Supplementary Material for "On the Impact of Channel Estimation on the Design and Analysis of IRSA based Systems"

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I. Introduction

In this document, we present the detailed steps involved in the proofs of the theorems and lemmas presented in our work. Firstly, we derive the channel estimates and error variances under the three estimation schemes. Then, we derive the signal to interference plus noise ratio (SINR) under the three schemes. Finally, we evaluate the SINR in the high antenna regime using the theory of deterministic equivalents.

Unless otherwise mentioned, the notation used in this document follows the notation used in the main work.

II. Proof of Theorem 1: Channel Estimation

1) MMSE: We first vectorize the signal as

$$\overline{\mathbf{y}}_t^k \triangleq \text{vec}(\mathbf{Y}_t^{\mathrm{p}k}) = (\mathbf{P}_t^{k*} \otimes \mathbf{I}_N)\mathbf{h}_t^k + \overline{\mathbf{n}}_t, \quad (1)$$

where $\mathbf{h}_t^k \triangleq \text{vec}(\mathbf{H}_t^k)$, $\overline{\mathbf{n}}_t \triangleq \text{vec}(\mathbf{N}_t^{\mathrm{p}})$, and $\otimes$ is the Kronecker product. The MMSE estimator is $\hat{\mathbf{h}}_t^k \triangleq \mathbb{E}_{\mathbf{z}}\left[\mathbf{h}_t^k\right]$, where $\mathbf{z} = \overline{\mathbf{y}}_t^k$. The error $\tilde{\mathbf{h}}_t^k \triangleq \hat{\mathbf{h}}_t^k - \mathbf{h}_t^k$ is uncorrelated with $\mathbf{z}$ and the estimate. The conditional statistics of a Gaussian random vector $\mathbf{x}$ are

$$\mathbb{E}_{\mathbf{z}}\left[\mathbf{x}\right] = \mathbb{E}\left[\mathbf{x}\right] + \mathbf{K}_{\mathbf{xz}}\mathbf{K}_{\mathbf{zz}}^{-1}\left(\mathbf{z} - \mathbb{E}\left[\mathbf{z}\right]\right), \quad (2)$$

$$\mathbf{K}_{\mathbf{xx}|\mathbf{z}} = \mathbf{K}_{\mathbf{xx}} - \mathbf{K}_{\mathbf{xz}}\mathbf{K}_{\mathbf{zz}}^{-1}\mathbf{K}_{\mathbf{zx}}. \quad (3)$$

Here, $\mathbf{K}_{\mathbf{xx}}$, $\mathbf{K}_{\mathbf{xx}|\mathbf{z}}$, and $\mathbf{K}_{\mathbf{xz}}$ are the unconditional covariance of $\mathbf{x}$, the conditional covariance of $\mathbf{x}$ conditioned on $\mathbf{z}$, and the cross-covariance of $\mathbf{x}$ & $\mathbf{z}$ respectively. From (2), the MMSE channel estimate $\hat{\mathbf{h}}_t^k$ can be calculated as

$$\hat{\mathbf{h}}_t^k = \mathbb{E}\left[\mathbf{h}_t^k\right] + \mathbb{E}\left[\mathbf{h}_t^k\overline{\mathbf{y}}_t^{kH}\right]\mathbb{E}[\overline{\mathbf{y}}_t^k\overline{\mathbf{y}}_t^{kH}]^{-1}(\overline{\mathbf{y}}_t^k - \mathbb{E}\left[\overline{\mathbf{y}}_t^k\right]). \quad (4)$$

The terms in the above expression can be evaluated as

$$\mathbb{E}\left[\mathbf{h}_t^k\overline{\mathbf{y}}_t^{kH}\right] = \mathbf{B}_t^k\mathbf{P}_t^{kT} \otimes \mathbf{I}_N,$$
$$\mathbb{E}[\overline{\mathbf{y}}_t^k\overline{\mathbf{y}}_t^{kH}] = (\mathbf{P}_t^{k*}\mathbf{B}_t^k\mathbf{P}_t^{kT} + N_0\mathbf{I}_\tau) \otimes \mathbf{I}_N,$$
$$\hat{\mathbf{h}}_t^k = (\mathbf{B}_t^k\mathbf{P}_t^{kT}(\mathbf{P}_t^{k*}\mathbf{B}_t^k\mathbf{P}_t^{kT} + N_0\mathbf{I}_\tau)^{-1} \otimes \mathbf{I}_N)\overline{\mathbf{y}}_t^k,$$

and thus, the MMSE estimate $\hat{\mathbf{H}}_t^k$ of $\mathbf{H}_t^k$ is

$$\hat{\mathbf{H}}_t^k = \mathbf{Y}_t^{pk}(\mathbf{P}_t^k\mathbf{B}_t^k\mathbf{P}_t^{kH} + N_0\mathbf{I}_\tau)^{-1}\mathbf{P}_t^k\mathbf{B}_t^k, \quad (5)$$

$$\stackrel{(a)}{=} \mathbf{Y}_t^{pk}\mathbf{P}_t^k\mathbf{B}_t^k(\mathbf{P}_t^{kH}\mathbf{P}_t^k\mathbf{B}_t^k + N_0\mathbf{I}_{M_t^k})^{-1}, \quad (6)$$

where (a) follows from $(\mathbf{AB}+\mathbf{I})^{-1}\mathbf{A} = \mathbf{A}(\mathbf{BA}+\mathbf{I})^{-1}$.

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2) LCMMSE: The LCMMSE estimator is $\hat{\mathbf{h}}_{tm}^k \triangleq \mathbb{E}_{\mathbf{z}}\left[\mathbf{h}_{tm}\right]$, where $\mathbf{z} = \mathbf{y}_{tm}^{\mathrm{p}k}$ is the received pilot signal. The error $\tilde{\mathbf{h}}_{tm}^k \triangleq \hat{\mathbf{h}}_{tm}^k - \mathbf{h}_{tm}$ is uncorrelated with the signal $\mathbf{y}_{tm}^{\mathrm{p}k}$ and the channel estimate $\hat{\mathbf{h}}_{tm}^k$. From (2), the LCMMSE channel estimate $\hat{\mathbf{h}}_{tm}^k$ can be calculated

$$\hat{\mathbf{h}}_{tm}^k = \mathbb{E}\left[\mathbf{h}_{tm}\mathbf{y}_{tm}^{\mathrm{p}kH}\right]\mathbb{E}[\mathbf{y}_{tm}^{\mathrm{p}k}\mathbf{y}_{tm}^{\mathrm{p}kH}]^{-1}\mathbf{y}_{tm}^{\mathrm{p}k}$$
$$= \frac{g_{tm}\beta_m\|\mathbf{p}_m\|^2\sigma_{\mathrm{h}}^2}{N_0\|\mathbf{p}_m\|^2 + \sum_{i\in\mathcal{S}_k}|\mathbf{p}_i^H\mathbf{p}_m|^2 g_{ti}\beta_i\sigma_{\mathrm{h}}^2}\mathbf{y}_{tm}^{\mathrm{p}k} \triangleq \eta_{tm}^k\mathbf{y}_{tm}^{\mathrm{p}k}.$$

3) MSBL: In each iteration of MSBL, two steps are performed. The first step, termed the E-step, updates the covariance $\mathbf{\Sigma}_{kt}^{j+1}$ and mean $\boldsymbol{\mu}_{ktn}^{j+1}$ of the posterior $p([\mathbf{Z}_t^k]_{:,n}|[\mathbf{Y}_t]_{:,n}, \boldsymbol{\gamma}_{kt}^j)$

$$\mathbf{\Sigma}_{kt}^{j+1} = \mathbf{\Gamma}_{kt}^j - \mathbf{\Gamma}_{kt}^j\mathbf{P}^{kH}(N_0\mathbf{I}_\tau + \mathbf{P}^k\mathbf{\Gamma}_{kt}^j\mathbf{P}^{kH})^{-1}\mathbf{P}^k\mathbf{\Gamma}_{kt}^j, \quad (7)$$

$$\boldsymbol{\mu}_{ktn}^{j+1} = N_0^{-1}\mathbf{\Sigma}_{kt}^{j+1}\mathbf{P}^{kH}[\overline{\mathbf{Y}}_t^{\mathrm{p}k}]_{:,n}, \ n \in [N]. \quad (8)$$

The second step, termed the M-step, updates the hyperparameter for the ith user in the tth RB as

$$[\boldsymbol{\gamma}_{kt}^{j+1}]_i = \frac{1}{N}\sum_{n=1}^{N}([\mathbf{\Sigma}_{kt}^{j+1}]_{i,i} + |[\boldsymbol{\mu}_{ktn}^{j+1}]_i|^2), \ i \in [M^k]. \quad (9)$$

This step estimates the variance of the channel of the ith user in the tth RB. Based on the estimate $\hat{g}_{ti}^k$ and the true g_{ti}, the set of users $[M^k]$ can be divided into four disjoint subsets

$$\mathcal{A}_t^k = \{i \in [M^k] \mid \hat{g}_{ti}^k g_{ti} = 1\}, \quad (10)$$
$$\mathcal{F}_t^k = \{i \in [M^k] \mid \hat{g}_{ti}^k(1 - g_{ti}) = 1\}, \quad (11)$$
$$\mathcal{M}_t^k = \{i \in [M^k] \mid (1 - \hat{g}_{ti}^k)g_{ti} = 1\}, \quad (12)$$
$$\mathcal{I}_t^k = \{i \in [M^k] \mid (1 - \hat{g}_{ti}^k)(1 - g_{ti}) = 1\}. \quad (13)$$

$\mathcal{A}_t^k$ is the set of true positive users, $\mathcal{F}_t^k$ is the set of false positive users, $\mathcal{M}_t^k$ is the set of false negative users, and $\mathcal{I}_t^k$ is the set of true negative users. False positive and false negative users form the errors in APM estimation. As the decoding iterations proceed, more users get decoded, and the errors in APM estimation decrease. The MSBL channel estimate $\hat{\mathbf{H}}_t^k = \mathbf{Y}_t^{\mathrm{p}k}\mathbf{P}^k\hat{\mathbf{\Gamma}}_{kt}(\mathbf{P}^{kH}\mathbf{P}^k\hat{\mathbf{\Gamma}}_{kt} + N_0\mathbf{I}_{M^k})^{-1}$ is output in the E-step from Algorithm 1, where $\hat{\mathbf{\Gamma}}_{kt} = \text{diag}(\boldsymbol{\gamma}_{kt}^{j_{\max}})$. The false negative users' channels do not get estimated even though they contribute towards $\mathbf{Y}_t^{\mathrm{p}k}$. The false positive users' channels get estimated even though they haven't transmitted, and thus, an erroneous channel estimate is output for those users. Since $[\gamma_{kt}]_i$ models the variance of the ith users signal in the tth RB, it models $g_{ti}\beta_i\sigma_{\mathrm{h}}^2$. Thus, the estimated hyperparameter $[\boldsymbol{\gamma}_{kt}^{j_{\max}}]_i$ would recover both $\hat{g}_{ti}^k$ and $\hat{\beta}_i^k$. Since the path loss is same across RBs, a higher quality estimate for the path loss can be estimated by averaging across RBs, and thus we obtain $\hat{\beta}_i^k = (\sum_{t=1}^T \hat{g}_{ti}^k [\gamma_{kt}^{j_{\max}}]_i)/(\sigma_{\mathtt{h}}^2 \sum_{t=1}^T \hat{g}_{ti}^k)$.

4) Error variances: The conditional covariance of $\mathbf{h}_{ti}$ is calculated conditioned on $\mathbf{z} = \hat{\mathbf{h}}_{ti}^k$. In MMSE, with $\mathbf{c}_{ti}^k = [\mathbf{C}_t^k]_{:,i}$ and $\mathbf{C}_t^k \triangleq \mathbf{P}_t^k \mathbf{B}_t^k (\mathbf{P}_t^{kH} \mathbf{P}_t^k \mathbf{B}_t^k + N_0 \mathbf{I}_{M_t^k})^{-1}$, we have

$$\begin{aligned}\mathbf{K}_{\mathbf{h}_{ti}\mathbf{h}_{ti}} &= \mathbb{E}[\mathbf{h}_{ti}\mathbf{h}_{ti}^H] = \beta_i \sigma_{\mathtt{h}}^2 \mathbf{I}_N,\\ \mathbf{K}_{\mathbf{h}_{ti}\mathbf{z}} &= \mathbb{E}[\mathbf{h}_{ti}\hat{\mathbf{h}}_{ti}^{kH}] = \mathbf{p}_i^H \mathbf{c}_{ti}^k g_{ti} \beta_i \sigma_{\mathtt{h}}^2 \mathbf{I}_N,\\ \mathbf{K}_{\mathbf{zz}} &= (N_0 \|\mathbf{c}_{ti}\|^2 + \textstyle\sum_{j\in\mathcal{S}_k} |\mathbf{p}_j^H \mathbf{c}_{ti}^k|^2 g_{tj} \beta_j \sigma_{\mathtt{h}}^2)\mathbf{I}_N.\end{aligned}$$

Thus, the conditional covariance is

$$\begin{aligned}&\mathbf{K}_{\mathbf{h}_{ti}\mathbf{h}_{ti}|\mathbf{z}} = \mathbf{K}_{\mathbf{h}_{ti}\mathbf{h}_{ti}} - \mathbf{K}_{\mathbf{h}_{ti}\mathbf{z}} \mathbf{K}_{\mathbf{zz}}^{-1} \mathbf{K}_{\mathbf{z}\mathbf{h}_{ti}}\\ &= \beta_i \sigma_{\mathtt{h}}^2 \left(\frac{N_0 \|\mathbf{c}_{ti}^k\|^2 + \sum_{j\in\mathcal{S}_k^i} |r_{jti}^k|^2 g_{tj} \beta_j \sigma_{\mathtt{h}}^2}{N_0 \|\mathbf{c}_{ti}^k\|^2 + \sum_{j\in\mathcal{S}_k} |r_{jti}^k|^2 g_{tj} \beta_j \sigma_{\mathtt{h}}^2} \right) \mathbf{I}_N \triangleq \delta_{ti}^k \mathbf{I}_N,\end{aligned}$$

where $r_{jti}^k \triangleq \mathbf{p}_j^H \mathbf{c}_{ti}^k$ and δ_{ti}^k accounts for pilot contamination. The conditional autocorrelation follows as

$$\begin{aligned}\mathbb{E}_{\mathbf{z}}[\mathbf{h}_{tm}\mathbf{h}_{tm}^H] &= \mathbf{K}_{\mathbf{h}_{tm}\mathbf{h}_{tm}|\mathbf{z}} + \mathbb{E}_{\mathbf{z}}[\mathbf{h}_{tm}]\mathbb{E}_{\mathbf{z}}[\mathbf{h}_{tm}]^H\\ &= \delta_{tm}^k \mathbf{I}_N + \hat{\mathbf{h}}_{tm}^k \hat{\mathbf{h}}_{tm}^{kH}. \end{aligned} \tag{14}$$

The unconditional and conditional means of the estimation error are $\mathbb{E}[\tilde{\mathbf{h}}_{tm}^k] = \mathbb{E}[\hat{\mathbf{h}}_{tm}^k - \mathbf{h}_{tm}] = 0$ and $\mathbb{E}_{\mathbf{z}}[\tilde{\mathbf{h}}_{tm}^k] = \mathbb{E}_{\mathbf{z}}[\hat{\mathbf{h}}_{tm}^k - \mathbf{h}_{tm}] = \hat{\mathbf{h}}_{tm}^k - \hat{\mathbf{h}}_{tm}^k = 0$. The conditional autocovariance of the error therefore simplifies as

$$\begin{aligned}&\mathbf{K}_{\tilde{\mathbf{h}}_{tm}^k \tilde{\mathbf{h}}_{tm}^k|\mathbf{z}} = \mathbb{E}_{\mathbf{z}}[\tilde{\mathbf{h}}_{tm}^k \tilde{\mathbf{h}}_{tm}^{kH}]\\ &= \mathbb{E}_{\mathbf{z}}[\mathbf{h}_{tm}\mathbf{h}_{tm}^H] - \hat{\mathbf{h}}_{tm}^k \hat{\mathbf{h}}_{tm}^{kH} = \delta_{tm}^k \mathbf{I}_N,\end{aligned} \tag{15}$$

and thus, δ_{tm}^k is also the variance of the estimation error. Substituting $\mathbf{C}_t^k = \mathbf{P}_t^k \ \mathrm{diag}(\eta_{ti_1}^k, \ldots, \eta_{ti_{M_t^k}}^k)$, we get the error variance for LCMMSE.

The MSBL estimate error is also uncorrelated with the estimate and the error variance can be derived similar to the MMSE scheme since the MSBL estimate is a "plug-in" MMSE estimate. Since only true positive users' channels are estimated, the error variance is calculated only for the subset of true positive users (users with $\hat{g}_{ti}^k g_{ti} = 1$), and thus, each g_{ti} is accompanied by $\hat{g}_{ti}^k$ similar to [1]. Further, since the error variance models the true interference from other true positive users, the true path loss coefficient accompanies $\hat{g}_{ti}^k g_{ti}$. Hence we define $\mathbf{C}_t^k \triangleq \mathbf{P}^k \mathbf{D}_t^k (\mathbf{P}^{kH} \mathbf{P}^k \mathbf{D}_t^k + N_0 \mathbf{I}_{M^k})^{-1}$ and $\mathbf{D}_t^k \triangleq \mathrm{diag}(d_{ti_1}^k, d_{ti_2}^k, \ldots, d_{ti_{M^k}}^k)$, with $d_{ti}^k = \hat{g}_{ti}^k g_{ti} \beta_i \sigma_{\mathtt{h}}^2$. Substituting for $\mathbf{C}_t^k$, we get the error variance for MSBL.

III. Proof of Theorem 2: SINR Evaluation

In order to evaluate the SINR, we first calculate the power of the received signal, which is calculated conditioned on the knowledge of the estimates $\mathbf{z} \triangleq \mathrm{vec}(\hat{\mathbf{H}}_t^k)$ as $\mathbb{E}_{\mathbf{z}}[|\tilde{y}_{tm}^k|^2] = \mathbb{E}_{\mathbf{z}}[|\sum_{i=1}^4 T_i|^2]$. Since noise is uncorrelated with data, $\mathbb{E}_{\mathbf{z}}[T_1 T_4^H] = \mathbb{E}_{\mathbf{z}}[T_2 T_4^H] = \mathbb{E}_{\mathbf{z}}[T_3 T_4^H] = 0$. Since MMSE channel estimates are uncorrelated with their errors [2], $\mathbb{E}_{\mathbf{z}}[T_1 T_2^H] = 0$. Computing the remaining power components requires the evaluation of $\mathbb{E}_{\mathbf{z}}[x_i x_j]$ for $i \neq j$ which can be calculated as $\mathbb{E}_{\mathbf{z}}[x_i x_j] = \mathbb{E}_{\mathbf{z}}[x_i]\mathbb{E}_{\mathbf{z}}[x_j] = 0$. Thus, all the four terms are uncorrelated and the power in the received signal is just a sum of the powers of the individual components $\mathbb{E}_{\mathbf{z}}[|\tilde{y}_{tm}^k|^2] = \sum_{i=1}^4 \mathbb{E}_{\mathbf{z}}[|T_i|^2]$. We now compute the powers of each of the components. The useful signal power is

$$\mathbb{E}_{\mathbf{z}}[|T_1|^2] = \mathbb{E}_{\mathbf{z}}[|\mathbf{a}_{tm}^{kH} \hat{\mathbf{h}}_{tm}^k g_{tm} x_m|^2] = P g_{tm}^2 |\mathbf{a}_{tm}^{kH} \hat{\mathbf{h}}_{tm}^k|^2. \tag{16}$$

The desired gain is written as

$$\mathtt{Gain}_{tm}^k \triangleq \frac{\mathbb{E}_{\mathbf{z}}[|T_1|^2]}{P\|\mathbf{a}_{tm}^k\|^2} = g_{tm} \frac{|\mathbf{a}_{tm}^{kH} \hat{\mathbf{h}}_{tm}^k|^2}{\|\mathbf{a}_{tm}^k\|^2}. \tag{17}$$

The power of the estimation error is expressed as

$$\mathbb{E}_{\mathbf{z}}[|T_2|^2] = \mathbb{E}_{\mathbf{z}}[|\mathbf{a}_{tm}^{kH} \tilde{\mathbf{h}}_{tm}^k g_{tm} x_m|^2] = P g_{tm}^2 \delta_{tm}^k \|\mathbf{a}_{tm}^k\|^2.$$

Next, the power of the inter-user interference term T_3 is

$$\begin{aligned}\mathbb{E}_{\mathbf{z}}[|T_3|^2] &= \mathbb{E}_{\mathbf{z}}\left[\left|\mathbf{a}_{tm}^{kH} \textstyle\sum_{i\in\mathcal{S}_k^m} g_{ti}\mathbf{h}_{ti} x_i\right|^2\right]\\ &= P \textstyle\sum_{i\in\mathcal{S}_k^m} g_{ti}^2 \mathbf{a}_{tm}^{kH} \mathbb{E}_{\mathbf{z}}[\mathbf{h}_{ti}\mathbf{h}_{ti}^H] \mathbf{a}_{tm}^k\\ &= P \textstyle\sum_{i\in\mathcal{S}_k^m} g_{ti}^2 \mathbf{a}_{tm}^{kH} (\delta_{ti}^k \mathbf{I}_N + \hat{\mathbf{h}}_{ti}^k \hat{\mathbf{h}}_{ti}^{kH}) \mathbf{a}_{tm}^k\\ &= P \textstyle\sum_{i\in\mathcal{S}_k^m} g_{ti}^2 (\|\mathbf{a}_{tm}^k\|^2 \delta_{ti}^k + |\mathbf{a}_{tm}^{kH} \hat{\mathbf{h}}_{ti}^k|^2).\end{aligned} \tag{18}$$

Here, $\mathbb{E}_{\mathbf{z}}[|T_2|^2] + \mathbb{E}_{\mathbf{z}}[|T_3|^2]$ represents the contribution of estimation errors and multi-user interference components of the other users. Since g_{ti} is binary, its powers are dropped. We now split the normalized version of the above into the sum of the error component $\mathtt{Est}_{tm}^k$ and the multi-user interference $\mathtt{MUI}_{tm}^k$ as follows

$$\mathtt{Est}_{tm}^k \triangleq \textstyle\sum_{i\in\mathcal{S}_k} g_{ti} \delta_{ti}^k, \ \mathtt{MUI}_{tm}^k \triangleq \sum_{i\in\mathcal{S}_k^m} g_{ti} \frac{|\mathbf{a}_{tm}^{kH} \hat{\mathbf{h}}_{ti}^k|^2}{\|\mathbf{a}_{tm}^k\|^2}. \tag{19}$$

The noise power is calculated as

$$\mathbb{E}_{\mathbf{z}}[|T_4|^2] = \mathbb{E}_{\mathbf{z}}[|\mathbf{a}_{tm}^{kH} \mathbf{n}_t|^2] = N_0 \|\mathbf{a}_{tm}^k\|^2. \tag{20}$$

A meaningful SINR expression can be written out by dividing the useful signal power from (17) by the sum of the interference and the noise powers (from (19), and (20)) [2]. Note that the interference component is comprised of the estimation error term and the signal powers of other users who have also transmitted in the same RB. For MMSE/LCMMSE, the corresponding SINR can be calculated by plugging in the channel estimates.

In MSBL, each of T_1, T_2, and T_3 is calculated among the subset of true positive users in the tth RB, i.e., users in $\mathcal{A}_t^k = \{i \in [M^k] | \hat{g}_{ti}^k g_{ti} = 1\}$. Hence, each of the powers previously derived for MMSE is accompanied by $\hat{g}_{ti}^k g_{ti}$. We need to account for false negative users, i.e., users in $\mathcal{M}_t^k = \{i \in [M^k] | (1 - \hat{g}_{ti}^k) g_{ti} = 1\}$. These users interfere with the decoding of other users and the SINR for such users is 0 since they will never get decoded. Such users' signals are uncorrelated with the other terms, and thus, their power is

$$\begin{aligned}\mathbb{E}_{\mathbf{z}}[|T_5|^2] &= \mathbb{E}_{\mathbf{z}}[|\textstyle\sum_{i\in\mathcal{S}_k^m\cap\mathcal{M}_t^k} \mathbf{a}_{tm}^{kH} \mathbf{h}_{ti} g_{ti} x_i|^2]\\ &\overset{(b)}{=} P \textstyle\sum_{i\in\mathcal{S}_k^m\cap\mathcal{M}_t^k} g_{ti}^2 \mathbf{a}_{tm}^{kH} \mathbb{E}[\mathbf{h}_{ti}\mathbf{h}_{ti}^H] \mathbf{a}_{tm}^k\\ &= P \textstyle\sum_{i\in\mathcal{S}_k^m\cap\mathcal{M}_t^k} g_{ti}^2 \mathbf{a}_{tm}^{kH} (\beta_i \sigma_{\mathtt{h}}^2 \mathbf{I}_N) \mathbf{a}_{tm}^k\\ &= P \textstyle\sum_{i\in\mathcal{S}_k^m\cap\mathcal{M}_t^k} g_{ti}^2 \beta_i \sigma_{\mathtt{h}}^2 \|\mathbf{a}_{tm}^k\|^2,\end{aligned} \tag{21}$$

where the conditional expectation is dropped in (b) since the BS does not have the knowledge of the channel estimates of false negative users. The normalised power of the false positive users is $\texttt{FNU}_{tm}^k \triangleq \sum_{i\in\mathcal{S}_k^m}(1-\hat{g}_{ti}^k)g_{ti}\beta_i\sigma_{\texttt{h}}^2$.

IV. Proof of Lemma 1: Deterministic Equivalent

It is known that, as the number of antennas gets large, both $\|\hat{\mathbf{h}}_{tm}^k\|^2$ and $|\hat{\mathbf{h}}_{tm}^{kH}\hat{\mathbf{h}}_{ti}^k|^2$ converge almost surely (a.s.) to their deterministic equivalents [3]. Evaluating the deterministic equivalents as in [3] and plugging into the SINR expression instead of the original terms, we can find an approximation to the SINR in the high antenna regime. As N gets large, the SINR with MRC converges almost surely ($\rho_{tm}^k \xrightarrow{\text{a.s.}} \overline{\rho}_{tm}^k$) to

$$\overline{\rho}_{tm}^k = \frac{N\texttt{Sig}_{tm}^k}{\epsilon_{tm}^k\left(N_0/P + \texttt{IntNC}_{tm}^k\right) + \texttt{IntC}_{tm}^k}, \tag{22}$$

where $\texttt{Sig}_{tm}^k$ is the desired gain, $\texttt{IntNC}_{tm}^k$ is the non-coherent interference, and $\texttt{IntC}_{tm}^k$ is the coherent interference. For LCMMSE, $\texttt{IntNC}_{tm}^k \triangleq g_{tm}\delta_{tm}^k + \sum_{i\in\mathcal{S}_k^m} g_{ti}\beta_i\sigma_{\texttt{h}}^2$, $\texttt{Sig}_{tm}^k \triangleq g_{tm}\beta_m^2\sigma_{\texttt{h}}^4\|\mathbf{p}_m\|^4$, $\texttt{IntC}_{tm}^k \triangleq N\sum_{i\in\mathcal{S}_k^m} g_{ti}\beta_i^2\sigma_{\texttt{h}}^4|\mathbf{p}_m^H\mathbf{p}_i|^2$, and $\epsilon_{tm}^k \triangleq N_0\|\mathbf{p}_m\|^2 + \sum_{i\in\mathcal{S}_k} g_{ti}\beta_i\sigma_{\texttt{h}}^2|\mathbf{p}_m^H\mathbf{p}_i|^2$. For MMSE, $\epsilon_{tm}^k \triangleq N_0\|\mathbf{c}_{tm}^k\|^2 + \sum_{i\in\mathcal{S}_k} g_{ti}\beta_i\sigma_{\texttt{h}}^2|\mathbf{c}_{tm}^{kH}\mathbf{p}_i|^2$, $\texttt{Sig}_{tm}^k \triangleq g_{tm}(\epsilon_{tm}^k)^2$, $\texttt{IntC}_{tm}^k \triangleq N\sum_{i\in\mathcal{S}_k^m} g_{ti}\beta_i^2\sigma_{\texttt{h}}^4|\mathbf{c}_{tm}^{kH}\mathbf{p}_i|^2$, $\texttt{IntNC}_{tm}^k \triangleq g_{tm}\delta_{tm}^k + \sum_{i\in\mathcal{S}_k^m} g_{ti}\beta_i\sigma_{\texttt{h}}^2$. For MSBL, $\epsilon_{tm}^k \triangleq N_0\|\mathbf{c}_{tm}^k\|^2 + \sum_{i\in\mathcal{S}_k} g_{ti}\beta_i\sigma_{\texttt{h}}^2|\mathbf{c}_{tm}^{kH}\mathbf{p}_i|^2$, $\texttt{IntNC}_{tm}^k \triangleq \hat{g}_{tm}^k g_{tm}\delta_{tm}^k + \sum_{i\in\mathcal{S}_k^m} g_{ti}\beta_i\sigma_{\texttt{h}}^2$, $\texttt{Sig}_{tm}^k \triangleq \hat{g}_{tm}^k g_{tm}(\epsilon_{tm}^k)^2$, and $\texttt{IntC}_{tm}^k \triangleq N\sum_{i\in\mathcal{S}_k^m} g_{ti}\beta_i^2\sigma_{\texttt{h}}^4|\mathbf{c}_{tm}^{kH}\mathbf{p}_i|^2$. Here, δ_{tm}^k and $\mathbf{c}_{tm}^k$ are obtained from Theorems 1 and 2, respectively, for the three estimation schemes. The above expressions are obtained by replacing each of the terms involving $\hat{\mathbf{h}}_{tm}^k$ in the SINR with their respective deterministic equivalents.

References

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