalizes the sensor when it fails to update the remote estimator. In this section, we derive an analytical characterization of AoII as well as MAE_S. Furthermore, we propose a weighted AoII to obtain an upper bound on the MAE.

We define $T_c(t) = \max\{s \in \mathbb{Z}_+ : s \le t, S(s) = 1\}$, i.e., the latest slot in time up to t at which a slope change occurred. The AoII is a function of slot index t defined as

$$A(t) = \begin{cases} 0, & \text{if } T_c(t) = T_{N(t)}, \\ A(t-1)+1, & \text{otherwise.} \end{cases}$$

The initial value $A(0)$ is assumed to be 0. Thus, $A(t)$ counts the number of slots for which the monitor has an incorrect estimate of $X(t)$. We now define the average AoII as

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}A(t),$$

where the expectation is over the randomness of the underlying PL signal as well as random packet losses (included by using $T_{N(t)}$ in the definition of $A(t)$) between the sensor and remote estimator. We also define an absolute slope error weighted AoII as $W(t) = \left[A(t) \times |\hat{M}(t) - M(t)|\right]$ and its time average

$$\text{WAoII} = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}W(t).$$

We have the following characterizations.

Proposition III.1. *The average AoII is*

$$(1-p_s)\left[1 + \left(1 + \frac{q}{p}\right)\frac{1-p_s}{p_s} + \frac{q}{p(p+q)}\right]. \qquad (1)$$

The proof is provided in Appendix A. We note that the above characterization of average AoII also holds for Markov sources which are considered in [6] or [11]. We also have the following characterizations for the MAE based metrics.

Proposition III.2. *Suppose M_0 and M_1 are two independent random variables with the slope distribution $f_M(m)$. Then, for $p_s \in (0,1)$, $MAE_S = \mathbb{E}[|M_1 - M_0|](1-p_s)$ and $MAE \le WAoII = \mathbb{E}[|M_1 - M_0|]\, AoII$.*

The proof is provided in Appendix B. We note that MAE_S is independent of the source parameters p and q. Also, for $M_i \sim U[-m_0, m_0]$, $\mathbb{E}[|M_1 - M_0|] = 2m_0/3$ (since M_1 and M_0 are independent). If $|M(t)| \le m_0$, then $|\hat{X}(t) - X(t)| \le 2m_0 A(t)$ and MAE is bounded above by $2m_0$ AoII. Thus, we have a tighter upper bound in the above result.

[5] We note that the probability of $\hat{X}(t) = X(t)$ or $\hat{M}(t) = M(t)$ without $D(t) = 1$ is zero.

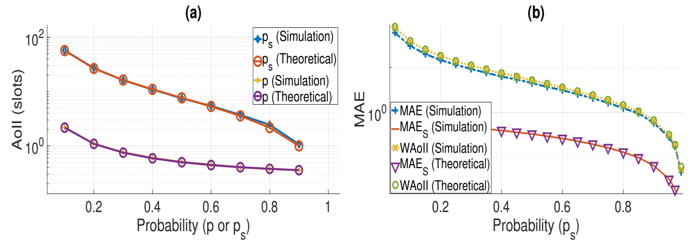

Fig. 2: (a) AoII (simulation vs. theoretical) as a function of p (or p_s) with $p_s = 0.8, q = 0.5$ (and $p = 0.1, q = 0.5$). (b) MAE and MAE_s vs. p_s, for $p = 0.1, q = 0.5$. The WAoII upper bound is also shown.

Simulation Results and discussion: In Figure 2(a), we compare the average AoII from (1) with that obtained from simulation as a function of (a) p with $p_s = 0.8$ and $q = 0.5$, and (b) p_s with $p = 0.1$ and $q = 0.5$. We see that the analysis and simulations match perfectly. We observe from (1) that sources with $g \downarrow 0$ have average AoII $\uparrow \infty$, due to the second term in the brackets. This is because, when there are infrequent transmissions, any unsuccessful transmission leads to a large increase in average AoII. We also note that within the set of sources with $g \approx 0$, those with $q \to 1$ (sources with slopes that are not clustered) have the largest average AoII. This is intuitive, since in comparison to unclustered slope changes, for clustered slope changes even if there is a slope change that was not known at the receiver due to an unsuccessful transmission, there is subsequent slope change (with high probability) which may possibly be successful due to which the estimation error resets to 0. In Figure 2(b) we compare the simulated MAE_S with the analytical characterization; they match perfectly. We also plot the simulated MAE with the upper bound WAoII. The upper bound is observed to be tight for p_s close to 1. We also observe that the simulated and theoretical WAoII match perfectly. Our analytical bound on the MAE can be used to identify the source statistics which can be transmitted over a channel with a given p_s, while satisfying a given bound on the MAE. Furthermore, in cases where the PL signal is obtained as the output of a compression algorithm (such as LTC), the MAE bound can be used to choose the parameters of the algorithm (e.g., the allowable error 'e' for LTC) so that the error in compression and the error in remote estimation can be traded off with each other.

Multiple access channels: In the above discussion, the success probability p_s was modelled as being independent of p and q. However, this does not always hold true, as in the case of multiple access channels. Consider a network consisting of $N \in \mathbb{Z}_+$ nodes where the nodes use slotted ALOHA (SA) as a multiple access protocol. Each node observes a PL signal modelled as an independent Markov chain as in Figure 1, with the same p and q across the nodes. The transmission from a node which has $S(t) = 1$ is successful only when all other $N-1$ nodes do not transmit or have $S(t) = 0$. Then, we have that $p_s = \left(\frac{q}{p+q}\right)^{N-1}$.

The following observations can be obtained for SA from Proposition III.1: (a) The average AoII for a node increases exponentially with the number of nodes N (as $N \to \infty$, then $p_s \to 0$ and the dominating term in average AoII is $1/p_s$).

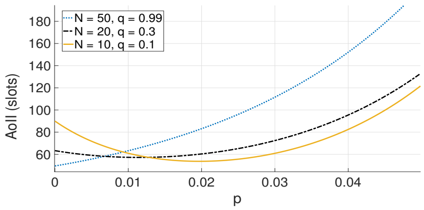

Fig. 3: AoII for SA as a function of p, with $N = 10$, $q = 0.1$.

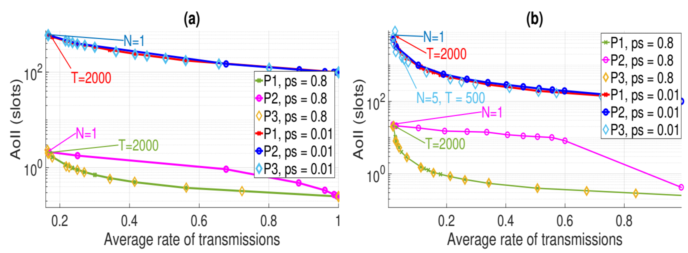

Fig. 4: AoII vs the average rate of transmissions for P1, P2, and P3 for $q = 0.5$, $p_s \in 0.8, 0.01$ and (a) $p = 0.1$, (b) $p = 0.01$. Plots are obtained by varying T (for P1), N (for P2), or both (for P3).

(b) For sources with rare changes in slope (i.e., $g \to 0$ for a fixed value of q) the average AoII converges to $(N-1)/q$ (this limit is obtained by approximating p_s as $1-(N-1)p/q$ when $p \approx 0$). (c) With SA, for a source with fixed q, there is an optimal value of $p > 0$ at which the minimum average AoII is achieved (see Figure 3). An interesting observation here is that compared to a channel with fixed p_s (for which the average AoII $\uparrow \infty$ as $g \downarrow 0$), for SA, since p_s decreases as g increases, sources with $g \downarrow 0$ have finite average AoII or MAE. Similar qualitative behavior is expected for other MAC protocols. Exact characterization of the average AoII for other MAC protocols is an area for future work.

Tradeoff of AoII with rate of transmission: In many scenarios, the average AoII can be reduced by retransmissions. However, this reduction in average AoII comes at the cost of an increase in the average rate (or in general, cost) of transmissions. Although the characterization of the optimal tradeoff is beyond the scope of this work, we illustrate this tradeoff for three families of policies in Figure 4. We first consider a family of policies (denoted by P1, parameterized by T) in which the sensor periodically samples and transmits every T slots after the previous transmission or when a slope change occurs, whichever is earlier. We also consider a family of policies (denoted by P2, parameterized by N) that transmits in N contiguous slots after a slope change or till the next slope change occurs whichever is earlier. Another family, P3, which combines features of P1 and P2 by transmitting in N contiguous slots periodically, with period T, is also considered. We observe that for higher p_s, P1 achieves a better tradeoff compared to P2, while for lower values of p_s, P1 and P2 have similar performance. As observed from Figure 4, in many instances (e.g., $N = 5, T = 500$), P3 achieves tradeoff points which are better than that achieved by both P1 and P2.

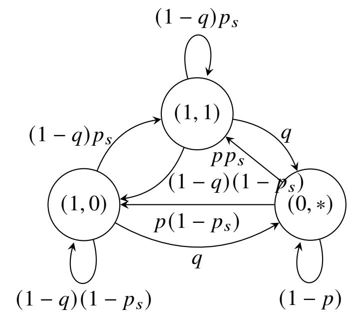

Fig. 5: Transition diagram for the $(S(t), D(t))$-Markov chain.

IV. Conclusions and Future Work

In this paper, we obtained an analytical characterization of average AoII and MAE for slope estimates in terms of the parameters of the source. We also obtained an upper bound on MAE using a weighted AoII. Then, we considered the case where the unreliable channel arises from slotted ALOHA. Future work can consider the characterization of MAE and average AoII for other multiple access protocols (for example, CSMA, IRSA) as well as the optimal AoII-cost tradeoff.

Appendix A
Proof of Proposition III.1

We define a Markov chain with state $(S(t), D(t))$ with transition probability diagram as in Figure 5. Note that if $S(t) = 0$, $D(t)$ is not defined; we denote this by $D(t) = *$. A renewal cycle can be defined as the total time taken by the chain to move from $(1, 1)$ to $(1, 0)$ and then from $(1, 0)$ back to $(1, 1)$. The first time to visit y starting from x is denoted as T_x^y. Then, the renewal cycle duration is $T_{(1,1)}^{(1,0)} + T_{(1,0)}^{(1,1)}$. We note that successive renewal cycle durations are IID. During the time $T_{(1,1)}^{(1,0)}$, there could be multiple visits to $(1, 1)$ but note that $A(t)$ and the errors in $X(t)$'s and $M(t)$'s estimates are zero. The absolute errors and AoII are nonzero during $T_{(1,0)}^{(1,1)}$. In the duration $T_{(1,0)}^{(1,1)}$, there could be multiple visits to $(1, 0)$ (at which the slope changes). The expected cumulative AoII over a renewal cycle is $T_{(1,0)}^{(1,1)}(T_{(1,0)}^{(1,1)} + 1)/2$. Then, the average AoII can be obtained from renewal reward theorem (RRT) [14] as the ratio of the expected cumulative AoII during the renewal cycle and the expected length $\mathbb{E}\left[T_{(1,1)}^{(1,0)} + T_{(1,0)}^{(1,1)}\right]$, so that

$$\text{AoII} = \mathbb{E}\left[T_{(1,0)}^{(1,1)}(T_{(1,0)}^{(1,1)} + 1)\right] / 2\mathbb{E}\left[T_{(1,1)}^{(1,0)} + T_{(1,0)}^{(1,1)}\right]. \tag{2}$$

From Figure 5 we have the following recursive relations. To compute $\mathbb{E}[T_{(1,0)}^{(1,1)}]$, we note that

$$\mathbb{E}[T_{(1,0)}^{(1,1)}] = (1-q)p_s + (1-q)(1-p_s)\mathbb{E}[1 + T_{(1,0)}^{(1,1)}] + q\mathbb{E}[1 + T_{(0,*)}^{(1,1)}] \tag{3}$$

and

$$\mathbb{E}[T_{(0,*)}^{(1,1)}] = pp_s + (1-p)\mathbb{E}[1 + T_{(0,*)}^{(1,1)}] + p(1-p_s)\mathbb{E}[1 + T_{(1,0)}^{(1,1)}]. \tag{4}$$

From (3) and (4), we get $\mathbb{E}[T_{(1,0)}^{(1,1)}] = (1 + q/p)/p_s$. Similarly, using recursive relations for $\mathbb{E}[T_{(1,1)}^{(1,0)}]$ and $\mathbb{E}[T_{(0,*)}^{(1,0)}]$ we have

$\mathbb{E}[T_{(1,1)}^{(1,0)}] = (1+q/p)/(1-p_s)$. We compute $\mathbb{E}[T_{(1,0)}^{(1,1)}]^2$ using similar recursive relations. We obtain $\mathbb{E}[T_{(1,0)}^{(1,1)}]^2$ as

$$(1-q)p_s+(1-q)(1-p_s)\mathbb{E}[(1+T_{(1,0)}^{(1,1)})^2]+q\mathbb{E}[(1+T_{(0,*)}^{(1,1)})^2] \tag{5}$$

and $\mathbb{E}[T_{(0,*)}^{(1,1)}]^2$ as

$$pp_s + (1-p)\mathbb{E}[(1+T_{(0,*)}^{(1,1)})^2] + p(1-p_s)\mathbb{E}[(1+T_{(1,0)}^{(1,1)})^2]. \tag{6}$$

From (3), (4), (5) and (6) we get

$$\mathbb{E}[T_{(1,0)}^{(1,1)}]^2 = \frac{1}{p_s}\left[1+\frac{q}{p}\left(1+\frac{2}{p}\right)+2\frac{(1-p_s)}{p_s}\left(1+\frac{q}{p}\right)^2\right].$$

From (2), the average AoII is

$$\overline{A} = \mathbb{E}\left[\left(T_{(1,0)}^{(1,1)}\right)^2 + T_{(1,0)}^{(1,1)}\right]\Big/2\mathbb{E}\left[T_{(1,1)}^{(1,0)} + T_{(1,0)}^{(1,1)}\right], \tag{7}$$

which can be simplified to

$$(1-p_s)\left[1+\left(1+\frac{q}{p}\right)\frac{1-p_s}{p_s}+\frac{q}{p(p+q)}\right],$$

as desired, which concludes the proof.

Appendix B
Proof of Proposition III.2

We reuse the notation from Appendix A. We consider the Markov chain in Figure 5 and the renewal process from Appendix A to characterize MAE_S. Note that $|\hat{M}(t)-M(t)|>0$ during $T_{(1,0)}^{(1,1)}$. Starting from the state $(1,0)$, let us assume that there are $N \in \mathbb{Z}_+$ visits to $(1,0)$ before visiting $(1,1)$ in the duration $T_{(1,0)}^{(1,1)}$. We denote the durations of these visit times by $C_1, C_2, \ldots, C_N$. Furthermore, let $C_{N+1} \triangleq T_{(1,0)}^{(1,1)} - \sum_{j=1}^{N} C_j$. During $T_{(1,0)}^{(1,1)}$, the expected cumulative absolute error in slope can be obtained from the sum of expected cumulative absolute errors over each duration C_i. The cumulative absolute error over a duration C_i is $C_i|M_i - M_0|$, where M_0 is the slope before the duration $T_{(1,0)}^{(1,1)}$ starts and M_i is the slope during C_i. The cumulative absolute error over all the renewal cycles is therefore $\sum_{i=1}^{N+1} C_i|M_i - M_0|$. Since the slopes are independent of C_is, the expected cumulative absolute error is $\mathbb{E}|M_1 - M_0|\mathbb{E}\sum_i C_i$, which is $\mathbb{E}|M_1-M_0|\mathbb{E}T_{(1,0)}^{(1,1)}$. Using the RRT [14], $\text{MAE}_S = \mathbb{E}|M_1 - M_0|\mathbb{E}T_{(1,0)}^{(1,1)}/(\mathbb{E}T_{(1,0)}^{(1,1)} + \mathbb{E}T_{(1,1)}^{(1,0)})$. We simplify using (3) and (4) to obtain $\text{MAE}_S = \mathbb{E}|M_1 - M_0|(1-p_s)$.

We characterize WAoII using RRT on the above renewal process, for which the expected cumulative weighted AoII during $T_{(1,0)}^{(1,1)}$ is required. This can be obtained from the sum of expected cumulative weighted AoII over each duration C_i as follows:

$$C_i\sum_{j<i} C_j|M_i - M_0| + C_i(C_i+1)|M_i - M_0|/2. \tag{8}$$

The cumulative weighted AoII over all the cycles C_i is therefore

$$\sum_{i=1}^{N+1}\left(\sum_{j<i} C_iC_j|M_i - M_0| + \frac{C_i(C_i+1)}{2}|M_i - M_0|\right).$$

The expected cumulative WAoII is therefore

$$\mathbb{E}\left[|M_1 - M_0|\right]\mathbb{E}\left[\sum_{i=1}^{N+1}\left(\sum_{j<i} C_iC_j + \frac{C_i(C_i+1)}{2}\right)\right], \tag{9}$$

since the slopes are independent of N as well as (C_i). We now consider the term $\sum_{i=1}^{N+1}\left(\sum_{j<i} C_iC_j + \frac{C_i(C_i+1)}{2}\right)$, which can be simplified to $\frac{1}{2}\left(\sum_{i=1}^{N+1} C_i\right)^2 + \frac{1}{2}\sum_{i=1}^{N+1} C_i$. This is $\frac{T_{(1,0)}^{(1,1)}(T_{(1,0)}^{(1,1)}+1)}{2}$. Then, the WAoII using RRT is

$$\mathbb{E}\left[|M_1 - M_0|\right]\mathbb{E}\left[T_{(1,0)}^{(1,1)}(T_{(1,0)}^{(1,1)}+1)\right]\Big/2\mathbb{E}\left[T_{(1,1)}^{(1,0)} + T_{(1,0)}^{(1,1)}\right].$$

From (7), we have that WAoII $= \mathbb{E}\left[|M_1 - M_0|\right]$ AoII.

We note that for a slot t in a duration $T_{(1,0)}^{(1,1)}$ and a cycle C_i within, $\hat{X}(t) - X(t) = \sum_{j<i}(M_j - M_0)C_j + (M_i - M_0)(t - \sum_{j<i} C_j)$. Applying the triangle inequality to $\mathbb{E}|\hat{X}(t) - X(t)|$, we get $\mathbb{E}|\hat{X}(t) - X(t)| \le \sum_{j<i}\mathbb{E}|M_j - M_0|C_j + \mathbb{E}|M_i - M_0|(t - \sum_{j<i} C_j)$. Since $\mathbb{E}|M_i - M_0|$ is the same for any i, the expected cumulative absolute error over a cycle is then bounded above by (9). We thus obtain the bound MAE $\le$ WAoII. The bound is expected to be tight for $p_s \approx 1$ since $T_{(1,0)}^{(1,1)}$ would consist of a single cycle with high probability, for which upper bound is equal to the error.

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