

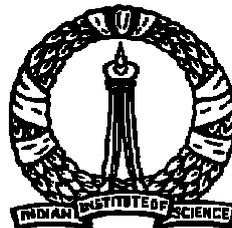
Finding a Subset of Non-defective Items from a Large Population: Fundamental Limits and Efficient Algorithms

A Thesis

Submitted for the Degree of
Doctor of Philosophy
in the Faculty of Engineering

by

Abhay Sharma



Electrical Communication Engineering
Indian Institute of Science, Bangalore
Bangalore – 560 012 (INDIA)

December 2014

TO

My Parents

Sh. M. G. Sharma & Smt. Suman Sharma

and

Parth & Upma

Acknowledgements

My Ph. D. journey has been an immensely enjoyable one. A milestone is about to be reached, and it would not have been possible without the able guidance of my adviser *Prof. Chandra Murthy*, encouragement of my family, and support from my friends and lab-mates.

First and foremost, I want to express my deepest gratitude to my **saarhi**, *Prof. Chandra Murthy*, whose guidance and leadership not only helped me reach my goal but also made the whole journey enjoyable. His approach of giving a lot of exploratory freedom while making sure we are not lost was instrumental in helping me achieve the twin goals I had set for myself: working on an interesting problem and broadening my knowledge base. His qualities of commitment, hard work and positive attitude during adverse moments will continue to inspire and guide me throughout my life. I would also like to extend my sincere appreciation to the faculty members at IISc, especially from the ECE, CSA and Math departments, where I attended several interesting courses.

I want to thank my family members for being such a spirited bunch of fellow co-passengers on my Ph. D. endeavor. I owe everything to my parents and no amount of words are sufficient to express my gratitude for them. The values they have instilled in me have been useful again and again during the course of my Ph. D. A very special thanks to my wife *Upma* without whose love, encouragement and unconditional support the dream of pursuing Ph. D. could not have been fulfilled. I want to thank my son *Parth* who was a source of such lovely distractions that provided the much needed refreshing breaks, especially during the times when research had hit the wall. I also want to thank *Ajay, Neeru* and *Adit*, for always being there in the background and providing dependable support that I could always count on.

I would like to thank my fellow lab-mates *Venu, Sanjeev, Ranjitha, Parthajit* and *Bharath* for making the last five years so memorable via a lot of technical and non-technical interactions. Also, I want to thank other friends from our lab and the adjoining lab, *Nanda, Gana, Saurabh, Mohit, Geethu, Vinnu, Vinuthna, Karthik P. N., Anuva, Karthik B., Shilpa, Sainath, Praveen, Jobin, Salil, Vikas, Ananya, Priyanka, Harsha, Suma* and *Nirmal*, with whom I have shared so many good moments during the course of my Ph. D. Special thanks to *Suma* for helping us with a variety of administrative tasks and also sharing with us interesting news about people around us.

I would like to thank an old friend *Subbu*, whose companionship at the start of my Ph. D. helped me immensely in easing into this ride. Finally, I would like to thank a close friend *Arun Hiregange*, who was a constant coffee-and-walk companion throughout the last five years. His unwavering enthusiasm to continuously learn about new and diverse things in life has been a source of inspiration for me.

Abstract

Consider a large population containing a small number of *defective* items. A commonly encountered goal is to identify the defective items, for example, to isolate them. In the classical non-adaptive group testing (NAGT) approach, one groups the items into subsets, or pools, and runs tests for the presence of a defective item on each pool. Using the outcomes the tests, a fundamental goal of group testing is to reliably identify the complete set of defective items with as few tests as possible. In contrast, this thesis studies a non-defective subset identification problem, where the primary goal is to identify a “subset” of “non-defective” items given the test outcomes. The main contributions of this thesis are:

- We derive upper and lower bounds on the number of nonadaptive group tests required to identify a given number of non-defective items with arbitrarily small probability of incorrect identification as the population size goes to infinity. We show that an impressive reduction in the number of tests is achievable compared to the approach of first identifying all the defective items and then picking the required number of non-defective items from the complement set. For example, in the asymptotic regime with the population size $N \rightarrow \infty$, to identify L non-defective items out of a population containing K defective items, when the tests are reliable, our results show that $O\left(\frac{K}{\log K} \frac{L}{N}\right)$ measurements are sufficient when $L \ll N - K$ and K is fixed. In contrast, the necessary number of tests using the conventional approach grows with N as $O\left(\frac{K}{\log K} \log \frac{N}{K}\right)$ measurements. Our results are derived using a general sparse signal model, by virtue of which, they are also applicable to other important sparse signal based applications such as compressive sensing.

- We present a bouquet of computationally efficient and analytically tractable non-defective subset recovery algorithms. By analyzing the probability of error of the algorithms, we obtain bounds on the number of tests required for non-defective subset recovery with arbitrarily small probability of error. By comparing with the information theoretic lower bounds, we show that the upper bounds on the number of tests are order-wise tight up to a $\log(K)$ factor, where K is the number of defective items. Our analysis accounts for the impact of both the additive noise (false positives) and dilution noise (false negatives). We also provide extensive simulation results that compare the relative performance of the different algorithms and provide further insights into their practical utility. The proposed algorithms significantly outperform the straightforward approaches of testing items one-by-one, and of first identifying the defective set and then choosing the non-defective items from the complement set, in terms of the number of measurements required to ensure a given success rate.
- We investigate the use of adaptive group testing in the application of finding a spectrum hole of a specified bandwidth in a given wideband of interest. We propose a group testing based spectrum hole search algorithm that exploits sparsity in the primary spectral occupancy by testing a group of adjacent sub-bands in a single test. This is enabled by a simple and easily implementable sub-Nyquist sampling scheme for signal acquisition by the cognitive radios. Energy-based hypothesis tests are used to provide an occupancy decision over the group of sub-bands, and this forms the basis of the proposed algorithm to find contiguous spectrum holes of a specified bandwidth. We extend this framework to a multi-stage sensing algorithm that can be employed in a variety of spectrum sensing scenarios, including non-contiguous spectrum hole search. Our analysis allows one to identify the sparsity and SNR regimes where group testing can lead to significantly lower detection delays compared to a conventional bin-by-bin energy detection scheme. We illustrate the performance of the proposed algorithms via Monte Carlo simulations.

Notation

Vectors are denoted by boldface lower case letters. Matrices are denoted using upper-case bold letters. Scalar random variables are represented by capital non-bold alphabets. Indexed random variables are denoted using the index set as sub-script, e.g., with $S = \{1, 3, 5, 8\}$, Z_S denotes a set of 4 random variables $\{Z_1, Z_3, Z_5, Z_8\}$. $Pr\{A\}$ denotes the probability of occurrence of an event A and $Pr\{A|B\}$ denotes the conditional probability of occurrence of event A given event B . The notation used in this thesis is listed in the table below:

$[N]$: $\{1, 2, \dots, N\}$ for any positive integer N
$\{\emptyset\}$: Null set
For sets A and B	
A^c	: Complement of set A
$ A $: Cardinality of set A
$A \setminus B$: $A \cap B^c$, i.e., elements of A that are not in B

For vectors \underline{a} and \underline{b}

$\underline{a}(i)$: i^{th} component of vector \underline{a}
$\text{supp}(\underline{a})$: Support of \underline{a} , i.e., the set $\{j : \underline{a}(j) > 0\}$
$\{\underline{a} = c\}$: $\{j : \underline{a}(j) = c\}$ for any c
\underline{a}^c	: Component wise boolean complement of a boolean vector \underline{a}
$\underline{a} \preceq \underline{b}$: Component-wise inequality, i.e., $\underline{a}(i) \leq \underline{b}(i) \forall i$
$\underline{a} \circ \underline{b}$: Component-wise product, i.e., $\forall i$ $(\underline{a} \circ \underline{b})(i) = \underline{a}(i)\underline{b}(i)$

$\underline{1}_n$: All-ones vector of size n
$\underline{0}_n$: All-zeros vector of size n
$\text{diag}(\underline{a})$: The diagonal matrix with diagonal entries given by the entries of \underline{a}

For a given matrix \mathbf{Y}

- $\underline{y}_i^{(r)}$: The i^{th} row of \mathbf{Y}
 \underline{y}_j : The j^{th} column of \mathbf{Y}
 $\mathbf{Y}(S, :)$: The sub-matrix of \mathbf{Y} that consists only of rows indexed by set S
 $\mathbf{Y}(:, S)$ or \mathbf{Y}_S : The sub-matrix of \mathbf{Y} that consists only of columns indexed by set S
-

Functions

- \mathbb{I}_A : Indicator function of set A . Returns 1 if the event A is true, and returns 0 otherwise
 $Q(\cdot)$: The standard Gaussian tail function
 $\text{sgn}(\cdot)$: The signum function
 $H_b(p)$: The binary entropy function, i.e., for any $0 \leq p \leq 1$
 $H_b(p) = -p \log(p) - (1 - p) \log(1 - p)$
 $E[\cdot]$: Expectation operator
-

Big-O notation

- $x(n) = O(y(n))$: Implies that $\exists B > 0$ and $n_0 > 0$, such that $|x(n)| \leq B|y(n)|$ for all $n > n_0$
 $x(n) = \Omega(y(n))$: Implies that $\exists B > 0$ and $n_0 > 0$, such that $|x(n)| \geq B|y(n)|$ for all $n > n_0$
-

- $\mathcal{N}(m, s^2)$: The Gaussian distribution with mean m and variance s^2
 $\chi^2(k)$: The Chi-squared distribution with k degrees of freedom
 $\mathcal{B}(q)$: The Bernoulli distribution with parameter q , $q \in [0, 1]$

Contents

Acknowledgements	i
Abstract	iii
Notation	v
1 Introduction	1
1.1 Group testing	2
1.1.1 Non-Adaptive Group Testing	3
1.2 Non-Defective Subset Identification	7
1.2.1 Non-Defective Subset Identification: Example Applications	7
1.2.2 Scope of Work	9
1.3 Organization of the Thesis and Summary of Main Contributions	11
2 Finding a Subset of Non-Defective Items: Fundamental Bounds	17
2.1 Problem Setup	21
2.2 Sufficient Number of Observations	27
2.2.1 Decoding scheme 1: Look into the Complement Set	28
2.2.2 Decoding Scheme 2: Find the Inactive Subset Directly, $K = 1$ case	29
2.2.3 Decoding Scheme 2: Find the Inactive Subset Directly, $K \geq 1$ case	31
2.3 Necessary Number of Observations	34
2.4 Finding Non-Defective Items Via Group Testing	37
2.4.1 Sufficiency Results when K Grows with N	40
2.5 Proofs of the Main Results	41

2.5.1	Proof of Theorem 1: Sufficient Number of Observations, $K = 1$ Case	42
2.5.2	Proof of Theorem 2: Sufficient Number of Observations, $K \geq 1$ Case	45
2.5.3	Proof of Theorem 3: Necessary Number of Observations	50
2.6	Conclusions	52
3	Finding a Subset of Non-defective Items: Computationally Efficient Algorithms	53
3.1	Signal Model	56
3.2	Algorithms and Main Results	60
3.2.1	Row Based Algorithm	60
3.2.2	Column Based Algorithm	62
3.2.3	Linear program relaxation based algorithms	64
3.2.4	Discussion on the theoretical guarantees	69
3.3	Proofs of the Main Results	73
3.3.1	Proof of Theorem 4 and Corollary 2	73
3.3.2	Proof of Theorem 5	77
3.3.3	Proof of Theorem 6	79
3.3.4	Proof Sketch for Theorem 7	82
3.4	Simulations	83
3.5	Conclusions	91
4	Group Testing Based Spectrum Hole Search for Cognitive Radios	92
4.1	Signal Acquisition Scheme	96
4.2	The M -Bin Group Test Design	99
4.3	Optimal Parameters for the M -Bin Test	105
4.4	Multi-Stage Sensing Algorithm	109
4.5	Simulations and Results	112
4.6	Conclusions	117
5	Conclusions and Future Work	123
5.1	Summary of Contributions	123

5.2	Future Work	125
A	Appendix for Chapter 2	126
A.1	Asymptotic Properties of the Upper Bound for Decoding Scheme 2	126
A.2	Order-Tight Results for Necessary and Sufficient Number of Tests with Group Testing	129
A.3	Proof of Lemma 1	130
B	Appendix for Chapter 3	132
B.1	Proof of Lemma 2	132
B.2	Proof of Proposition 2	133
B.3	Proof of Proposition 4	133
B.4	Discussion on the theoretical guarantees for RoLpAl++	133
B.5	Chernoff Bounds	136
C	Appendix for Chapter 4	137
C.1	Proof of Proposition 5	137
C.2	Proof of Proposition 6	138
C.3	Proof of Proposition 7	139
C.4	Proof of Proposition 8	141
	Bibliography	143

List of Figures

1.1	A toy example to demonstrate noiseless non-adaptive group testing setup.	5
2.1	Sparse signal model: An equivalent random codebook based channel coding model.	23
2.2	Sufficiency bounds on the number of observations required to find L inactive variables.	35
3.1	Impact of different types of noise on the group testing signal model. . .	59
3.2	Average probability of error (APER) <i>vs.</i> number of tests M	87
3.3	Number of tests <i>vs.</i> size of non-defective subset.	88
3.4	Comparison of CoLpAI with the <i>scaled</i> information theoretic lower bounds.	89
3.5	Average probability of error variation with noise.	90
4.1	Block diagram of the wideband signal acquisition system.	99
4.2	Family of $\Gamma(\eta, K)$ functions.	107
4.3	$D_{avg}(K)$ is convex in K	109
4.4	A pictorial description for the multi-stage algorithm.	113
4.5	D_{avg} <i>Vs.</i> Sparsity in the AWGN case.	118
4.6	D_{avg} <i>Vs.</i> SNR in the AWGN case.	119
4.7	Good match with theoretical results, D_{avg} <i>Vs.</i> Sparsity in the AWGN case.	119
4.8	Contiguous hole search, with fading and shadowing, D_{avg} <i>Vs.</i> SNR. . . .	120
4.9	Contiguous hole search, AWGN, D_{avg} <i>Vs.</i> SNR.	120
4.10	Normalized throughput <i>Vs.</i> the sensing time.	121
4.11	Search durations for different multi-stage instantiations.	121
4.12	Spectrum hole search and frequency hopping.	122

A.1 Behavior of $\Gamma_{u_2}(\alpha)$ with N 128

List of Tables

2.1	Finding a subset of L non-defective items: Order results for sufficient number of group tests which hold asymptotically as $N \rightarrow \infty, L \rightarrow \infty, \frac{L}{N} \rightarrow \alpha_0$ with $0 < \alpha_0 < 1$ and K fixed.	39
2.2	Finding a subset of L non-defective items: Order results for necessary number of group tests which hold asymptotically as $N \rightarrow \infty, L \rightarrow \infty, \frac{L}{N} = \alpha_0$ with $0 < \alpha_0 < 1$ and K fixed.	39
3.1	Finding a subset of L non-defective items: Order results for necessary number of group tests which hold asymptotically as $N \rightarrow \infty$ (see Theorem 3, Chapter 2). “Fixed- K ” regime: $\frac{L}{N} = \alpha_0, 0 < \alpha_0 < 1$ and fixed K . “Increasing- K ” regime: $\frac{K}{N} = \beta_0, \frac{L}{N} = \alpha_0, \alpha_0 + \beta_0 < 1, 0 < \beta_0 < \alpha_0 < 1$ and $\phi_0 \triangleq \frac{\alpha_0}{1-\beta_0}$	73
3.2	Robustness of the non-defective subset identification algorithms in the presence of uncertainty in the knowledge of K	87
4.1	Percentage D_{avg} reduction compared to $M = 1$	117
4.2	Average number of tests for varying sparsity at SNR = 9dB.	117
4.3	Achievable throughput, R/B bits/s: Comparison of the M -bin test with the detector in [1].	118

Chapter 1

Introduction

Identification of “defective” members of a population of “items” is an interesting and challenging task especially when the population size is large. Consider a population consisting of N items, of which an unknown subset of $K (\ll N)$ items is defective. Suppose the items possess a certain testable property, e.g., the presence of an antigen in a blood sample, presence of a pollutant in an air sample, etc. One pertinent goal in this context is to identify the subset of defective items in the population. A straightforward approach is to test all N items one-by-one, but this would lead to prohibitively high testing times (or high hardware cost for testing items in parallel), when N is large. However, if it is possible to group multiple items together in a test, e.g., pooling blood samples from multiple individuals, mixing the pollutant gases from multiple environmental areas etc., intuitively speaking, we can expect to achieve a significant reduction in the number of tests, especially if most of the items being tested are non-defective. This simple idea forms the basis of a powerful testing framework called *group testing*.

1.1 Group testing

Group testing [2–4] is an efficient way of identifying a small number of defective items from a large population of items. A defining notion of this framework is that items are *grouped* or *pooled* together, and a test is run on the aggregate pooled sample. The outcome of the test is a binary indication as to whether or not the property of interest is present collectively in the group. A *negative* indication implies that *none* of the tested items possess this property. That is, none of the items tested in that group are defective. A *positive* outcome implies that *at least one* of the items possesses the given property. Using the outcomes of multiple such group tests, a basic goal of group testing is to identify the defective set of items with as few tests as possible. Intuitively, the merit in the idea of group testing lies in the following fact: Since a test with negative outcome implies (when the tests are reliable) that all the items included in the test are non-defective, no further testing is required for items tested in pools with negative outcomes, leading to a reduction in number of tests. The reduction is particularly dramatic when the number of defective items is small compared to the size of the population, and the number of items tested in each pool is chosen judiciously.

Group testing was proposed in 1943 by Dorfman [2], as a strategy to reduce the number of blood tests required to identify the men infected with syphilis among the prospective entrants into the US military service during the World War II. Instead of testing blood sample for each inductee for the infection, Dorfman suggested that the blood samples of multiple individuals be pooled and analyzed together. Thus, the blood samples of all the inductees denote the set of “items”, an infected blood sample denotes a “defective” item and the test performed to detect the presence of the

syphilitic antigen in the pooled sample corresponds to the “group test”. By analyzing the outcomes of the tests performed on each of the pools, the goal was to identify all the syphilis infected inductees. Interestingly, group testing was never used in the application for which it was proposed and the concept lay dormant until Sobel and Groll [5] proposed its use for industrial testing design.¹ Since then, the framework of group testing has found applications in diverse engineering fields such as industrial testing [5,7], data pattern mining [8], DNA sequencing [3,4,9], medical screening [4,10], multi-access communications [4,11], data streaming [12,13], food contamination testing [14], viral epidemiology [15], Graph constrained group testing [16].

1.1.1 Non-Adaptive Group Testing

Group testing can be broadly classified into two categories: adaptive and non-adaptive group testing. In the adaptive framework the tests are designed to be run sequentially, and the future tests take into account the outcomes of the previous tests. In the non-adaptive framework, all tests (pools) are specified in advance and can be conducted simultaneously, i.e., the tests do not use the information provided by the outcome of any other test. In many applications, e.g., in molecular biology, DNA library screening etc., the test procedure is long and can take several hours or even days, and the sequential testing associated with the adaptive framework leads to prohibitively high testing time. Similarly, in many other applications, e.g., in spectrum hole search for cognitive radios, the overall test (search) time is of critical importance and needs to be kept as small as possible. For such applications, non-adaptive group testing (NGT) framework is the

¹In fact, the term group testing was coined by Sobel and Groll [5]. For more interesting historical facts on group testing, the reader is referred to [6].

preferred choice, and will be the main focus of the work presented in this thesis.

A non-adaptive group testing setup can be described using a boolean valued test matrix (see Figure 1.1) where the rows are labeled by tests/pools and the columns are labeled by items. The i^{th} row describes the items included in the i^{th} group test, and these correspond to the columns that have a 1 in the i^{th} row. The j^{th} column of the test matrix describes the pools in which j^{th} item is included. Each test is associated with a boolean outcome, with a 1 indicating that at least one defective item is present in the pool and a 0 indicating that none of the items included in the pool are defective. Using the test outcome vector and the test matrix, a primary goal in non-adaptive group testing is to identify the defective set. A toy example demonstrating a typical non-adaptive group testing setup is presented in Figure 1.1.

An important aspect of the non-adaptive group testing is the pooling strategy, i.e., how to determine the set of individuals that go into each group test. Two main approaches exist [4]: First, a combinatorial approach, see e.g., [17–19], which considers explicit constructions, e.g., using superimposed codes, to design test matrices/pools with properties² that lead to guaranteed detection of a small number of defective items. The second pooling strategy for non-adaptive group testing is to use a probabilistic pooling approach, see e.g., [4, 15, 20]. Here, the items included in the group test are chosen uniformly at random from the population. Further, the items in different pools are chosen independently of each other, leading to a test matrix with random i.i.d. entries.

²One such property is disjunctness [4]. A test matrix, with tests indexing the rows and items indexing the columns, is said to be k -disjunct if the boolean sum of every k columns does not equal any other column in the matrix.

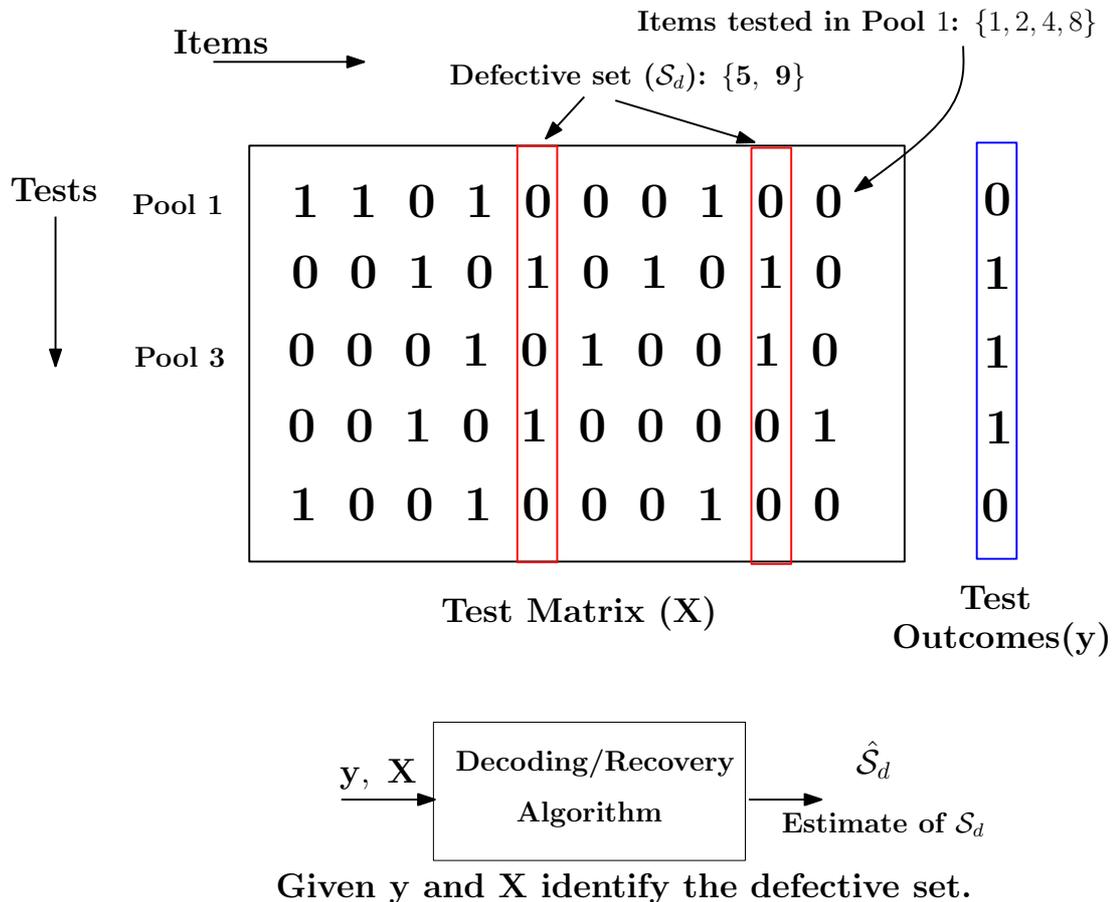


Figure 1.1: A toy example demonstrating the noiseless non-adaptive group testing setup: The number of items is $N = 10$, the number of defective items is $K = 2$, the defective set is $\{5, 9\}$, and the number of tests is $M = 5$. Pools are described by rows of the test matrix X . For example, Pool 1 is described by the first row, and indicates that items 1, 2, 4 and 8 are included in test 1. Also, column 1 indicates that item 1 is tested in Pools 1 and 5. The outcome vector y denotes the outcome of each test. For example, Pool 1 does not include any defective item and hence its output is 0. Pool 2 includes two defective items and one non-defective item. Its output is 1, indicating that there is at least one defective item included in the pool. The goal of the recovery algorithm is to use the test outcomes and the knowledge of the test matrix to identify the defective set.

Note that, with random constructions, there is always a (small) chance that a given instance of the test matrix will not be successful in identifying a defective set, even in the noiseless case. The combinatorial approach highlights the “design” approach towards the group testing. Explicit constructions are attractive as they are efficient and often lead to simpler decoding algorithms. However, only a few such constructions exist, see e.g., [4, Chapter 5]. Our main interest lies in the random pooling designs since these can always be constructed. Moreover, the random pooling design is amenable to comprehensive analysis, leading to critical insights on the impact of system parameters such as the number of tests on the performance of the group tests. However, a key issue with the random construction of the test matrix is the design of computationally efficient algorithms for defective set identification, given the set of test outcomes.

The example in Figure 1.1 shows a setup where the test outcomes are noiseless, i.e., the outcome is always a 1 if the pool includes at least one defective item, and, 0 otherwise. However, in real world situations, due to practical impairments, the group test outcomes can be noisy and unreliable. Two type of noise effects are typically considered in the group testing literature: (a) Additive noise: This noise accounts for the false alarms (or false positives) in the test outcomes. That is, a test output can be 1 even if none of the defective items are included in the pool. (b) Dilution noise: This noise accounts for the missed detection (or false negatives) in the test outcomes. It refers to a case when a defective item that is included in a test appears as absent. If all the defective items in a given test get diluted then the outcome will come out to be negative, thereby leading to a false conclusion that the pool does not contain any defectives. Historically, the name dilution noise is motivated by the dilution of blood samples when many samples are

pooled together. When the concentration of the infectious agents (or the corresponding antigen) goes below detectable levels due to the dilution, the test outcome would appear negative, even though the pool contains defective items.

With the above background, we are now ready to describe the central problem considered in this thesis.

1.2 Non-Defective Subset Identification

Consider the general test setup consisting of a set of N items, where an unknown subset of $K (\ll N)$ items are defective. Let us denote the defective subset by \mathcal{S}_d . Its complement set, i.e., $\mathcal{S}_h \triangleq [N] \setminus \mathcal{S}_d$, is referred to as the “non-defective” or “healthy” subset. Note that $|\mathcal{S}_h| = N - K$. In many applications, as described below, the items of interest belong to the non-defective set (as opposed to the defective set) and we wish to identify a subset of items of size $L (\leq N - K)$ belonging to \mathcal{S}_h . We refer to this as the *non-defective subset identification problem*, and is the main focus of the work presented in this thesis.

1.2.1 Non-Defective Subset Identification: Example Applications

We now present several practical scenarios where the primary goal is to identify a subset of non-defective items.

- **Spectrum hole search in cognitive radio networks:** Consider the spectrum hole search problem in a cognitive radio (CR) network setup. It is known that the primary user occupancy is sparse in the frequency domain, over a wide band of interest [21,22]. This is equivalent to having a small subset of defective items (bins with primary occupancy) embedded in a large set of candidate frequency bins.

The secondary users do not need to identify all the frequency bins occupied by the primary users (defective set identification); they only need to discover small unoccupied sub-bands to setup the secondary communications. This, in turn, is a non-defective subset identification problem when the bins to be tested for primary occupancy can be pooled together into group tests [23]. For example, consider a CR network being setup on the Bluetooth band with a requirement of 8 MHz for its operations. The Bluetooth band consists of 80 1-MHz channels (items). In this case, a defective item is equivalent to a channel occupied by a Bluetooth user. At a given location and time, if, say, 4 Bluetooth users are active, the cognitive network needs to find a set of 8 non-defective items from a population of 80 items, out of which 4 are defective.

- **Data stream sketch queries:** This is an example from the data stream domain [12, 24]. We receive a high volume SMS data stream as the response to a trivia contest run during a television show. The SMS data stream is processed to ascertain whether the answer is correct. The outcome is streamed to the server at the TV studio as $\langle phone.number, flag \rangle$, where $flag$ is a binary valued variable, with $flag = 1$ indicating a correct answer and $flag = -1$ indicating a wrong answer. Owing to the simplicity of trivia questions, typically, a large majority of the $flag$ variables are equal to 1. That is, each processed SMS record is an item, and the records with $flag = -1$ are considered to be defective. Due to the large number of received records and severe memory constraints, the data stream is often summarized using a small number of “sketches” using measurement/test matrices, and this sketch vector is equivalent to the test outcome vector in the non-adaptive

group testing setup [13,24]. The objective is to use the sketch vector to identify a small group of responders with correct answers, i.e., the winners of the contest, and is thus a *non-defective subset identification problem*, since the goal is to find a small *subset* of non-defectives from a large population of items.

- **Food contamination:** Consider a food contamination testing setup where food samples, e.g., sprout samples, from different vendors is being tested for contamination by bacteria of a specific type [14]. A small subset of vendors with uncontaminated food samples (non-defective items) has to be identified for an urgent food consignment delivery. Once again, the interest here is in the identification of a subset of the non-defective items using as few group tests as possible.

1.2.2 Scope of Work

In classical group testing, the defective set identification problem has been studied in detail, see e.g., [4,6] and references therein. There exist bounds on the number of tests required to identify the defective set, both in the noiseless and noisy settings, and both for computationally efficient decoding algorithms as well as under general information theoretic models [4, 13, 15, 17–20, 25–30]. However, to the best of our knowledge, the problem of non-defective subset identification has not been studied before. For example, no results are available that relate the parameter L , the size of non-defective subset, to the number of tests required to identify it.

The defective set identification problem is a special case of the non-defective subset identification problem: we obtain the former by setting $L = N - K$ in the latter. Further, we note that a straightforward way to identify a set of $L(\leq N - K)$ non-defective

items is to first identify all the defective items and subsequently choose any L items from the complement set. Thus, a solution to defective set identification is also a (sub-optimal) solution to the non-defective subset identification. Hence, the number of tests that are sufficient for defective set identification is an upper bound on the number of tests that are sufficient for non-defective subset identification. However, intuitively, a much smaller number of tests should suffice for non-defective subset identification, since it is not necessary to identify *all* the defective items. A formalization of the above intuition by quantifying the number of tests required to identify an L sized subset of non-defective items is one of the key highlights of this work. In particular, we have studied the following two aspects in detail:

- (a) **Fundamental Limits:** Here, our goal is to derive information theoretic upper and lower bounds on the number of tests required for identifying L non-defective items in a non-adaptive framework. The utility of deriving such bounds is two-fold: First, it aids comparison with the existing bounds for *defective set identification* problem and thereby substantiates the need to separately study the non-defective subset identification problem. Second, such analysis provides a rigorous benchmark against which the performance of computationally efficient and practically useful algorithms can be compared.

- (b) **Computationally Efficient and Analytically Tractable Algorithms:** Here we study computationally efficient algorithms to recover/identify a set of L non-defective items given the set of noisy outcomes in a non-adaptive group testing setup. Another key requirement for these algorithms is analytical tractability. Thus, an important aspect of this work is the derivation of sample complexity guarantees for each

algorithm, i.e., the number of tests required to identify a subset of non-defective items with arbitrarily small probability of error.

In addition to studying the non-defective subset identification problem in a general mathematical framework, we have investigated it in the context of a specific application. We have studied the use of group testing for finding a spectrum hole of a specified bandwidth in a given wideband of interest. An interesting aspect of this work is the design of the group test, i.e., a test that operates on multiple items (sub-bands in this case) and provides a binary decision regarding the presence of the primary in the any of the sub-bands included in the test. The proposed test is based on a simple and easily implementable sub-Nyquist sampling scheme for signal acquisition by the cognitive radios. Since running multiple tests in parallel would entail a prohibitively high hardware cost, we study this problem in an adaptive framework.

1.3 Organization of the Thesis and Summary of Main Contributions

The remainder of this dissertation is organized as follows:

In Chapter 2, we study the information theoretic aspects of the non-defective subset identification problem. We derive mutual information based upper and lower bounds on the number of nonadaptive group tests required to identify a given number of “non-defective” items from a large population containing a small number of “defective” items with arbitrarily small probability of incorrect identification as the population size goes to infinity. We formulate the problem of non-defective subset identification as a detection problem using observations obtained from a general sparse signal model.

Using multiple observations from this framework, we propose and analyze decoding schemes to identify L non-defective items. The upper bounds are derived by analyzing the error probability of these decoding schemes. We use a modified form of Fano's inequality to derive a lower bound on the number of observations required to identify L inactive variables with arbitrarily small probability of error. We show that an impressive reduction in the number of tests is achievable compared to the approach of first identifying all the defective items and then picking the required number of non-defective items from the complement set. For example, in the asymptotic regime with the population size $N \rightarrow \infty$, to identify L non-defective items out of a population containing K defective items, when the tests are reliable, our results show that $O\left(\frac{K}{\log K} \frac{L}{N}\right)$ measurements are sufficient when $L \ll N - K$ and K is fixed. In contrast, the necessary number of tests using the conventional approach grows with N as $O\left(\frac{K}{\log K} \log \frac{N}{K}\right)$ measurements. As mentioned earlier, a highlight of our analysis is that it is based on a very general sparse signal model, by virtue of which, the results obtained are also applicable to other important sparse signal based applications such as compressive sensing.

In Chapter 3, the main focus is on developing non-defective subset recovery algorithms. We present a bouquet of computationally efficient and analytically tractable algorithms for non-defective subset identification in the noisy, non-adaptive group testing with random pooling framework. We analyze the probability of error of the algorithms and obtain bounds on the number of tests required for non-defective subset recovery with arbitrarily small probability of error. Our analysis also helps to determine the parameters of the algorithms that offer the best performance in terms of optimizing the error upper bounds. By comparing with the information theoretic lower bounds

presented in Chapter 2, we show that the upper bounds on the number of tests are order-wise tight up to a $\log(K)$ factor, where K is the number of defective items. We also provide extensive simulation results that compare the relative performance of the different algorithms and provide further insights into their practical utility.

In Chapter 4, we investigate the use of *adaptive* group testing for finding a spectrum hole of a specified bandwidth in a given wide band of interest. We propose a group testing based search algorithm that is enabled by a simple and easily implementable sub-Nyquist sampling scheme for signal acquisition. The sampling scheme deliberately introduces aliasing during signal acquisition, resulting in a signal that is the sum of signals from adjacent sub-bands. The acquisition scheme entails only a minimal hardware change, compared to the narrowband energy detector. The proposed group test, which is an energy-based hypothesis test and provides an occupancy decision over the group of adjacent sub-bands, forms the basis of the proposed algorithm to find contiguous spectrum holes of a specified bandwidth. We also propose a multi-stage sensing algorithm that can be employed in a variety of spectrum sensing scenarios, e.g., spectrum sensing with frequency hopping primaries, non-contiguous spectrum hole search etc. Further, we theoretically analyze the detection delay behavior of the algorithm, and use it to optimize the parameters (group size, samples per test, and detection thresholds) of the search algorithm. Our analysis allows one to identify the sparsity and SNR regimes where group testing can lead to significantly lower detection delays compared to a conventional bin-by-bin energy detection scheme. We provide extensive simulation results that corroborate our theoretical results.

The appendices containing supplementary material for each chapter (Appendix A for

Chapter 2, Appendix B for Chapter 3 and Appendix C for Chapter 4) are included at the end of this dissertation.

List of Publications from this Thesis

Journal Papers

- Abhay Sharma and C. R. Murthy, “ **Computationally Tractable Algorithms for Finding a Subset of Non-defective Items from a Large Population**”, *Submitted to IEEE Transactions on Information Theory*, Nov., 2014.
- Abhay Sharma and C. R. Murthy, “**On Finding a Subset of Non-Defective Items from a Large Population**”, *Submitted to IEEE Transactions On Information Theory*, Sept., 2014.
- Abhay Sharma and C. R. Murthy, “**Group Testing Based Spectrum Hole Search for Cognitive Radios**”, *IEEE Tran. On Vehicular Technology*, vol. 63, no. 8, pp. 3794-3805, Oct., 2014.

Conference Papers

- Abhay Sharma and C. R. Murthy, “**On Finding a Subset of Non-Defective Items from A Large Population Using Group Tests: Recovery Algorithms and Bounds**”, *Submitted to IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, 2015.
- Abhay Sharma and C. R. Murthy, “**On Finding a Set of Healthy Individuals**

From a Large Population", *Information Theory and Applications Workshop*, San Diego, CA, USA, 2013 (Invited paper).

- Abhay Sharma and C. R. Murthy, "**A Group Testing Based Spectrum Hole Search Using a Simple Sub-Nyquist Sampling Scheme**", *Proc. IEEE Global Communications Conference (GLOBECOM)*, 2012.

Chapter 2

Finding a Subset of Non-Defective Items: Fundamental Bounds

In this chapter, we derive information theoretic upper and lower bounds on the number of nonadaptive group tests required to identify a given number of non-defective items from a large population containing a small number of defective items. We adopt a general sparse signal model used in the literature, that is applicable in a variety of areas such as compressive sensing [31], group testing [2,4], signal de-noising [32], subset selection [33], etc. In the sparse signal model, out of a given number N of input variables, only a small subset of size $K (\ll N)$ contributes to the observed output, and the output is independent of the other variables. For example, in a non-adaptive group testing setup, the output is independent of whether or not any items from the non-defective set participate in the group test. Similarly, in a compressive sensing setup, the output signal is a set of random projections of a sparse signal. Hence, the output only depends on the non-zero entries (support set) of the input vector. This *salient* subset of inputs is referred to by different names, e.g., defective items, sick individuals, support set, etc. In this chapter, we will refer to it as *the active set*, and its complement as *the inactive set*.

In this chapter, we address the issue of the *inactive subset recovery*. That is, we focus on the task of finding an $L (\leq N - K)$ sized subset of *the inactive set* (of size $N - K$), given the observations from a sparse signal model with N inputs, out of which $K (\ll N)$ are active. The problem of finding a subset of items belonging to the inactive set is of interest in many applications, and we refer the reader to Section 1.2.1 for a discussion of example applications.

Related work: In the group testing literature, the problem of bounding the number of tests required to identify the defective items in a large pool has been studied, both in the noiseless and noisy settings, both for tractable decoding algorithms as well as under general information theoretic models [13, 15, 17–20, 25–30]. A combinatorial approach has been adopted in [17–19], where explicit constructions for the test matrices are used, e.g., using superimposed codes, to design matrices with properties that lead to guaranteed detection of a small number of defective items. Two such properties were considered: disjunctness and separability [4].¹ A probabilistic approach was adopted in [13, 15, 25, 26], where random test matrix designs were considered, and upper and lower bounds on the number of tests required to satisfy the properties of disjunctness or separability with high probability were derived. In particular, [15] analyzed the performance of group testing under the so-called dilution noise. A recent study [20] uses random test designs, and develops computationally efficient algorithms for identifying defective items from the noisy test outcomes by exploiting the connection with compressive sensing. The authors also show that the algorithms are near-optimal in terms

¹A test matrix, with tests indexing the rows and items indexing the columns, is said to be k -disjunct if the boolean sum of every k columns does not equal any other column in the matrix. Also, a test matrix is said to be k -separable if the boolean sum of every set of k columns is unique.

of the number of tests required for a guaranteed performance. A general sparse signal model for studying group testing problems, that turns out to be very useful in dealing with noisy settings, was proposed and used in [27–30]. In this framework, the group testing problem was formulated as a detection problem and a one-to-one correspondence was established with a communication channel model. Using information theoretic arguments, mutual information based expressions (that are easily computable for a wide variety of noisy channels) for upper and lower bounds on the number of tests were obtained [30]. In the related field of compressive sensing, an active line of research has focused on the conditions under which reliable signal recovery from observations drawn from a linear sparse signal model is possible, for example, conditions on the number of measurements required and on isometry properties of the measurement matrix ([34, 35], and references therein). In particular, there exists a good understanding of the bounds on the number of measurements required for support recovery from noisy linear projections (e.g., [36–40]).

Thus, to the best of our knowledge, fundamental bounds on the number of tests needed to find L non-defective items, which is the focus in this chapter, have not been derived in the existing literature. A recent work [41] studies the problem of finding zeros in a sparse vector in the framework of compressive sensing. The authors propose computationally efficient recovery algorithms and study their performance through simulations. In contrast, our work builds on our earlier work [42], and focuses on deriving information theoretic upper and lower bounds on the number of measurements needed for identifying a given number of inactive items in a large population with arbitrarily small error probability.

For the fundamental bounds, we consider the general sparse signal model employed in [27, 30] in the context of the problem of defective set recovery. The model consists of N input variables, out of which, an unknown subset S of size K is “active”; in the sense that, only the variables in the set S are relevant to the output. Mathematically, this is modeled by assuming that, conditioned on the active set S , the output Y is independent of remaining input variables. The probability distribution of the output conditioned on a given active set is assumed to be known for all possible active sets. Given multiple observations from this model, we propose and analyze decoding schemes to identify a set of L inactive variables. We compare two alternative schemes: (a) Decode the active set and then choose L variables randomly from the complement set, and, (b) Decode the inactive subset directly from the observations. Our main contributions are as follows:

1. We analyze the average probability of error for both the decoding schemes. We use the analysis to obtain mutual information based upper bounds on the number of observations required to identify a set of L inactive variables with the probability of error decreasing exponentially with the number of observations.
2. We specialize the above bounds to various noisy non-adaptive group testing scenarios, and characterize the number of tests required to identify L non-defective items, in terms of L , N and K .
3. Using Fano’s inequality, we also derive a lower bound on the number of observations required to identify L inactive variables with arbitrarily small error probability.

Our results show that, compared to the conventional approach of identifying an inactive subset by first decoding the active set, directly searching for an L -sized inactive subset offers a substantial reduction in the number of observations (tests/measurements), especially when L is small compared to $N - K$. For example, in the asymptotic regime as $N \rightarrow \infty, L \rightarrow \infty, \frac{L}{N} = \alpha_0$ and fixed K , when the tests are reliable, $O\left(\frac{K}{\log K} \frac{L}{N}\right)$ observations are sufficient when α_0 is small. In contrast, the number of observations required by the conventional approach grows as $O\left(\frac{K}{\log K} \log \frac{N}{K}\right)$ [30].

The rest of the chapter is organized as follows. Section 2.1 describes the signal model and problem setup. We present our upper and lower bounds on the number of observations in Sections 2.2 and 2.3, respectively. An application of the bounds to group testing is described in Section 2.4. The proofs for the main results are provided in Section 2.5, and concluding remarks are offered in Section 2.6.

2.1 Problem Setup

In this section, we describe the signal model and problem setup. Let $X_{[N]} = [X_1, X_2, \dots, X_N]$ denote a set of N independent and identically distributed input random variables (or *items*). Let each X_j belong to a finite alphabet denoted by \mathcal{X} and be distributed as $\Pr\{X_j = x\} = Q(x), x \in \mathcal{X}, j = 1, 2, \dots, N$. For a group of input variables, e.g., $X_{[N]}$, $Q(X_{[N]}) = \prod_{j \in [N]} Q(X_j)$ denotes the joint distribution for all the input variables. We consider a sparse signal model where only a subset of the input variables are *active* (or *defective*), in the sense that only a subset of the input variables contribute to the output. Let $S \subset [N]$ denote the set of input variables that are active, with $|S| = K$. Let $S^c \triangleq [N] \setminus S$ denote the set of variables that are *inactive* (or *non-defective*). Let the output

belong to a finite alphabet denoted by \mathcal{Y} . We assume that Y is generated according to a known conditional distribution $P(Y|X_{[N]})$. Then, in our observation model, we assume that given the active set, S , the output signal, Y , is independent of the other input variables. That is, $\forall Y \in \mathcal{Y}$,

$$P(Y|X_{[N]}) = P(Y|X_S). \quad (2.1)$$

We observe the outputs corresponding to M independent realizations of the input variables, and denote the inputs and the corresponding observations by $\{\mathbf{X}, \underline{\mathbf{y}}\}$. Here, \mathbf{X} is an $M \times N$ matrix, with its i^{th} row representing the i^{th} realization of the input variables, and $\underline{\mathbf{y}}$ is an $M \times 1$ vector, with its i^{th} component representing the i^{th} observed output. Note that, the independence assumption across the input variables and across different observations implies that each entry in \mathbf{X} is independent and identically distributed (i.i.d.). Let $L \leq N - K$. We consider the problem of finding a set of L inactive variables given the observation set, $\{\mathbf{X}, \underline{\mathbf{y}}\}$. That is, we wish to find an index set $S_H \subset S^c$ such that $|S_H| = L$. In particular, our goal is to derive information theoretic bounds on the number of observations (measurements/group tests) required to find a set of L inactive variables with the probability of error exponentially decreasing with the number of observations. Here, an error event occurs if the chosen inactive set contains one or more active variables. Now, one way to find L inactive variables is to find all the active variables and then choose any L variables from the complement set. Thus, existing bounds on M for finding the active set are an upper bound on the number of observations required for solving our problem. However, intuitively speaking, fewer observations should suffice to find L inactive variables, since we do not need to find

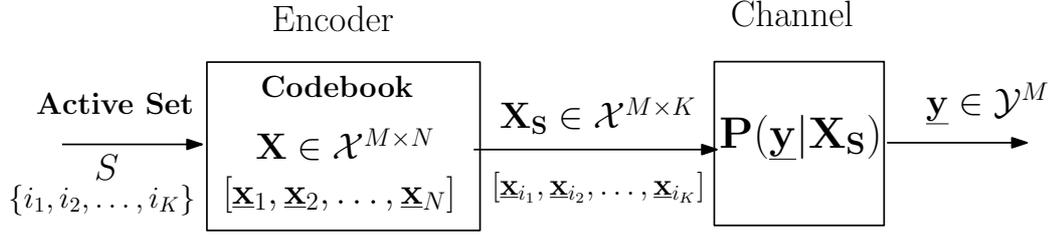


Figure 2.1: Sparse signal model: An equivalent random codebook based channel coding model.

the full active set. This is confirmed by our results presented in the next section.

The above signal model can be equivalently described using Shannon's random codebook based channel coding framework. The active set S , that corresponds to one of the $\binom{N}{K}$ possible active sets with K variables, constitutes the input message. Let $\mathbf{X} \in \mathcal{X}^{M \times N}$ be a random codebook consisting of N codewords of length M ; each associated with one of the N input variables. Let $\underline{\mathbf{x}}_i$ denote the codeword associated with i^{th} input variable. The encoder outputs the length- M message $\mathbf{X}_S \in \mathcal{X}^{M \times K}$, that comprises of the K codewords chosen according to the index set S from \mathbf{X} . That is, $\mathbf{X}_S = [\underline{\mathbf{x}}_{i_1} \ \underline{\mathbf{x}}_{i_2} \ \dots \ \underline{\mathbf{x}}_{i_K}]$, where $S = \{i_1, i_2, \dots, i_K\}$. The encoded message is transmitted through a discrete memoryless channel [43,44], denoted by $(\mathcal{X}^M, P(\underline{\mathbf{y}}|\mathbf{X}_S), \mathcal{Y}^M)$, where $P(\underline{\mathbf{y}}|\mathbf{X}_S) = \prod_{i=1}^M P(\underline{\mathbf{y}}(i)|\mathbf{X}_S^{(i)})$ and the distribution function $P(\underline{\mathbf{y}}(i)|\mathbf{X}_S^{(i)})$ is known for each active set S . Here, $\mathbf{X}_S^{(i)}$ denotes the i^{th} row of the matrix \mathbf{X}_S , and $\underline{\mathbf{y}}(i)$ denotes the i^{th} component of $\underline{\mathbf{y}}$. Given the codebook \mathbf{X} and the output message $\underline{\mathbf{y}}$, our goal is to find a set of L variables *not* belonging to the active set S . See Fig. 2.1 for a pictorial representation of the channel model.

We also note that our signal model, proposed and used earlier in [27,30], is a generalization of the signal models employed in some of the popular non-adaptive measurement systems such as compressed sensing² and non-adaptive group testing. Thus, the general mutual information based bounds on number of observations to find a set of inactive items obtained using the above model are applicable in a variety of practical scenarios. We elaborate on this in the following paragraphs.

We now discuss the above signal model in context of a specific non-adaptive measurement system, namely the random pooling based, noisy non-adaptive group testing framework [4,30]. In a group testing framework [4,27,30], we have a population of N items, out of which K are defective. Let $\mathcal{G} \subset [N]$ denote the defective set, such that $|\mathcal{G}| = K$. The group tests are defined by a boolean matrix, $\mathbf{X} \in \{0,1\}^{M \times N}$, that assigns different items to the M group tests (pools). In the i^{th} test, the items corresponding to the columns with 1 in the i^{th} row of \mathbf{X} are tested. Thus, M tests are specified. As in [30], we consider an i.i.d. random Bernoulli measurement matrix, where each $X_{ij} \sim \mathcal{B}(p)$ for some $0 < p < 1$. Here, p is a design parameter that controls the average group size. If the tests are completely reliable, then the output of the M tests is given by the boolean OR of the columns of \mathbf{X} corresponding to the *defective set* \mathcal{G} . In group testing, two different noise models are considered [15,30]: (a) An *additive* noise model, where there is a probability, $q \in (0, 1]$, that the outcome of a group test containing only non-defective items comes out positive; (b) A *dilution* model, where there is a probability, $u \in (0, 1]$, that a given item does not participate in a given group test. Let $\underline{d}_i \in \{0, 1\}^M$. Let $\underline{d}_i(j) \sim \mathcal{B}(1 - u)$ be chosen independently for all $j = 1, 2, \dots, M$ and

²Although we focus on models with finite alphabets in this work, our results easily extend to models with continuous alphabets, following along the lines of [45,46].

for all $i = 1, 2, \dots, N$. Let $\mathbf{D}_i \triangleq \text{diag}(\underline{d}_i)$. Let “ \vee ” denote the boolean OR operation. The output vector $\underline{y} \in \{0, 1\}^M$ can be represented as

$$\underline{y} = \bigvee_{i \in \mathcal{G}} \mathbf{D}_i \underline{x}_i \vee \underline{w}, \quad (2.2)$$

where $\underline{x}_i \in \{0, 1\}^M$ is the i^{th} column of \mathbf{X} , $\underline{w} \in \{0, 1\}^M$ is the additive noise with the i^{th} component $\underline{w}(i) \sim \mathcal{B}(q)$. Note that, for the noiseless case, $u = 0, q = 0$. In an additive model, $u = 0, q > 0$. In a dilution model, $u > 0, q = 0$.

The above “logical-OR” signal model arises in many practical non-adaptive group testing measurement systems. For example, consider a medical screening application, where a large number of individuals need to be tested for the presence of a specific antigen in their blood. The blood samples drawn from the different individuals are pooled together, according to a randomly generated test matrix \mathbf{X} (as described above), into multiple pools. Each pool is tested for the presence of the antigen. This test is well modeled by the OR-operation described above, i.e., when the tests are reliable, a test outcome is positive if one or more samples in the pool contain the antigen, and, a test outcome is negative only if none of the samples in the pool contain the antigen. Note that, given the knowledge of the set of individuals having the antigen, the test outcome is independent of whether the blood sample from any other individual is included in the pool or not. Another interesting concrete example of the above model, from the domain of viral epidemiology, is provided in [15]. The goal is to identify virally-infected people in a large population using a few number of group tests. A group test is realized through an “agent”, who is a person that makes contact with multiple people chosen randomly from the large population. The basic idea is that when the agent contacts an

infected person, there is a chance that the agent also gets infected. Multiple such agents comb a given area and at the end of the “mingling” procedure, each agent is tested for the viral infection. It is easy to see that the “OR-model” captures the process of viral infection for the agent. If the agent is infected, then it implies that it has made contact with one or more infected individuals from the population. Note that, even when the agent comes in contact with the infected person there is a chance that agent does not get infected, which can be modeled using the dilution noise model. Several other use case scenarios for the above described measurement system can be found in [4, 8, 11, 24].

We now relate the non-adaptive group testing model with the general sparse signal model described in (2.1). Note that, $\mathcal{X} = \{0, 1\}$, $\mathcal{Y} = \{0, 1\}$. Each item in the group testing framework corresponds to one of the N input variables. The i^{th} row of the test matrix corresponds to the i^{th} realization of the input variables, which also specifies the i^{th} random pool. From (2.2), we note that, given the defective set \mathcal{G} , the i^{th} test outcome $\underline{y}(i)$ is independent of values of input variables from the set $[N] \setminus \mathcal{G}$. That is, for the test outcome, it is *irrelevant* whether or not the items from the set $[N] \setminus \mathcal{G}$ are included in the pool. Thus, \mathcal{G} corresponds to the active set S . The probability distribution function $P(\underline{y} | \mathbf{X}_{\mathcal{G}})$ for any \mathcal{G} , is fully determined from (2.2) and the statistical models for the dilution and additive noise. Thus, the group testing framework is a special case of the general sparse model that we have considered; and the number of group tests corresponds to the number of observations in the sparse model.

We now define two quantities that are very useful in the development to follow. Let S be a given active set. For any $1 \leq j \leq K$, let $S^{(j)}$ and $S^{(K-j)}$ represent a partition of S

such that $S^{(j)} \cup S^{(K-j)} = S$, $S^{(j)} \cap S^{(K-j)} = \{\emptyset\}$ and $|S^{(j)}| = j$. Define

$$E_0(\rho, j, n) = -\log \sum_{Y \in \mathcal{Y}} \sum_{X_{S^{(K-j)}} \in \mathcal{X}^{K-j}} \left\{ \sum_{X_{S^{(j)}} \in \mathcal{X}^j} Q(X_{S^{(j)}}) (P(Y, X_{S^{(K-j)}} | X_{S^{(j)}}))^{\frac{1}{1+\rho n}} \right\}^{1+\rho n} \quad (2.3)$$

for any positive integer n and any $\rho \in [0, 1]$. Define $I^{(j)} \triangleq I(Y, X_{S^{(K-j)}}; X_{S^{(j)}})$ as the mutual information between $\{Y, X_{S^{(K-j)}}\}$ and $X_{S^{(j)}}$ [43,44]. Mathematically,

$$I^{(j)} = \sum_{Y \in \mathcal{Y}} \sum_{X_{S^{(K-j)}} \in \mathcal{X}^{K-j}} \sum_{X_{S^{(j)}} \in \mathcal{X}^j} P(Y, X_{S^{(K-j)}} | X_{S^{(j)}}) Q(X_{S^{(j)}}) \log \frac{P(Y, X_{S^{(K-j)}} | X_{S^{(j)}})}{P(Y, X_{S^{(K-j)}})}. \quad (2.4)$$

Using the independence assumptions in the signal model, for a given j , $E_0(\rho, j, n)$ and $I^{(j)}$ are independent of the specific choice of S , and of the specific partitions of S . It is easy to verify that $\frac{dE_0(\rho, j, n)}{d\rho} \Big|_{\rho=0} = nI^{(j)}$. Furthermore, it can be shown that $E_0(\rho, j, n)$ is a concave function of ρ [43].

2.2 Sufficient Number of Observations

We first present results on the sufficient number of observations to find a set of L inactive variables. The general methodology used to find the upper bounds is as follows: (a) Given a set of inputs and observations, $\{\mathbf{X}, \underline{\mathbf{y}}\}$, we first propose a decoding algorithm to find an L -sized inactive set, S_H ; (b) For the given decoding scheme, we find (or upper bound) the average probability of error, where the error probability is averaged over the random set $\{\mathbf{X}, \underline{\mathbf{y}}\}$ as well as over all possible choices for the active set. An error event occurs when the decoded set of L inactive variables contains one or more active variables. That is, with S as the active set and S_H as the *decoded* inactive set, an

error occurs if $S \cap S_H \neq \{\emptyset\}$; (c) We find the relationships between M, N, L and K that will drive the average probability of error to zero. Section 2.2.1 describes the straightforward decoding scheme where we find the inactive variables by first isolating the active set followed by choosing the inactive set randomly from the complement set. This is followed by the analysis of a new decoding schemes we propose in Section 2.2.2 and 2.2.3 , where we directly search for an inactive subset of the required cardinality.

2.2.1 Decoding scheme 1: Look into the Complement Set

One way to find a set of inactive (or non-defective) variables is to first decode the active (defective) set and then pick a set of L variables uniformly at random from the complement set. Here, we employ maximum likelihood based optimal decoding [30] to find the active set. Note that, intuitively, even if we choose a wrong active set, there is still a nonzero probability of picking a correct inactive set, since there remain only a few active variables in the complement set. The probability of error in identifying the active set was analyzed in [30]. The error probability when the same decoding scheme is employed to identify a inactive subset is similar, with an extra term to account for the probability of picking an incorrect set of L variables from the complement set. For this decoding scheme, we present the following result, without proof, as a corollary to (Theorem III.I, [30]).

Corollary 1. *Let N, M, L and K be as defined above. Let P_e be the average probability of error in finding L inactive variables. If*

$$M > \max_{1 \leq j \leq K} \frac{\log \left[\binom{N-K}{j} \binom{K}{j} C_0(L, N, K, j) \right]}{I^{(j)}}, \quad (2.5)$$

where

$$C_0(L, N, K, j) \triangleq \frac{\sum_{i=1}^j \binom{N-K-j}{L-i} \binom{j}{i}}{\binom{N-K}{L}}, \quad (2.6)$$

then there exists a positive constant, ϵ , such that $P_e \leq K \exp(-M\epsilon)$. That is, for all fixed $K \geq 1$, P_e approaches zero exponentially with the number of observations.

Thus, we have obtained a bound that gives us the sufficient number of observations to find a set of L inactive variables. Since $C_0 \leq 1$, this bound is tighter than the bound obtained by using the same number of observations as is required to find the active set [30]. Can we do better? The answer is yes. The key idea, as we discuss in the next subsection, is to look at the problem independently of the problem of finding the full active set.

2.2.2 Decoding Scheme 2: Find the Inactive Subset Directly, $K = 1$ case

For simplicity of exposition, we describe this decoding scheme in two stages: First, we present the result for the $K = 1$ case, i.e., when there is only one active variable. This case brings out the fundamental difference between finding active and inactive variables. We then generalize our decoding scheme to $K \geq 1$.

We start by proposing the following decoding scheme:

- Given $\{\mathbf{X}, \mathbf{y}\}$, compute $P(\mathbf{y}|\underline{\mathbf{x}}_i)$ for all $i \in [N]$ and sort them in descending order. Since $K = 1$, we know $P(Y|X_i)$ for all $i \in [N]$, and hence $P(\mathbf{y}|\underline{\mathbf{x}}_i)$ can be computed using the independence assumption across different observations.
- Pick the last L indices in the sorted array as the set of L inactive variables.

Note that, in contrast to finding active set, the problem of finding L inactive variables does not have unique solution (except for $L = N - K$). The proposed decoding scheme provides a way to pick a solution, and the probability of error analysis takes into account the fact that an error event happens only when the inactive set chosen by the decoding algorithm contains an active variable.

Theorem 1. *Let N, M, L and K be as defined above. Let $K = 1$. Let E_0 and $I^{(j)}$ be as defined in (2.3) and (2.4). Let $\rho \in [0, 1]$. With the above decoding scheme, the average probability of error, P_e , in finding L inactive variables is upper bounded as*

$$P_e \leq \exp \left[-M \left(E_0(\rho, 1, N - L) - \frac{\rho \log \binom{N-1}{L-1}}{M} \right) \right]. \quad (2.7)$$

Further, if

$$M > \frac{\log \binom{N-1}{L-1}}{(N - L)I^{(1)}}, \quad (2.8)$$

then there exists a positive constant, ϵ , such that $P_e \leq \exp(-M\epsilon)$, i.e., P_e approaches zero exponentially with the number of observations.

We make the following observations:

- (a) Figure 2.2(a) compares the above bound on the number of observations with the bounds in (2.5) and in Theorem III.I [30], for the $K = 1$ case.
- (b) Consider the case $L = N - 1$, i.e., we want to find all the inactive variables. This task is equivalent to finding the active variable. The above decoding scheme for finding $N - 1$ inactive variables is equivalent³ to the maximum likelihood criterion

³The decoding schemes are equivalent in the sense that an error in finding K active variables implies an error in finding $N - K$ inactive variables, and vice-versa.

based decoding scheme used in Theorem III.I in [30] for finding 1 active variable. This is also reflected in the above result, as the number of observations sufficient for finding $N - 1$ inactive variables matches exactly with the number of observations sufficient for finding 1 active variable (see Figure 2.2(a)).

We now consider the $K \geq 1$ case and establish that there exist decoding schemes that achieve similar gains as in the $K = 1$ case.

2.2.3 Decoding Scheme 2: Find the Inactive Subset Directly, $K \geq 1$ case

First, it is easy to see that the decoding scheme for $K = 1$ does not directly extend to the $K > 1$ case. For example, let us arrange $P(\underline{y}|\mathbf{X}_{S_d})$ in decreasing order for all $S_d \subset [N]$ such that $|S_d| = K$. Since the different S_d are not necessarily disjoint, it is possible that the sets towards the end of the sorted list have overlapping entries. Thus, it is not clear how many sets from the end of the list we need to choose, in order to find at least L distinct inactive variables.⁴ Hence, we propose to use a multi-stage algorithm where we collect the required inactive variables in a *greedy* fashion. We start with considering all N variables as candidate inactive variables. Then at each stage, we find K inactive variables, which are then removed from the candidate set for the next stage.

Decoding Scheme:

1. Initialize $T_1 = [N]; S_H = []$;
2. For $i = 1, 2, \dots, \lceil \frac{L}{K} \rceil$ do:

⁴Clearly, since the sets are distinct, picking the last L sets is sufficient. However, this leads to a weaker bound than the scheme proposed in this section.

- Given $\{\mathbf{X}, \underline{\mathbf{y}}\}$, compute $P(\underline{\mathbf{y}}|\mathbf{X}_{S_\omega})$ for all $S_\omega \subset T_i$ and $|S_\omega| = K$. Find:

$$S_\omega^{(i)} = \underset{S_\omega \subset T_i, |S_\omega|=K}{\operatorname{argmin}} P(\underline{\mathbf{y}}|\mathbf{X}_{S_\omega}). \quad (2.9)$$

- Set $S_H = [S_H S_\omega^{(i)}]$ and $T_{i+1} = T_i \setminus S_\omega^{(i)}$.

End do.

3. Output S_H as the decoded set of inactive variables.

The probability of error analysis of the above multi-stage algorithm leads to a sufficient condition on M . We summarize the result in the following theorem; we provide its proof in Sec. 2.5.

Theorem 2. Let N , M , L and K be as defined above. Let $N_{\text{stg}} \triangleq \lceil \frac{L}{K} \rceil$, $L_j \triangleq (N - K) - (N_{\text{stg}}K - j)$ and $C_2(L, N, K, j) \triangleq \binom{N-K}{L_j} \binom{K N_{\text{stg}} - j}{K-j} \binom{K}{1} \binom{K-1}{j-1}$. Let $\rho \in [0, 1]$. With the above decoding scheme, the average probability of error, P_e , in finding L inactive variables is upper bounded as

$$P_e \leq \sum_{j=1}^K \exp \left[-M \left(E_0(\rho, 1, L_j) - \frac{\rho \log C_2(L, N, K, j)}{M} \right) \right]. \quad (2.10)$$

Further, if

$$M > \max_{1 \leq j \leq K} \frac{\log C_2(L, N, K, j)}{L_j I^{(1)}}, \quad (2.11)$$

then there exists a positive constant, ϵ , such that $P_e \leq K \exp(-M\epsilon)$, i.e., for all fixed $K \geq 1$, P_e approaches zero exponentially with the number of observations.

We make following observations about the above result:

- (a) Let $\Gamma_{u2} \triangleq \max_{1 \leq j \leq K} \frac{\log C_2(L, N, K, j)}{L_j}$, $\Gamma_{ud} \triangleq \log \left[\binom{N-K}{1} \binom{K}{1} \right]$, and let $\Gamma_{u1} \triangleq \Gamma_{ud} + \log \left[\frac{\binom{N-K-1}{L-1}}{\binom{N-K}{L}} \right]$. The bounds in (2.5), (2.11) and in Theorem III.I [30] are all of the form $\frac{\Gamma_u}{I^{(1)}}$, with Γ_{u1} , Γ_{u2} and Γ_{ud} being the shorthand notation for Γ_u for each mentioned scheme, respectively.⁵ Figure 2.2(b) presents a comparison between these multiplicative factors. We see that the approach of directly finding an L -sized inactive set, especially for small L , requires far fewer number of observations compared to the other approaches.
- (b) We now characterize the limiting behavior of the bounds, as $N \rightarrow \infty$, on the sufficient number of observations required to find L inactive variables, with K fixed. We derive the exact scaling laws in the regime where K grows linearly with N in the next section, in the context of group testing. For the purpose of this discussion, all the limits are with $N \rightarrow \infty$, e.g., $a \rightarrow a_0$ implies $\lim_{N \rightarrow \infty} a = a_0$. Let $\alpha \triangleq \frac{L-1}{N-K}$, and note that $\alpha \in [0, 1)$. We consider the following scaling regimes for L as $N \rightarrow \infty$:
- (i) $\frac{L}{N} \rightarrow 0$, K fixed: That is, L is varying sub-linearly with N , which includes the case where L is fixed. Using (A.1) in Lemma 3, Appendix A.1, we note that since $\alpha \rightarrow 0$, $\lim_{N \rightarrow \infty} \Gamma_{u2} = 0$, i.e., we do not need any observations. This is intuitive, as, with $N \rightarrow \infty$, we can choose any L set of variables, and it will constitute an inactive set with high probability. In contrast, $\Gamma_{u1} = O(\log(KL))$ is positive and non-decreasing in N , while $\Gamma_{ud} = O(\log(KN))$ increases with N .
 - (ii) $\frac{L}{N} = \alpha_0$ for any $0 < \alpha_0 < 1$, K fixed: That is, L scales linearly with N . Note that, $\alpha \rightarrow \alpha_0$. From Lemma 3, we note that, for a given α_0 , Γ_{u2} decreases

⁵Note that $\Gamma_{u1}/I^{(1)}$ and $\Gamma_{ud}/I^{(1)}$ in fact further lower bound the sufficient number of observations given in (2.5) and Theorem III.I [30], respectively.

monotonically with N (see Figure A.1 in Appendix A.1), and $\lim_{N \rightarrow \infty} \Gamma_{u2} \leq \frac{H_b(\alpha_0)}{1-\alpha_0}$, where $H_b(\cdot)$ denotes the binary entropy function. In contrast, both $\Gamma_{u1} = O(\log(KL))$ and $\Gamma_{ud} = O(\log(KN))$, i.e., both increase as $\log(N)$. It is interesting to note that for large N , Γ_{u2} depends on N , L and K *only* as a function of α . Hence, the size of inactive set, L , impacts the sufficient number of observations only through α , the fraction of inactive variables that need to be found.

- (c) We note that the decoding scheme for the $K \geq 1$ is a generalization of the scheme for $K = 1$ case. Due to this, the bound in (2.11) reduces to the bound in Theorem 1 when $K = 1$. However, for $L = N - K$, unlike the $K = 1$ case, the decoding scheme for finding L inactive variables with $K > 1$ is not equivalent to the maximum likelihood criterion based decoding scheme for finding K active variables. Our focus in this work is on low to moderate values of α (or L), for which the proposed decoding scheme gives excellent results. Nonetheless, it would be an interesting challenge to find decoding schemes to identify inactive variables that are equivalent to the schemes for finding all active variables directly when $L = N - K$, and which also perform as well as or better than the scheme proposed in this paper, for smaller L .

2.3 Necessary Number of Observations

In this section, we derive lower bounds on the number of observations M required to find a set of L inactive variables. Here, we need to lower bound the probability of error

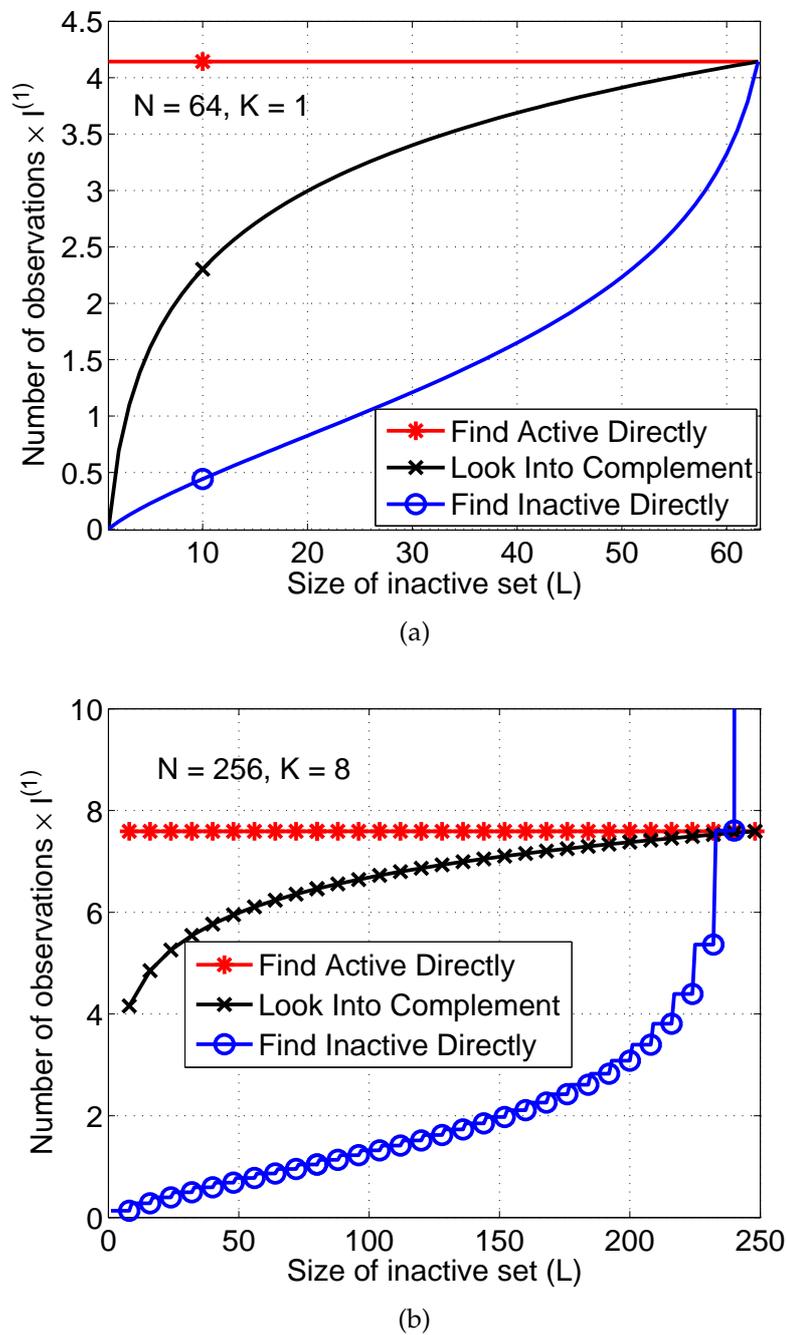


Figure 2.2: Sufficiency bounds on the number of observations required to find L inactive variables. Panel (a) $K = 1$ case; Panel (b) $K = 8$ case. The comparison is presented with respect to the value of $MI^{(1)}$, as the application-dependent mutual information term $I^{(1)}$ is common to all the bounds. The approach of finding the L inactive variables directly, especially for small values of L , requires significantly fewer number of observations compared to the approach of finding the inactive variables indirectly, after first identifying the active variables. The plot corresponding to the curve labeled `Find Active Directly` refers to the number of observations that are sufficient for finding the K active variables [30].

in choosing a set of L inactive variables. To do so, we employ an adaptation of Fano's inequality [43,44].

Let $\mathcal{I}^d \triangleq \{\omega_1, \omega_2, \dots, \omega_{\binom{N}{K}}\}$ be the collection of all K sized subsets of $[N]$ such that $|S_{\omega_i}| = K \forall i = 1, 2, \dots, \binom{N}{K}$. For each $\omega \in \mathcal{I}^d$ let us associate a collection of sets, $\mathcal{I}_\omega^h \triangleq \{\alpha_1, \alpha_2, \dots, \alpha_{\binom{N-K}{L}}\}$, such that $|S_{\alpha_i}| = L$ and $S_{\alpha_i} \cap S_\omega = \{0\}, i = 1, 2, \dots, \binom{N-K}{L}$. That is, \mathcal{I}_ω^h is the collection of all L -sized subsets of all-inactive variables with S_ω representing the active set. Also, let \mathcal{I}^H denote the set of all L -sized subsets of $[N]$. Note that $|\mathcal{I}^H| = \binom{N}{L}$. Given the observation vector, $\underline{\mathbf{y}} \in \mathcal{Y}^M$, let $\phi : \mathcal{Y}^M \rightarrow \mathcal{I}^H$ denote a decoding function, such that $\hat{\alpha} = \phi(\underline{\mathbf{y}})$ is the decoded set of L inactive variables. Given an active set ω and an observation vector $\underline{\mathbf{y}}$, an error occurs if $\hat{\alpha} \notin \mathcal{I}_\omega^h$. Define

$$P_e = P(\hat{\alpha} \notin \mathcal{I}_\omega^h). \quad (2.12)$$

Define a binary error RV, E , as $E \triangleq \mathbb{I}_{\{\hat{\alpha} \notin \mathcal{I}_\omega^h\}}$. Note that $P_e = \Pr(E = 1)$. We state the necessary condition on the number of observations in the following theorem.

Theorem 3. *Let N, M, L and K be as defined before. Let $I^{(j)}$ and P_e be as defined in (2.4) and (2.12), respectively. A necessary condition on the number of observations M required to find L inactive variables with asymptotically vanishing probability of error, i.e., $\lim_{N \rightarrow \infty} P_e = 0$, is given by*

$$M \geq \max_{1 \leq j \leq K} \frac{\Gamma_l(L, N, K, j)}{I^{(j)}}, \text{ where } \Gamma_l(L, N, K, j) \triangleq \log \left[\frac{\binom{N-K+j}{j}}{\binom{N-K+j-L}{j}} \right]. \quad (2.13)$$

That is, any sequence of random codebooks that achieves $\lim_{N \rightarrow \infty} P_e = 0$, must satisfy the above bound on the length of the codewords. Let $\Gamma_{l1} \triangleq \Gamma_l(L, N, K, 1)$. From (2.13), we note that $\Gamma_{l1}/I^{(1)}$ further lower bounds necessary number of observations, and thus,

by comparing Γ_{l1} with Γ_{u2} , we can characterize the gap between necessary and sufficient number of observations. Consider the scaling regime where L varies linearly with N , i.e., $L, N \rightarrow \infty$, $\frac{L}{N} = \alpha_0$ and K fixed. We note that, $\lim_{N \rightarrow \infty} \Gamma_{l1} = \log \frac{1}{1-\alpha_0}$. Thus, using Lemma 3 in Appendix A.1, it is easy to see that the gap is asymptotically smaller than $\frac{f(\alpha_0)}{I^{(1)}}$, where $f(\alpha) \triangleq \frac{\alpha}{1-\alpha} \log \left(\frac{1}{\alpha} \right)$. The function $f(\alpha)$ is bounded, with $f(\alpha) < 1$, $\forall \alpha \in (0, 1)$. Thus, the gap between the necessary and sufficient number of observations is asymptotically smaller than $\frac{1}{I^{(1)}}$. Given a specific application, we can bound $I^{(j)}$ for each $j = 1, 2, \dots, K$, and thereby obtain an even tighter characterization of the gap between the necessary and sufficient number of observations, as we show in the next section.

2.4 Finding Non-Defective Items Via Group Testing

In this section, we apply the above mutual information based results to the specific case of non-adaptive group testing, and characterize the number of tests to identify a subset of non-defective items of a large population. In particular, we consider a random pooling based noisy non-adaptive group testing framework [4, 30], as described in Section 2.1. Our goal here is to find upper and lower bounds on the number of tests required to identify an L sized subset belonging to $[N] \setminus \mathcal{G}$ using the observations \underline{y} , with vanishing probability of error as $N \rightarrow \infty$. In this subsection, we consider the parameter regime where L varies linearly with N , i.e., $L, N \rightarrow \infty$, $\frac{L}{N} = \alpha_0$ for a fixed $\alpha_0 \in (0, 1)$ and K fixed; in the next subsection, we extend these results to the scaling regime where K grows with N .

To compute the bounds on the number of tests, we need to bound the mutual information term, $I^{(j)}$, for the group testing signal model given in (2.2). Using the bounds on $I^{(j)}$ [47], with⁶ $p = \frac{1}{K}$ and $u \leq 0.5$, we summarize the order-accurate upper and lower bounds on the number of tests to find a set of L non-defective items in Tables 2.1 and 2.2, respectively. A brief sketch of the derivation of these results is provided in Appendix A.2. Here, for simplicity, we have considered L to be an integer multiple of K . The second column in both the tables presents an approximation of the bounds that is valid for small values of α , which shows that the upper and lower bounds match to a first order of approximation. It is clear that our approach of directly looking for non-defective items offers a significant advantage over the indirect but straightforward approach of finding the defective set, and then picking the requisite number of non-defective items from its complement. For example, for small values of α , with the direct approach, $O\left(\frac{K}{\log K} \frac{L}{N}\right)$ tests are sufficient to ensure a vanishing probability of error. In contrast, the indirect approach will require at least $O\left(\frac{K}{\log K} \log \frac{N}{K}\right)$ [30] tests, without which the probability of error is bounded strictly away from zero. Similar observations can be made in the noisy settings also, i.e., the direct approach leads to a significantly smaller number of tests being required to find L non-defective items compared to the indirect approach [15, 30].

Some additional observations are as follows:

1. If L is small, it is seemingly reasonable to test items one by one. In this case, the number of tests scales linearly with L , with slope greater than or equal to 1. Our sufficiency results indicate that, with large N , group tests will offer a reduction in

⁶In the noiseless case, it can be seen that $p \approx 1/K$ is optimal by setting $\frac{dI^{(1)}}{dp} = 0$.

Table 2.1: Finding a subset of L non-defective items: Order results for sufficient number of group tests which hold asymptotically as $N \rightarrow \infty$, $L \rightarrow \infty$, $\frac{L}{N} \rightarrow \alpha_0$ with $0 < \alpha_0 < 1$ and K fixed.

	$0 \leq \alpha_0 < 1$	Small α , e.g., $\alpha \leq 0.5$
No Noise	$O\left(\frac{K}{\log K} \frac{H_b(\alpha_0)}{1 - \alpha_0}\right)$	$O\left(\frac{K\alpha_0}{\log K}\right)$
Dilution Noise	$O\left(\frac{K}{(1-u)\log K} \frac{H_b(\alpha_0)}{1 - \alpha_0}\right)$	$O\left(\frac{K\alpha_0}{(1-u)\log K}\right)$
Additive Noise	$O\left(\frac{K}{\log \frac{1}{q}} \frac{H_b(\alpha_0)}{1 - \alpha_0}\right)$	$O\left(\frac{K\alpha_0}{\log \frac{1}{q}}\right)$

Table 2.2: Finding a subset of L non-defective items: Order results for necessary number of group tests which hold asymptotically as $N \rightarrow \infty$, $L \rightarrow \infty$, $\frac{L}{N} = \alpha_0$ with $0 < \alpha_0 < 1$ and K fixed.

	$0 \leq \alpha_0 < 1$	Small α_0 , e.g., $\alpha_0 \leq 0.5$
No Noise	$\Omega\left(\frac{K}{\log K} \log \frac{1}{1 - \alpha_0}\right)$	$\Omega\left(\frac{K\alpha_0}{\log K}\right)$
Dilution Noise	$\Omega\left(\frac{K}{(1-u)\log K} \log \frac{1}{1 - \alpha_0}\right)$	$\Omega\left(\frac{K\alpha_0}{(1-u)\log K}\right)$
Additive Noise	$\Omega\left(\frac{K}{\log \frac{1}{q}} \log \frac{1}{1 - \alpha_0}\right)$	$\Omega\left(\frac{K\alpha_0}{\log \frac{1}{q}}\right)$

the number of tests whenever $\frac{3K}{N} < 1$, since it ensures that the sufficient number of observations will increase with L at a rate strictly less than 1.

2. Although $I^{(1)}$ depends on p , the value of p that minimizes the number of tests *does not depend upon* L . This is easily seen from the expressions for the sufficient number of tests, since L only appears in the combinatorial term Γ_{u2} , which is independent of p . Hence, the p that minimizes the sufficient number of tests does not depend on the number of non-defective items we want to find, which is a desirable feature from a practical point of view.

2.4.1 Sufficiency Results when K Grows with N

In comment (b), Section 2.2.3, we characterized the asymptotic behavior of the sufficient number of observations for the case when K is fixed as $L, N \rightarrow \infty$. In this subsection, we extend these results, in the context of group testing, to the scaling regime when both K and L grow with N . The overall average probability of error, as given in (2.10), is a sum of K terms. In contrast to the fixed K case, it is no longer sufficient to drive each term in the summation to zero, since the number of terms grows with N . Hence, we need stronger conditions on M to drive the overall probability of error to zero. As an example, we consider the noiseless group testing scenario and derive the sufficient number of tests required to find L non-defective items under the asymptotic regime when K grows *linearly* with N . The following lemma presents the bound on M required to drive the overall probability of error to zero.

Lemma 1. *Let P_e be the average probability of error in finding L inactive variables under the decoding scheme described in Section 2.2.3, which is upper bounded by (2.10). Let L_j and $C_2(L, N, K, j)$ be as defined in Theorem 2. Let $L < N - 2K$ and let $K \geq K_0$, where K_0 is some positive constant. Define $C_3 \triangleq -\log \left[1 - \left(1 - \frac{1}{K_0} \right)^{K_0} + \exp(-2) \right]$. For the noiseless group testing case, if*

$$M > \frac{1}{C_3} K \max_{1 \leq j \leq K} \frac{\log C_2(L, N, K, j)}{L_1} + \frac{\log K}{C_3}, \quad (2.14)$$

then there exists a positive ϵ such that $P_e \leq \exp(-M\epsilon)$, and hence, $\lim_{N \rightarrow \infty} P_e = 0$.

Note that, in contrast with the above result, the bound on the probability of error in Theorem 2 was of the form $P_e \leq K \exp(-M\epsilon)$, due to which, it required K to be

fixed. The result in the above Lemma is valid even when K grows linearly with N . To prove the result, we analyze the behavior of $E_0(\rho, j, n)$ for $\rho > 0$ and show, by using a lower bound, that the negative exponent in the probability of error can be made strictly greater than 0 when (2.14) is satisfied. We provide an outline of the proof in Appendix A.3.

Let us characterize the above bound as $N \rightarrow \infty$. Let $K = \beta N$ and $L = \gamma(N - K)$, where β and γ are *fixed* constants and $0 < \beta, \gamma < 1$. Let $\alpha \triangleq \frac{L-1}{N-K}$. We note that $\alpha \rightarrow \gamma$ as $N \rightarrow \infty$. Using Lemma 3 in Appendix A.1 to bound $\max_{1 \leq j \leq K} \frac{\log C_2(L, N, K, j)}{L_1}$, we arrive at an upper bound for the right hand side (R.H.S.) of (2.14), which we denote by M_u . It can be verified that

$$\lim_{N \rightarrow \infty} M_u = \frac{K}{C_3} \frac{H_b(\gamma)}{(1-\gamma)} + \frac{K}{C_3(1-\gamma)} \left[\frac{\beta}{1-\beta} \left(\log \frac{\gamma(1-\beta)}{\beta} + 2 \right) \right] + C'_3 \log K, \quad (2.15)$$

where $C'_3 = \frac{1}{C_3} \left[1 + \frac{\beta}{(1-\beta)(1-\gamma)} \right]$.

Compared to the fixed K case (see Table 2.1), there are extra terms that contribute towards the number of tests required to find L inactive variables. The second and third terms above arise due to the fact that K increases linearly with N . The second term is of the same order as the first term; and its contribution to M_u depends upon β and the ratio L/K .

2.5 Proofs of the Main Results

We now present the proofs of Theorems 1, 2 and 3, which constitute the main results in this chapter.

2.5.1 Proof of Theorem 1: Sufficient Number of Observations, $K = 1$ Case

At the heart of the proof of this theorem is the derivation of an upper bound on the average probability of error in finding L inactive variables using the decoding scheme described in Section 2.2.2. In turn, the upper bound is obtained by characterizing the error exponents on the average probability of error [43]. Without loss of generality, due to the symmetry in the model, we can assume that the RV X_1 is active. Given that X_1 is the active variable, the decoding algorithm will make an error if $P(\underline{\mathbf{y}}|\mathbf{X}_1)$ falls within the last L entries of the sorted array generated as described in the decoding scheme. Let $\mathcal{E} \triangleq \{\text{error}|X_1 \text{ is active, } \mathbf{X}_1, \underline{\mathbf{y}}\}$. The overall average probability of error, P_e , can be expressed as

$$P_e = \sum_{\underline{\mathbf{y}}, \mathbf{X}_1} P(\underline{\mathbf{y}}|\mathbf{X}_1)Q(\mathbf{X}_1)\Pr(\mathcal{E}). \quad (2.16)$$

Let $S_z \subset [N] \setminus 1$ such that $|S_z| = N - L$. Let \mathcal{S}_z denote a set of all possible S_z . Further, let $\mathcal{A}_{S_z} \subset \{\mathbf{X}_{S_z}\}$ be such that, $\mathcal{A}_{S_z} = \{\mathbf{X}_{S_z} : P(\underline{\mathbf{y}}|\mathbf{X}_j) \geq P(\underline{\mathbf{y}}|\mathbf{X}_1) \forall j \in S_z\}$. It is easy to see that $\mathcal{E} \subset \mathcal{A} \triangleq \bigcup_{S_z \in \mathcal{S}_z} \mathcal{A}_{S_z}$, i.e., an error event implies that there exists a set of $N - L$ variables, S_z , such that $P(\underline{\mathbf{y}}|\mathbf{X}_j) \geq P(\underline{\mathbf{y}}|\mathbf{X}_1) \forall j \in S_z$. Thus, $\Pr(\mathcal{E}) \leq \Pr(\mathcal{A})$. Let s be a optimization variable such that $0 \leq s \leq 1$. The following set of inequalities upper

bound $\Pr(\mathcal{E})$ [43, Section 5.6]:

$$\begin{aligned}
\Pr(\mathcal{E}) &\leq \sum_{S_z \in \mathcal{S}_z} \sum_{\mathbf{X}_{S_z} \in \mathcal{A}_{S_z}} Q(\mathbf{X}_{S_z}) \\
&\stackrel{(a)}{\leq} \sum_{S_z \in \mathcal{S}_z} \sum_{\mathbf{X}_{S_z} \in \mathcal{A}_{S_z}} Q(\mathbf{X}_{S_z}) \prod_{j \in S_z} \left[\frac{P(\underline{\mathbf{y}}|\mathbf{X}_j)}{P(\underline{\mathbf{y}}|\mathbf{X}_1)} \right]^s \\
&\stackrel{(b)}{\leq} \sum_{S_z \in \mathcal{S}_z} \sum_{\mathbf{X}_{S_z}} \prod_{j \in S_z} Q(\mathbf{X}_j) \left[\frac{P(\underline{\mathbf{y}}|\mathbf{X}_j)}{P(\underline{\mathbf{y}}|\mathbf{X}_1)} \right]^s \\
&\stackrel{(c)}{=} \sum_{S_z \in \mathcal{S}_z} \prod_{j \in S_z} \sum_{\mathbf{X}_j} Q(\mathbf{X}_j) \left[\frac{P(\underline{\mathbf{y}}|\mathbf{X}_j)}{P(\underline{\mathbf{y}}|\mathbf{X}_1)} \right]^s \\
&\stackrel{(d)}{=} \binom{N-1}{L-1} \left\{ \sum_{\mathbf{X}_j} Q(\mathbf{X}_j) \left[\frac{P(\underline{\mathbf{y}}|\mathbf{X}_j)}{P(\underline{\mathbf{y}}|\mathbf{X}_1)} \right]^s \right\}^{N-L}. \tag{2.17}
\end{aligned}$$

In the above, (a) follows since we are multiplying with terms that are all greater than 1 and (b) follows since we are adding extra nonnegative terms by summing over all \mathbf{X}_{S_z} . We also use the independence of \mathbf{X}_{S_z} across variables, i.e., $Q(\mathbf{X}_{S_z}) = \prod_{j \in S_z} Q(\mathbf{X}_j)$. (c) uses the fact that $P(\underline{\mathbf{y}}|\mathbf{X}_j)$ is independent of $P(\underline{\mathbf{y}}|\mathbf{X}_k)$ for all $j \neq k$ and $j, k \in S_z$. (d) follows since the value of the expression inside the product term does not depend upon any particular j .

Let $0 \leq \rho \leq 1$. If the R.H.S. in (2.17) is less than 1, then raising it to the power ρ makes it bigger, and if it is greater than 1, it remains greater than 1 after raising it to the power ρ . Thus, using the Gallager bounding technique [43, Section 5.6], we get the following upper bound on $\Pr(\mathcal{E})$:

$$\Pr(\mathcal{E}) \leq \binom{N-1}{L-1}^\rho \left\{ \sum_{\mathbf{X}_j} Q(\mathbf{X}_j) \left[\frac{P(\underline{\mathbf{y}}|\mathbf{X}_j)}{P(\underline{\mathbf{y}}|\mathbf{X}_1)} \right]^s \right\}^{\rho(N-L)}. \tag{2.18}$$

Substituting this into (2.16) and simplifying, we get

$$P_e \leq \binom{N-1}{L-1}^\rho \sum_{\underline{\mathbf{y}}} \sum_{\mathbf{X}_1} Q(\mathbf{X}_1) P(\underline{\mathbf{y}}|\mathbf{X}_1)^{1-\rho(N-L)s} \left\{ \sum_{\mathbf{X}_j} Q(\mathbf{X}_j) P(\underline{\mathbf{y}}|\mathbf{X}_j)^s \right\}^{\rho(N-L)}. \quad (2.19)$$

Putting $s = 1/(1 + \rho(N - L))$, we get

$$P_e \leq \binom{N-1}{L-1}^\rho \sum_{\underline{\mathbf{y}}} \left\{ \sum_{\mathbf{X}_j} Q(\mathbf{X}_j) P(\underline{\mathbf{y}}|\mathbf{X}_j)^{\frac{1}{1+\rho(N-L)}} \right\}^{1+\rho(N-L)}. \quad (2.20)$$

Finally, using the independence across observations and using the definition of $E_0(\rho, j, n)$

from (2.3) with $j = 1$ and $n = N - L$, we get

$$\begin{aligned} P_e &\leq \binom{N-1}{L-1}^\rho \left[\sum_{Y \in \mathcal{Y}} \left\{ \sum_{X_j \in \mathcal{X}} Q(X_j) P(Y|X_j)^{\frac{1}{1+\rho(N-L)}} \right\}^{1+\rho(N-L)} \right]^M \\ &= \exp^{[-MF(\rho)]}, \text{ where } F(\rho) = E_0(\rho, 1, N-L) - \frac{\rho \log \binom{N-1}{L-1}}{M}. \end{aligned} \quad (2.21)$$

Hence (2.7) follows.

For the following discussion, we treat F and E_0 as functions of ρ only and all the derivatives are with respect to ρ . The average probability of error will approach zero exponentially with M if $F(\rho) > \epsilon > 0$ for some $0 < \rho \leq 1$ and $\epsilon > 0$. Note that $F'(\rho) = E_0'(\rho) - \frac{\log \binom{N-1}{L-1}}{M}$. It is easy to see that $E_0(0) = 0$ and hence $F(0) = 0$. With some calculation (see, e.g., [30], [43, Section 5.6]), we get,

$$E_0'(\rho)|_{\rho=0} = (N-L) \sum_{Y,X} P(Y,X) \log \frac{P(Y|X)}{P(Y)} = (N-L)I^{(1)}. \quad (2.22)$$

Thus, the condition on M given in (2.7) ensures that $F'(\rho) > 0$ at $\rho = 0$. Hence, from the mean value theorem [48], \exists a sufficiently small $\rho > 0$ such that $F(\rho) > 0$. This completes

the proof.

2.5.2 Proof of Theorem 2: Sufficient Number of Observations, $K \geq 1$ Case

Let $N_{\text{stg}} \triangleq \lceil \frac{L}{K} \rceil$ be the number of stages in the decoding algorithm. With a slight abuse of notation, let \mathcal{E} denote the event that the multi-stage algorithm makes an error. Let \mathcal{E}'_i denote the event that the algorithm makes an error in the i^{th} stage, given that no errors were made in any of the previous $i - 1$ stages. In other words, \mathcal{E}'_i denotes the event that an error occurs for the first time at i^{th} stage. It is easy to verify that \mathcal{E} can be expressed as

$$\mathcal{E} = \bigcup_{i=1}^{N_{\text{stg}}} \mathcal{E}'_i. \quad (2.23)$$

Since up to K errors can occur at any stage, $\mathcal{E}'_i = \bigcup_{j=1}^K \mathcal{E}'_{ij}$, where \mathcal{E}'_{ij} denotes the event that exactly j ($1 \leq j \leq K$) errors occur at the i^{th} stage, given that no errors were made in previous stages. Let $\mathcal{E}''_j \triangleq \bigcup_{i=1}^{N_{\text{stg}}} \mathcal{E}'_{ij}$. Thus, from (2.23), $\mathcal{E} = \bigcup_{j=1}^K \mathcal{E}''_j$. Let P_{ej} denote the average probability of the event \mathcal{E}''_j . The overall average probability of error, P_e , in finding L inactive variables via the multi-stage decoding algorithm can thus be upper bounded as

$$P_e \leq \sum_{j=1}^K P_{ej}. \quad (2.24)$$

In the above, the probability is averaged over all possible instantiations of $\{\mathbf{X}, \underline{\mathbf{y}}\}$ as well as over all possible active (or defective) sets.

We now upper bound P_{ej} for $j = 1, 2, \dots, K$. By symmetry, the average probability

of error is the same for all active sets. Hence, we fix the active set and then compute average probability of error with this set. Let $S_1 \subset [N]$ be the active set such that $|S_1| = K$. We note that \mathcal{E}_j'' is the union of events where exactly j errors occur, and the error occurs for the first time. Let this first erroneous set of variables be denoted by S_ω , where $|S_\omega| = K$. Define $S_{1\omega} \triangleq S_1 \cap S_\omega$, $S_{1^c\omega} \triangleq S_1^c \cap S_\omega$ and $S_{1\omega^c} \triangleq S_1 \cap S_\omega^c$. Since there are j errors, $|S_{1\omega}| = j$. Further, let $S_{1\omega}$ be partitioned as $S_{1\omega} = S_{1d} \cup S_{2d}$ with $S_{1d} \cap S_{2d} = \{\emptyset\}$, $|S_{1d}| = 1$ and $|S_{2d}| = j - 1$. Define

$$\mathcal{B} \triangleq \{\mathcal{E}_j'' | S_1 \text{ is the active set, } \underline{\mathbf{y}}, \mathbf{X}_{S_1}, \mathbf{X}_{S_{1^c\omega}}\}. \quad (2.25)$$

With this notation, P_{ej} can be expressed as

$$\begin{aligned} P_{ej} &= \sum_{\underline{\mathbf{y}}} \sum_{\mathbf{X}_{S_1}} P(\underline{\mathbf{y}} | \mathbf{X}_{S_1}) Q(\mathbf{X}_{S_1}) \sum_{\mathbf{X}_{S_{1^c\omega}}} Q(\mathbf{X}_{S_{1^c\omega}}) [\Pr(\mathcal{B})] \\ &= \sum_{\underline{\mathbf{y}}} \sum_{\mathbf{X}_{S_{1^c\omega}}} \sum_{\mathbf{X}_{S_1}} P(\underline{\mathbf{y}}, \mathbf{X}_{S_{1^c\omega}} | \mathbf{X}_{S_1}) Q(\mathbf{X}_{S_1}) [\Pr(\mathcal{B})] \\ &= \sum_{\underline{\mathbf{y}}} \sum_{\mathbf{X}_{S_{1^c\omega}}} \sum_{\mathbf{X}_{S_{1\omega}}} \sum_{\mathbf{X}_{S_{1\omega^c}}} P(\underline{\mathbf{y}}, \mathbf{X}_{S_{1^c\omega}}, \mathbf{X}_{S_{1\omega^c}} | \mathbf{X}_{S_{1\omega}}) Q(\mathbf{X}_{S_{1\omega}}) [\Pr(\mathcal{B})] \\ &= \sum_{\underline{\mathbf{y}}} \sum_{\mathbf{X}_{S_{1^c\omega}}} \sum_{\mathbf{X}_{S_{1d}}} \sum_{\mathbf{X}_{S_{2d}}} \sum_{\mathbf{X}_{S_{1\omega^c}}} P(\underline{\mathbf{y}}, \mathbf{X}_{S_{1^c\omega}}, \mathbf{X}_{S_{2d}}, \mathbf{X}_{S_{1\omega^c}} | \mathbf{X}_{S_{1d}}) Q(\mathbf{X}_{S_{1d}}) [\Pr(\mathcal{B})]. \end{aligned} \quad (2.26)$$

In the above equation, we have used the fact that given the active set S_1 , $\underline{\mathbf{y}}$ is independent of other input variables. Thus, $P(\underline{\mathbf{y}}, \mathbf{X}_{S_{1^c\omega}} | \mathbf{X}_{S_1}) = P(\underline{\mathbf{y}} | \mathbf{X}_{S_1}) Q(\mathbf{X}_{S_{1^c\omega}})$. Define $S_z \subset [N] \setminus S_1$ such that $|S_z| = L_j$, where $L_j = (N - K) - (N_{\text{stg}}K - j)$. Note that, when the algorithm terminates with j errors, there are at most L_j inactive variables in the complement of the set output by the algorithm. Further, let \mathcal{S}_z be the collection of all

possible S_z . It is easy to see that $|S_z| = \binom{N-K}{L_j}$. Define $\mathcal{A}_{S_z} \subset \{\mathbf{X}_{S_z}\}$ and \mathcal{A} as follows:

$$\mathcal{A}_{S_z} = \{\mathbf{X}_{S_z} : P(\underline{\mathbf{y}}|\mathbf{X}_{S_{1c_\omega}}, \mathbf{X}_{S_{2d}}, \mathbf{X}_{S_\alpha}) \geq P(\underline{\mathbf{y}}|\mathbf{X}_{S_{1c_\omega}}, \mathbf{X}_{S_{2d}}, \mathbf{X}_{S_{1d}}) \forall S_\alpha \subset S_z \text{ and } |S_\alpha| = 1\}, \quad (2.27)$$

$$\mathcal{A} = \bigcup_{S_z \in \mathcal{S}_z} \mathcal{A}_{S_z}. \quad (2.28)$$

Proposition 1. $\mathcal{B} \subset \mathcal{A}$.

Proof. We will show that the event \mathcal{B} implies the event \mathcal{A} .

Let $S_{iz} \subset [N] \setminus S_1$ be such that $|S_{iz}| = L_{ij}$, where $L_{ij} = (N - K) - (iK - j)$. Suppose the event $\mathcal{B}_i \triangleq \{\mathcal{E}'_{ij} | S_1 \text{ is active set, } \underline{\mathbf{y}}, \mathbf{X}_{S_1}, \mathbf{X}_{S_{1c_\omega}}\}$ occurs for some $i = 1, 2, \dots, N_{\text{stg}}$. This implies that there are no errors in the first $i-1$ stages, and hence, in those stages, $(i-1)K$ inactive variables are found. Since, at the i^{th} stage, the chosen set of K variables contains j variables from the active set S_1 , there exists $\mathbf{X}_{S_{iz}}$ such that $P(\underline{\mathbf{y}}|\mathbf{X}_{S_{1c_\omega}}, \mathbf{X}_{S_{2d}}, \mathbf{X}_{S_\alpha}) \geq P(\underline{\mathbf{y}}|\mathbf{X}_{S_{1c_\omega}}, \mathbf{X}_{S_{2d}}, \mathbf{X}_{S_{1d}}) \forall S_\alpha \subset S_{iz} \text{ and } |S_\alpha| = 1$. Since $L_{ij} \geq L_j$ for all $i = 1, 2, \dots, N_{\text{stg}}$, this implies that $\exists S_z \subset S_{iz}$ with $|S_z| = L_j$ such that

$$P(\underline{\mathbf{y}}|\mathbf{X}_{S_{1c_\omega}}, \mathbf{X}_{S_{2d}}, \mathbf{X}_{S_\alpha}) \geq P(\underline{\mathbf{y}}|\mathbf{X}_{S_{1c_\omega}}, \mathbf{X}_{S_{2d}}, \mathbf{X}_{S_{1d}}) \quad \forall S_\alpha \subset S_z \text{ and } |S_\alpha| = 1.$$

That is, the event \mathcal{A} occurs. □

Using the above proposition, we can upper bound $\Pr(\mathcal{B})$ with $\Pr(\mathcal{A})$ in (2.26). Next, we upper bound $\Pr(\mathcal{A})$ as follows:

$$\Pr(\mathcal{A}) \leq \sum_{S_z \in \mathcal{S}_z} \sum_{S_{1c_\omega} \in \mathcal{S}_{1c_\omega}} \sum_{S_{1d} \in \mathcal{S}_{1d}} \sum_{S_{2d} \in \mathcal{S}_{2d}} q_j, \quad (2.29)$$

where $q_j \triangleq \Pr\{\mathcal{A}_{S_z} | S_1 \text{ is active set, } \underline{\mathbf{y}}, \mathbf{X}_{S_1}, \mathbf{X}_{S_1^{c_\omega}}, S_z, S_{1d}, S_{2d}, S_{1^{c_\omega}}\}$. Here, the randomness comes from the set of variables in S_z , i.e., \mathbf{X}_{S_z} . Let s be such that $0 \leq s \leq 1$.

Following similar steps as in (2.17), we have

$$\begin{aligned}
q_j &\leq \sum_{\mathbf{X}_{S_z} \in \mathcal{A}_{S_z}} Q(\mathbf{X}_{S_z}) \\
&\leq \sum_{\mathbf{X}_{S_z} \in \mathcal{A}_{S_z}} Q(\mathbf{X}_{S_z}) \prod_{S_\alpha \in S_z} \left[\frac{P(\underline{\mathbf{y}} | \mathbf{X}_{S_1^{c_\omega}}, \mathbf{X}_{S_{2d}}, \mathbf{X}_{S_\alpha})}{P(\underline{\mathbf{y}} | \mathbf{X}_{S_1^{c_\omega}}, \mathbf{X}_{S_{2d}}, \mathbf{X}_{S_{1d}})} \right]^s \\
&\leq \sum_{\mathbf{X}_{S_z}} Q(\mathbf{X}_{S_z}) \prod_{S_\alpha \in S_z} \left[\frac{P(\underline{\mathbf{y}} | \mathbf{X}_{S_1^{c_\omega}}, \mathbf{X}_{S_{2d}}, \mathbf{X}_{S_\alpha})}{P(\underline{\mathbf{y}} | \mathbf{X}_{S_1^{c_\omega}}, \mathbf{X}_{S_{2d}}, \mathbf{X}_{S_{1d}})} \right]^s \\
&\leq \prod_{l=1}^{L_j} \sum_{\mathbf{X}_{S_\alpha}} Q(\mathbf{X}_{S_\alpha}) \left[\frac{P(\underline{\mathbf{y}} | \mathbf{X}_{S_1^{c_\omega}}, \mathbf{X}_{S_{2d}}, \mathbf{X}_{S_\alpha})}{P(\underline{\mathbf{y}} | \mathbf{X}_{S_1^{c_\omega}}, \mathbf{X}_{S_{2d}}, \mathbf{X}_{S_{1d}})} \right]^s \\
&= \left\{ \sum_{\mathbf{X}_{S_\alpha}} Q(\mathbf{X}_{S_\alpha}) \left[\frac{P(\underline{\mathbf{y}} | \mathbf{X}_{S_1^{c_\omega}}, \mathbf{X}_{S_{2d}}, \mathbf{X}_{S_\alpha})}{P(\underline{\mathbf{y}} | \mathbf{X}_{S_1^{c_\omega}}, \mathbf{X}_{S_{2d}}, \mathbf{X}_{S_{1d}})} \right]^s \right\}^{L_j} \\
&= \left\{ \sum_{\mathbf{X}_{S_\alpha}} Q(\mathbf{X}_{S_\alpha}) \left[\frac{P(\underline{\mathbf{y}}, \mathbf{X}_{S_1^{c_\omega}}, \mathbf{X}_{S_{2d}} | \mathbf{X}_{S_\alpha})}{P(\underline{\mathbf{y}}, \mathbf{X}_{S_1^{c_\omega}}, \mathbf{X}_{S_{2d}} | \mathbf{X}_{S_{1d}})} \right]^s \right\}^{L_j}. \tag{2.30}
\end{aligned}$$

From the above equation, q_j depends on values of \mathbf{X}_{S_z} , $\mathbf{X}_{S_1^{c_\omega}}$, $\mathbf{X}_{S_{1d}}$ and $\mathbf{X}_{S_{2d}}$ and not on the particular index sets, S_z , $S_1^{c_\omega}$, S_{2d} and S_{1d} , respectively. For a given S_1 , there are $\binom{K}{1}$ and $\binom{K-1}{j-1}$ different permutations of S_{1d} and S_{2d} for a given value of $\mathbf{X}_{S_{1d}}$ and $\mathbf{X}_{S_{2d}}$, respectively. As already mentioned there are $\binom{N-K}{L_j}$ ways of choosing S_z , and there are $\binom{KN_{\text{stg}}-j}{K-j}$ different permutations of $S_1^{c_\omega}$ for a given value of $\mathbf{X}_{S_1^{c_\omega}}$. Let $C_2 \triangleq \binom{N-K}{L_j} \binom{KN_{\text{stg}}-j}{K-j} \binom{K}{1} \binom{K-1}{j-1}$. Thus, from (2.29), for some $0 \leq \rho \leq 1$, we get

$$\Pr(\mathcal{A}) \leq \left[C_2 \left\{ \sum_{\mathbf{X}_{S_\alpha}} Q(\mathbf{X}_{S_\alpha}) \left[\frac{P(\underline{\mathbf{y}} | \mathbf{X}_{S_1^{c_\omega}}, \mathbf{X}_{S_{2d}}, \mathbf{X}_{S_\alpha})}{P(\underline{\mathbf{y}} | \mathbf{X}_{S_1^{c_\omega}}, \mathbf{X}_{S_{2d}}, \mathbf{X}_{S_{1d}})} \right]^s \right\}^{L_j} \right]^\rho. \tag{2.31}$$

We note, as in the previous theorem, that by raising the expression in above equation to a power $0 < \rho \leq 1$, we still get an upper bound for $\Pr(\mathcal{A})$ [43, Section 5.6]. Substituting

into (2.26) and averaging out $\mathbf{X}_{S_{1\omega^c}}$, since the upper bound expression for $\Pr(\mathcal{A})$ does not depend upon $\mathbf{X}_{S_{1\omega^c}}$, we get

$$P_{ej} \leq \sum_{\underline{\mathbf{y}}} \sum_{\mathbf{X}_{S_{1\omega^c}}} \sum_{\mathbf{X}_{S_{2d}}} \sum_{\mathbf{X}_{S_{1d}}} P(\underline{\mathbf{y}}, \mathbf{X}_{S_{1\omega^c}}, \mathbf{X}_{S_{2d}} | \mathbf{X}_{S_{1d}}) Q(\mathbf{X}_{S_{1d}}) \Pr(\mathcal{A}) \quad (2.32)$$

$$\leq C_2^\rho \sum_{\underline{\mathbf{y}}} \sum_{\mathbf{X}_{S_{1\omega^c}}} \sum_{\mathbf{X}_{S_{2d}}} \left\{ \sum_{\mathbf{X}_{S_\alpha}} Q(\mathbf{X}_{S_\alpha}) P(\underline{\mathbf{y}}, \mathbf{X}_{S_{1\omega^c}}, \mathbf{X}_{S_{2d}} | \mathbf{X}_{S_\alpha})^{\frac{1}{1+\rho L_j}} \right\}^{1+\rho L_j} \quad (2.33)$$

$$\leq \exp \left[-M \left(E_0(\rho, 1, L_j) - \frac{\rho \log C_2}{M} \right) \right]. \quad (2.34)$$

The second inequality above is obtained by first using $s = \frac{1}{1+\rho L_j}$, then further simplifying using independence across different observations and writing the bound in the exponential form, as in the $K = 1$ case. Using the above bound on P_{ej} in (2.24), we get the upper bound on P_e given in (2.10).

To obtain (2.11), we note that, for any fixed K , the overall P_e can be driven to zero by driving each P_{ej} to zero. As in $K = 1$ case, using the mean value theorem, it follows that $\lim_{N \rightarrow \infty} P_{ej} \rightarrow 0$ whenever $M > \frac{\log C_2}{E'_0(\rho, 1, L_j)|_{\rho=0}}$. The result now follows by noting that $E'_0(\rho, 1, L_j)|_{\rho=0} = L_j I^{(1)}$. ■

One remark related to the above proof is as follows. We have used the set \mathcal{A}_{S_z} in (2.27) for upper bounding P_e . Note that \mathcal{A}_{S_z} can be chosen in different ways, and will lead to different bounds on the sufficient number of observations. For example, consider another intuitive definition of \mathcal{A}_{S_z} , with the usual definitions above: $\mathcal{A}_{S_z} = \{\mathbf{X}_{S_z} : P(\underline{\mathbf{y}} | \mathbf{X}_{S_{1\omega^c}}, \mathbf{X}_{S_\alpha}) \geq P(\underline{\mathbf{y}} | \mathbf{X}_{S_{1\omega^c}}, \mathbf{X}_{S_{1\omega}}) \forall S_\alpha \subset S_z \text{ and } |S_\alpha| = j\}$. Following the steps in the above proof, we can show that this choice of \mathcal{A}_{S_z} leads to the following bound on the

number of observations:

$$M > \max_{1 \leq j \leq K} \frac{\log \left[\binom{N-K}{L_j} \binom{KN_{\text{stg}}-j}{K-j} \binom{K}{j} \right]}{\lfloor L_j/j \rfloor I^{(j)}}. \quad (2.35)$$

The above bound depends upon the variation of $I^{(j)}$ with j . For the applications considered in this paper, the bound in (2.35) turns out to be weaker than the one in (2.11).

2.5.3 Proof of Theorem 3: Necessary Number of Observations

For the purpose of this proof, recall that P_e was defined in (2.12). We need to prove that $\lim_{N \rightarrow \infty} P_e = 0$ implies the bound on the number of observations as given by (2.13). Towards that end, we first find, by lower bounding P_e , the conditions on M that will lead to error probability being bounded away from zero. We consider a genie-aided lower bound, where we assume that the active set is partially known. Let us define a partition for S_ω as $S_\omega = S^{(j)} \cup S^{(K-j)}$, where $|S^{(j)}| = j$ and $|S^{(K-j)}| = K - j$ and $S^{(j)} \cap S^{(K-j)} = \{\emptyset\}$. We assume that $S^{(K-j)}$ (and hence, for a given code, the matrix $\mathbf{X}_{S^{(K-j)}}$) is known to us. Now consider $H(\omega, E | \underline{\mathbf{y}}, S^{(K-j)})$:

$$H(\omega, E | \underline{\mathbf{y}}, \mathbf{X}_{S^{(K-j)}}) = H(E | \underline{\mathbf{y}}, \mathbf{X}_{S^{(K-j)}}) + H(\omega | E, \underline{\mathbf{y}}, \mathbf{X}_{S^{(K-j)}}) \quad (2.36)$$

$$\stackrel{(a)}{\leq} H_b(P_e) + (1 - P_e) H(\omega | E = 0, \underline{\mathbf{y}}, \mathbf{X}_{S^{(K-j)}}) + P_e H(\omega | E = 1, \underline{\mathbf{y}}, \mathbf{X}_{S^{(K-j)}}) \quad (2.37)$$

$$\stackrel{(b)}{\leq} H_b(P_e) + (1 - P_e) \log \binom{N - K + j - L}{j} + P_e H(\omega | \mathbf{X}_{S^{(K-j)}}) \quad (2.38)$$

$$\stackrel{(c)}{\leq} H_b(P_e) + (1 - P_e) \log \binom{N - K + j - L}{j} + P_e \log \binom{N - K + j}{j}. \quad (2.39)$$

In the above, (a) follows since E is a binary RV and $H(E | \underline{\mathbf{y}}, \mathbf{X}_{S^{(K-j)}}) \leq H(E) = H_b(P_e) \leq 1$. Since the entropy of any RV is bounded by the logarithm of the alphabet

size, (b) follows by considering the cardinality of the remaining number of outcomes conditioned on the outcome of E . For example, when $E = 0$, i.e., when there is no error, the number of ways of choosing the set $S^{(j)}$ is given by $\binom{N-K+j-L}{j}$. (c) follows by using a trivial bound on $H(\omega|\mathbf{X}_{S^{(K-j)}})$. Also, note that

$$H(\omega, E|\underline{\mathbf{y}}, \mathbf{X}_{S^{(K-j)}}) = H(\omega|\underline{\mathbf{y}}, \mathbf{X}_{S^{(K-j)}}) + H(E|\omega, \underline{\mathbf{y}}, \mathbf{X}_{S^{(K-j)}}) = H(\omega|\underline{\mathbf{y}}, \mathbf{X}_{S^{(K-j)}}). \quad (2.40)$$

For a given \mathbf{X} , the mapping from ω to \mathbf{X}_{S_ω} is one-one and onto. Thus, $H(\omega|\mathbf{X}_{S^{(K-j)}}) = H(\mathbf{X}_{S_\omega}|\mathbf{X}_{S^{(K-j)}})$ and similarly $H(\omega|\underline{\mathbf{y}}, \mathbf{X}_{S^{(K-j)}}) = H(\mathbf{X}_{S_\omega}|\underline{\mathbf{y}}, \mathbf{X}_{S^{(K-j)}})$. Using the above and the fact that $H(\omega|\mathbf{X}_{S^{(K-j)}}) = \log \binom{N-K+j}{j}$ in (2.39) and (2.40), we get

$$\log \binom{N-K+j}{j} = H(\mathbf{X}_{S_\omega}|\underline{\mathbf{y}}, \mathbf{X}_{S^{(K-j)}}) + I(\mathbf{X}_{S_\omega}; \underline{\mathbf{y}}|\mathbf{X}_{S^{(K-j)}}) \quad (2.41)$$

$$\begin{aligned} &\leq H_b(P_e) + \log \binom{N-K+j-L}{j} \\ &\quad + P_e \Gamma_l(L, N, K, j) + I(\mathbf{X}_{S_\omega}; \underline{\mathbf{y}}|\mathbf{X}_{S^{(K-j)}}). \end{aligned} \quad (2.42)$$

Note that $I(\mathbf{X}_{S_\omega}; \underline{\mathbf{y}}|\mathbf{X}_{S^{(K-j)}}) = I(\mathbf{X}_{S^{(j)}}; \underline{\mathbf{y}}|\mathbf{X}_{S^{(K-j)}})$ and using basic properties of entropy, mutual information and the i.i.d. assumption across observations [30], it can be shown that:

$$I(\mathbf{X}_{S^{(j)}}; \underline{\mathbf{y}}|\mathbf{X}_{S^{(K-j)}}) \leq MI(X_{S^{(j)}}; Y|X_{S^{(K-j)}}) = MI^{(j)}. \quad (2.43)$$

Thus, we get a genie aided lower bound on the probability of error as

$$P_e \geq 1 - \frac{H_b(P_e) + MI^{(j)}}{\Gamma_l(L, N, K, j)}, \quad \forall j = 1, 2, \dots, K. \quad (2.44)$$

This further implies

$$M \geq \frac{(1 - P_e)\Gamma_l(L, N, K, j) - H_b(P_e)}{I^{(j)}}, \quad \forall j = 1, 2, \dots, K. \quad (2.45)$$

The above equation holds for all $j = 1, 2, \dots, K$ and thus, the lower bound on the number of observations follow easily by noting that $H_b(P_e) \rightarrow 0$ as $P_e \rightarrow 0$. Hence the proof.

2.6 Conclusions

In this chapter, we considered the problem of identifying L non-defective items out of a large population of N items containing K defective items in a general sparse signal modeling setup. We contrasted two approaches: identifying the defective items using the observations followed by picking L items from the complement set, and directly identifying non-defective items from the observations. We derived upper and lower bounds on the number of observations required for identifying the L non-defective items. We showed that an impressive gain in the number of observations is obtainable by directly identifying the non-defective items. We also applied the results in a nonadaptive group testing setup. We characterized the number of tests that are sufficient to identify a subset of non-defective items in a large population under both dilution and additive noise models. We showed the order-wise tightness of the upper and lower bounds. Our results were information theoretic in nature, without considering the practicability of the decoding algorithms. In the next chapter, we present several computationally efficient algorithms for directly identifying a subset of inactive variables.

Chapter 3

Finding a Subset of Non-defective Items: Computationally Efficient Algorithms

In this chapter, we develop computationally efficient algorithms for non-defective subset identification in the noisy, non-adaptive group testing with random pooling (NNGT-R) framework. Recall that, in Chapter 2, using information theoretic arguments, we showed that compared to the conventional approaches of identifying the non-defective subset by first identifying the defective set or by testing individual items one-by-one, directly searching for an L -sized non-defective subset offers a significant reduction in the number of tests, especially when L is small compared to $N - K$. The achievability results in the previous chapter were obtained by analyzing the performance of the exhaustive search based algorithms, which are not practically implementable due to their high computational complexity. This motivates the need for computationally efficient algorithms that offer performance comparable to the exhaustive search based algorithms.

We also recall, that the problem of non-defective subset identification is a generalization of the defective set identification problem, in the sense that, when $L = N - K$, the non-defective subset identification problem is identical to that of identifying the K defective items. Hence, by setting $L = N - K$, the algorithms presented in this chapter can be related to algorithms for finding the defective set; see [4] for an excellent collection of existing results and references. In general, for the NNGT-R framework, three broad approaches have been adopted for defective set recovery [20]. First, the row based approach, also frequently referred to as the “naive” decoding algorithm, finds the defective set by finding *all* the non-defective items. For example, the survey in [49], lists many variants of this algorithm for finding defective items. More recently, the **CoCo** algorithm was studied in [20], where an interesting connection of the naive decoding algorithm with the classical coupon-collector problem was established for the noiseless case. The second popular decoding approach is based on the idea of finding defective items iteratively (or greedily) by “appropriately” matching the column of the test matrix corresponding to a given item with the test outcome vector [4,15,20,50]. For example, in [15], column matching consists of taking set differences between the set of pools where the item is tested and the set of pools with positive outcomes. Another variant of matching is considered in [20], where, for a given column, the ratio of number of times an item is tested in pools with positive and negative outcomes is computed and compared to a threshold. A recent work, [41], investigates the problem of finding zeros in a sparse vector in the compressive sensing framework, and also proposes a

greedy algorithm based on correlating the sensing matrix column (i.e., column matching) with the output vector.¹ The connection between defective set identification in group testing and the sparse recovery in compressive sensing was further highlighted in [20,51], where linear programming relaxation based algorithms have been proposed for defective set identification in group testing. A class of linear programs to solve the defective set identification problem was proposed by letting the boolean variables take real values (between 0 and 1) and setting up inequality or equality constraints to model the outcome of each pool.

In this chapter, we propose novel algorithms for identifying a non-defective subset in an NNGT-R framework. We present a probability of error analysis for each algorithm and derive non-asymptotic upper bounds on the average error rate. The derivation leads to a theoretical guarantee on the sample complexity, i.e., the number of tests required to identify a subset of non-defective items with arbitrarily small probability of error. We summarize our main contributions as follows:

- We propose a bouquet of computationally simple and analytically tractable algorithms for identifying a non-defective subset of given size in a NNGT-R framework: **RoAI** (row based), **CoAI** (column based) and **RoLpAI**, **RoLpAI++**, **CoLpAI** (Linear Program (LP) relaxation based) algorithms.
- We derive bounds on the number of tests that guarantee successful non-defective subset recovery for each algorithm. The derived bounds are a function of the system parameters, namely, the number of defective items, the size of non-defective

¹Note that directly computing correlations between column vector for an item and the test outcome vector will not work in case of group testing as both the vectors are boolean. Furthermore, positive and negative pools have asymmetric roles in the group testing problem.

subset, the population size and the noise parameters. Further,

- The presented bounds on the number of tests for different algorithms are within $O(\log K)$ factor, where K is the number of defective items, of the information theoretic lower bounds which were derived in Chapter 2.
- We analytically determine the parameters of the algorithms that offer the best performance in terms of optimizing the error upper bounds.
- For our suite of LP based algorithms, we present a novel analysis technique based on characterizing the recovery conditions via the dual variables associated with the LP, which may be of interest in its own right.
- Finally, we present numerical simulations to compare the relative performance of the algorithms. The results also illustrate the significant benefit in finding non defective items directly, compared to using the existing defective set recovery methods or testing items one-by-one, in terms of the number of group tests required.

The rest of the chapter is organized as follows. Section 3.1 describes the NNGT-R framework and the problem setup. The proposed algorithms and the main analytical results are presented in Section 3.2. The proofs of the main results are provided in Section 3.3. Section 3.4 discusses the numerical simulation results, and the conclusions are presented in Section 3.5.

3.1 Signal Model

In our setup, we have a population of N items, out of which K are defective. Let $\mathcal{G} \subset [N]$ denote the defective set, such that $|\mathcal{G}| = K$. We consider a non-adaptive group

testing framework with random pooling designs [4,20,27,30], where all the group tests are decided a priori and the items to be pooled in a given test are chosen randomly. The group tests are defined by a boolean matrix, $\mathbf{X} \in \{0, 1\}^{M \times N}$, that assigns different items to the M group tests (pools). The j^{th} pool tests the items corresponding to the columns with 1 in the j^{th} row of \mathbf{X} . We consider an i.i.d. random Bernoulli measurement matrix [30], where each $X_{ij} \sim \mathcal{B}(p)$ for some $0 < p < 1$. Thus, M randomly generated pools are specified. In the above, p is a design parameter that controls the average group size, i.e., the average number of items being tested in a single group test. In particular, we choose $p = \frac{\alpha}{K}$, and a specific value of α is chosen based on the analysis of different algorithms.²

If the tests are completely reliable, then the output of the M tests is given by the boolean OR of the columns of \mathbf{X} corresponding to the defective set \mathcal{G} . In group testing, two different noise models are considered [15,20,30]: (a) An *additive* noise model, where there is a probability, $q \in (0, 0.5]$, that the outcome of a group test containing only non-defective items turns out to be positive (Fig. 3.1); (b) A *dilution* model, where there is a probability, $u \in (0, 0.5]$, that a given item does not participate in a given group test (Fig. 3.1). Let $\underline{d}_i \in \{0, 1\}^M$. Let $\underline{d}_i(j) \sim \mathcal{B}(1 - u)$ be chosen independently for all $j = 1, 2, \dots, M$ and for all $i = 1, 2, \dots, N$. Let $\mathbf{D}_i \triangleq \text{diag}(\underline{d}_i)$. Let “ \vee ” denote the boolean

²The above parametrized form of p has been motivated by our results from Chapter 2, where one of the conclusions, based on information theoretic arguments, was that the optimal value of p that minimizes the number of tests required for “finding a non-defective subset” and “finding the defective set” are the same. The form α/K approximates this optimal value very well and has been widely used in literature [15,20,36] when probabilistic constructions are employed for designing the test matrices.

OR operation. The output vector $\underline{y} \in \{0, 1\}^M$ can be represented as

$$\underline{y} = \bigvee_{i=1}^N \mathbf{D}_i \underline{x}_i \mathbb{I}_{\{i \in \mathcal{G}\}} \bigvee \underline{w}, \quad (3.1)$$

where $\underline{x}_i \in \{0, 1\}^M$ is the i^{th} column of \mathbf{X} , $\underline{w} \in \{0, 1\}^M$ is the additive noise with the i^{th} component $w(i) \sim \mathcal{B}(q)$. Note that, for the noiseless case, $u = 0, q = 0$. The signal model considered in (3.1) is an abstraction of the testing setup used in many practical non-adaptive group testing applications. For example, consider the classical application of the screening of blood samples from a large number of individuals for the presence or absence of a certain antigen. The blood samples from a randomly chosen subset of individuals are pooled together in a test, and the test outcome is well described by the logical-OR operation specified in (3.1). That is, the test outcome is positive if *any* of the pooled samples contains the antigen, and is negative only if *none* of the pooled samples contain the antigen. Given the test output vector, \underline{y} , our goals are as follows:

- (a) To find computationally tractable algorithms to identify L non-defective items, i.e., an L -sized subset belonging to $[N] \setminus \mathcal{G}$.
- (b) To analyze the performance of the proposed algorithms with the objective of (i) finding the number of tests and (ii) choosing the appropriate design parameters that leads to non-defective subset recovery with high probability of success.

In the literature on defective set recovery in group testing or on sparse vector recovery in compressed sensing, there exist two type of recovery results: (a) *Non-uniform/Per-Instance recovery results*: These state that a randomly chosen test matrix leads to non-defective subset recovery with high probability of success for a given fixed defective

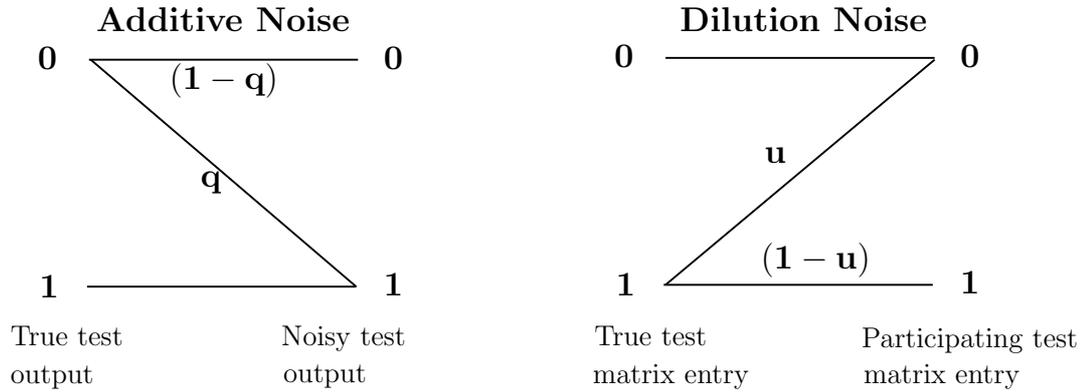


Figure 3.1: Impact of different types of noise on the group testing signal model.

set and, (b) *Uniform/Universal recovery results*: These state that a random draw of the test matrix leads to a successful non-defective subset recovery with high probability for all possible defective sets. Since it is possible to easily extend non-uniform results to the uniform case using union bounds, we mainly focus on non-uniform recovery results, and demonstrate the extension to the uniform case for one of the proposed algorithms (see Corollary 2). Note that the non-uniform scenario is equivalent to the uniform recovery scenario when the defective set is chosen uniformly at random from a set of $\binom{N}{K}$ possible choices. For the latter scenario, information theoretic lower bounds on the number of tests, based on Fano's inequality, for the non-defective subset recovery problem were derived in Chapter 2. We use these bounds in assessing the performance of the proposed algorithms (see Section 3.2.4). For the ease of reference, we summarize these results in Table 3.1.

For later use, we summarize some key facts pertaining to the above signal model in the lemma below. For any $l \in [M]$ and $k \in [N]$, let X_{lk} denote the $(l, k)^{\text{th}}$ entry of the test matrix \mathbf{X} and let $Y_l \triangleq \underline{y}(l)$ denote the l^{th} test output. With u , q and p as defined above, let $\Gamma \triangleq (1 - q)(1 - (1 - u)p)^K$ and $\gamma_0 \triangleq \frac{u}{(1 - (1 - u)p)}$. Then it follows that,

Lemma 2. (a) $\mathbb{P}(Y_l = 0) = \Gamma$.

(b) For any $j \notin S_d$, $\mathbb{P}(Y_l | X_{lj}) = \mathbb{P}(Y_l)$.

(c) For any $i \in S_d$, $\mathbb{P}(Y_l = 0 | X_{li} = 1) = \gamma_0 \Gamma$ and $\mathbb{P}(Y_l = 0 | X_{li} = 0) = \frac{\Gamma}{1 - (1-u)^p}$.

(d) Given Y_l , X_{li} is independent of X_{lj} for any $i \in S_d$ and $j \notin S_d$.

The proof is provided in Appendix B.1. We further note that, using Bayes rule, it is easy to compute the posterior probabilities of $\mathbb{P}(X_{li} | Y_l)$. For example, for any $i \in S_d$, it follows that $\mathbb{P}(X_{li} = 1 | Y_l = 0) = p\gamma_0$.

3.2 Algorithms and Main Results

We now present several algorithms for non-defective/healthy subset recovery. Each algorithm takes the observed noisy output vector $\underline{y} \in \{0, 1\}^M$ and the test matrix $\mathbf{X} \in \{0, 1\}^{M \times N}$ as inputs, and outputs a set of L items, \hat{S}_L , that have been declared non-defective. The recovery is successful if the declared set does not contain any defective item, i.e., $\hat{S}_L \cap S_d = \{\emptyset\}$. For each algorithm, we derive expressions for the upper bounds on the average probability of error, which are further used in deriving the number of tests required for successful non-defective subset recovery.

3.2.1 Row Based Algorithm

Our first algorithm to find non-defective items is also the simplest and the most intuitive one. We make use of the basic fact of group testing that, in the noiseless case, if the test outcome is negative, then all the items being tested are non-defective.

RoAI (Row based algorithm):

- Compute $\underline{z} = \sum_{j \in \text{supp}(\underline{y}^e)} \underline{x}_j^{(r)}$, where $\underline{x}_j^{(r)}$ is the j^{th} row of the test matrix.
- Order entries of \underline{z} in descending order.
- Declare the items indexed by the top L entries as the non-defective subset.

That is, declare the L items that have been tested most number of times in pools with negative outcomes as non-defective items. The above decoding algorithm proceeds by only considering the tests with negative outcomes. Note that, since the test outcomes are noisy, there is a non-zero probability of declaring a defective item as non-defective. In particular, the dilution noise can lead to a test containing defective items in the pool being declared negative, leading to a possible mis-classification of the defective items. On the other hand, since the algorithm only considers tests with negative outcomes, additive noise does not lead to mis-classification of defective items as non-defective. However, the additive noise does lead to an increased number of tests as the algorithm has to possibly discard many of the pools that contain only non-defective items.

We note that existing row based algorithms for finding defective set [4, 20] can be obtained as a special case of the above algorithm by setting $L = N - K$, i.e., by looking for all non-defective items. However, the analysis in the past work does not quantify the impact of the parameter L and that is our main goal here. We characterize the number of tests, M , that are required to find L non-defective items with high probability of success using **RoAI** in the following theorem:

Theorem 4. (Non-Uniform recovery with **RoAI**) Let N, L, M, p, u and q be as defined above.

Define $\gamma_0 \triangleq \frac{u}{(1-(1-u)p)}$. Let p be chosen as $\frac{\alpha}{K}$ with $\alpha = \frac{1}{3(1-u)}$. There exists an absolute constant $C_0 > 0$ such that, if the number of tests is chosen as

$$M \geq \frac{C_0 K (1-u)}{(1-q)(1-\gamma_0)^2} \left(\frac{\log [K \binom{N-K}{L-1}]}{(N-K) - (L-1)} \right), \quad (3.2)$$

then for a given defective set there exist positive constants c_0, c_1 , such that **RoAI** finds L non-defective items with probability exceeding $1 - \exp(-Mc_0) - \exp(-Mc_1)$.

The following corollary extends Theorem 4 to uniform recovery of a non-defective subset using **RoAI**.

Corollary 2. (Uniform recovery with **RoAI**) Define $\gamma_0 \triangleq \frac{u}{(1-(1-u)p)}$. Let p be chosen as $\frac{\alpha}{K}$ with $\alpha = \frac{1}{3(1-u)}$. There exist absolute constants $C_0 > 0$ and $C_1 > 0$ such that, if the number of tests is chosen as

$$M = \max \left\{ \frac{C_0 K (1-u)}{(1-q)(1-\gamma_0)^2} \left(\frac{\log [N \binom{N-K}{L-1}]}{(N-K) - (L-1)} \right), \frac{C_1 \log \binom{N}{K}}{(1-q)} \right\}, \quad (3.3)$$

then for **any** defective set there exist positive constants $c_0, c_1 > 0$ such that the algorithm (**RoAI**) finds L non-defective items with probability exceeding $1 - \exp(-Mc_0) - \exp(-Mc_1)$.

The proofs of the above theorem and corollary are presented in Section 3.3.1.

3.2.2 Column Based Algorithm

The column based algorithm is based on matching the columns of the test matrix with the test outcome vector. A non-defective item does not impact the output and hence the corresponding column in the test matrix should be “uncorrelated” with the output. On the other hand, “most” of the pools that test a defective item should test positive.

This forms the basis of distinguishing a defective item from a non-defective one. The specific algorithm is as follows:

CoAI (Column based algorithm): Let $\psi_{cb} > 0$ be some constant.

- For each $i = 1, \dots, N$, compute

$$\mathcal{T}(i) = \underline{x}_i^T \underline{y}^c - \psi_{cb}(\underline{x}_i^T \underline{y}), \quad (3.4)$$

where \underline{x}_i is the i^{th} column of \mathbf{X} .

- Sort $\mathcal{T}(i)$ in descending order.
- Declare the items indexed by the top L entries as the non-defective subset.

We note that, in contrast to the row based algorithm, **CoAI** works with pools of both the negative and positive test outcomes. Similar to the row based algorithm, by analyzing the probability of error, we can derive the sufficient number of tests required to achieve arbitrarily small error rates. In the above algorithm, the factor ψ_{cb} has been introduced for mathematical convenience; it helps in optimizing (minimizing) the upper bounds. We summarize the main result in the following theorem:

Theorem 5. (Non-Uniform recovery with **CoAI**) Let N, L, M, p, u and q be as defined above. Let $\Gamma \triangleq (1 - q)(1 - (1 - u)p)^K$ and $\gamma_0 \triangleq \frac{u}{(1 - (1 - u)p)}$. Let p be chosen as $\frac{1}{3(1 - u)K}$. Let $\psi_0 \triangleq \frac{\Gamma(1 + \gamma_0)}{2(1 - p)}$ and choose $\psi_{cb} = \psi_0$. There exists an absolute constant $C_2 > 0$ such that, if the number

of tests is chosen as

$$M \geq \frac{C_2 K (1 - u)}{(1 - \gamma_0)^2 (1 + \psi_0) (1 - q)} \left(\frac{\log [K \binom{N-K}{L-1}]}{(N - K) - (L - 1)} \right), \quad (3.5)$$

then for a given defective set **CoAI** finds L non-defective items with probability exceeding $1 - \exp(-M c_0)$, for some $c_0 > 0$.

The proof of the above theorem is presented in Section 3.3.2. It is tempting to compare the performance of **RoAI** and **CoAI** by comparing the required number of tests in (3.2) and (3.5), respectively. However, such comparisons must be done keeping in mind that the required number of observations in (3.2) and (3.5) are based on an upper bound on the average probability of error. The main objective of these results is to provide a guarantee on the number of tests required for non-defective subset recovery and highlight the order-wise dependence of the number of tests on the system parameters. For the comparison of the relative performance of the algorithms, we refer the reader to Section 3.4, where we present numerical results obtained from simulations. From the simulations, we observe that **CoAI** performs better than **RoAI** for most scenarios of interest.

3.2.3 Linear program relaxation based algorithms

In this section, we consider linear program (LP) relaxations to the non-defective subset recovery problem and identify the conditions under which such LP relaxations lead to recovery of a non-defective subset with high probability of success. These algorithms are inspired by analogous algorithms studied in the context of defective set recovery in the literature [20, 51]. However, past analysis on the number of tests for the defective

set recovery do not carry over to the non-defective subset recovery because the goals of the algorithms are very different. Let $Y_z \triangleq \{\underline{y} = 0\}$, i.e., Y_z is the set of all the pools with negative outcomes and $M_z \triangleq |Y_z|$. Similarly, let $Y_p \triangleq \{\underline{y} = 1\}$ and $M_p \triangleq |Y_p|$. Define the following linear program, with optimization variables $\underline{z} \in \mathbb{R}^N$ and $\underline{\eta} \in \mathbb{R}^{M_z}$:

$$\underset{\underline{z}, \underline{\eta}}{\text{minimize}} \quad \underline{1}_{M_z}^T \underline{\eta} \quad (3.6)$$

$$\text{(LP0)} \quad \text{subject to} \quad \mathbf{X}(Y_z, :)(\underline{1}_N - \underline{z}) - \underline{\eta} = \underline{0}_{M_z}, \quad (3.7)$$

$$\underline{0}_N \preceq \underline{z} \preceq \underline{1}_N, \quad \underline{\eta} \succeq \underline{0}_{M_z},$$

$$\underline{1}_N^T \underline{z} \leq L.$$

Consider the following algorithm³:

RoLpAl (LP relaxation with negative outcome pools only)

- Setup and solve **LP0**. Let $\hat{\underline{z}}$ be the solution of **LP0**.
- Sort $\hat{\underline{z}}$ in descending order.
- Declare the items indexed by the top L entries as the non-defective subset.

The above program relaxes the combinatorial problem of choosing L out of N items by allowing the boolean variables to acquire “real” values between 0 and 1 as long as the constraints imposed by negative pools, specified in (3.7), are met. Intuitively, the variable \underline{z} (or the variable $[\underline{1}_N - \underline{z}]$) can be thought of as the confidence with which an item is being declared as non-defective (or defective). The constraint $\underline{1}_N^T \underline{z} \leq L$ forces the

³In general other algorithms presented in this sub-section, namely **RoLpAl++** and **CoLpAl**, will have the same structure and will differ only in the linear program being solved.

program to assign high values (close to 1) for “approximately” the top L entries only, which are then declared as non-defective.

For the purpose of analysis, we first derive sufficient conditions for the non-defective subset recovery with **RoLpAI** in terms of the dual variables of **LP0**. We then derive the number of tests required to satisfy these sufficiency conditions with high probability. The following theorem summarizes the performance of the above algorithm:

Theorem 6. (Non-Uniform recovery with **RoLpAI**) *Let N, L, M, p, u and q be as defined above. Let p be chosen as $\frac{1}{3(1-u)K}$. If the number of tests is chosen as (3.2), then for a given defective set there exist positive constants c_0, c_1 , such that **RoLpAI** finds L non-defective items with probability exceeding $1 - \exp(-Mc_0) - \exp(-Mc_1)$.*

Note that **LP0** operates only on the set of pools with negative outcomes and is, thus, sensitive to the dilution noise which can lead to a misclassification of a defective item as non-defective. To combat this, we can leverage the information available from the pools with positive outcomes also, by incorporating constraints for variables involved in these tests. Consider the following linear program with optimization variables $\underline{z} \in \mathbb{R}^N$ and $\underline{\eta}_z \in \mathbb{R}^{M_z}$:

$$\text{minimize}_{\underline{z}, \underline{\eta}_z} \quad \mathbf{1}_{M_z}^T \underline{\eta}_z \quad (3.8)$$

(LP1)

$$\text{subject to} \quad \mathbf{X}(Y_z, :)(\mathbf{1}_N - \underline{z}) - \underline{\eta}_z = \mathbf{0}_{M_z}$$

$$\mathbf{X}(Y_p, :)(\mathbf{1}_N - \underline{z}) \succcurlyeq (1 - \epsilon_0)\mathbf{1}_{M_p} \quad (3.9)$$

$$\mathbf{0}_N \preccurlyeq \underline{z} \preccurlyeq \mathbf{1}_N, \quad \underline{\eta}_z \succcurlyeq \mathbf{0}_{M_z}$$

$$\mathbf{1}_N^T \underline{z} \leq L.$$

In the above, $0 < \epsilon_0 \ll 1$ is a small positive constant. Note that (3.9) attempts to model, in terms of real variables, a boolean statement that at least one of the items tested in tests with positive outcomes is a defective item. We refer to the algorithm based on **LP1** as **RoLpAl++**. We expect **RoLpAl++** to outperform **RoLpAl**, as the constraint (3.9) can provide further differentiation between items that are indistinguishable just on the basis of negative pools. Note that, due to the constraint $\mathbf{1}_N^T \mathbf{z} \leq L$, the entries of $\hat{\mathbf{z}}$ in $[N] \setminus \hat{S}_L$ are generally assigned small values. Hence, when L is small, for many of the positive pools, the constraint (3.9) may not be active. Thus, we expect **RoLpAl++** to perform better than **RoLpAl** as the value of L increases; this will be confirmed via simulation results in Section 3.4. Due to the difficulty in obtaining estimates for the dual variables associated with the constraints (3.9), it is difficult to derive theoretical guarantees for **RoLpAl++**. However, we expect the guarantees for **RoLpAl++** to be similar to **RoLpAl**, and we refer the reader to Appendix B.4 for a discussion regarding the same.

Motivated by the connection between **RoAl** and **RoLpAl**, as revealed in the proof of Theorem 6 (see Section 3.3.3), we now propose another LP based non-defective subset recovery algorithm that incorporates both positive and negative pools. By incorporating (3.9) in an unconstrained form and by using the *same* weights for all the associated Lagrangian multipliers in the optimization function, we get:

$$\begin{aligned}
 & \underset{\mathbf{z}}{\text{minimize}} && \mathbf{1}_{M_z}^T \mathbf{X}(Y_z, :)(\mathbf{1}_N - \mathbf{z}) - \psi_{lp} \left[\mathbf{1}_{M_p}^T \mathbf{X}(Y_p, :)(\mathbf{1}_N - \mathbf{z}) \right] && (3.10) \\
 \text{(LP2)} & \text{subject to} && \mathbf{0}_N \preceq \mathbf{z} \preceq \mathbf{1}_N, \\
 & && \mathbf{1}_N^T \mathbf{z} \leq L,
 \end{aligned}$$

where $\psi_{lp} > 0$ is a positive constant that provides appropriate weights to the two different type of cumulative errors. Note that, compared to **LP1**, we have also eliminated the equality constraints in the above program. The basic intuition is that by using (3.9) in an unconstrained form, i.e., by maximizing $\sum_{j \in Y_z} \mathbf{X}(j, :)(\mathbf{1}_N - \underline{z})$, the program will tend to assign higher values to $(1 - \hat{z}(i))$ (and hence lower values to $\hat{z}(i)$) for $i \in S_d$ since for random test matrices with i.i.d. entries, the defective items are likely to be tested more number of times in the pools with positive outcomes. Also, in contrast to **LP1** where different weightage is given to each positive pool via the value of the associated dual variable, **LP2** gives the same weightage to each positive pool, but it adjusts the overall weightage of positive pools using the constant ψ_{lp} . We refer to the algorithm based on **LP2** as **CoLpAI**. The theoretical analysis for **CoLpAI** follows on similar lines as **RoLpAI** and we summarize the main result in the following theorem:

Theorem 7. (*Non-Uniform recovery with CoLpAI*) Let N, L, M, α, u and q be as defined above. Let p be chosen as $\frac{1}{3(1-u)K}$. Let $\Gamma \triangleq (1-q)(1-(1-u)p)^K$ and $\gamma_0 \triangleq \frac{u}{(1-(1-u)p)}$. Let $\psi'_0 \triangleq \min\left(\frac{\Gamma(1+\gamma_0)}{2}, \frac{\Gamma}{2(1-\Gamma)}\right)$ and choose $\psi_{lp} = \psi'_0$. There exists an absolute constant $C_4 > 0$ such that, if the number of tests is chosen as

$$M \geq \frac{C_4 K (1-u)}{(1-\gamma_0)^2 (1-q)} \left(\frac{\log [K \binom{N-K}{L-1}]}{(N-K) - (L-1)} \right), \quad (3.11)$$

then for a given defective set **CoLpAI** finds L non-defective items with probability exceeding $1 - \exp(-Mc_0) - \exp(-Mc_1)$, with $c_0, c_1 > 0$.

3.2.4 Discussion on the theoretical guarantees

We now present some interesting insights by analyzing the number of tests required for correct non-defective subset identification by the proposed recovery algorithms in the asymptotic regime, as N gets large. We note that the expressions in (3.2), (3.5) and (3.11) are similar, and differ only on account of the constants involved. This allows us to present a unified analysis for all the algorithms. Also, recall, $\gamma_0 = \frac{u}{(1-(1-u)\frac{\alpha}{K})}$, and for large K , $\gamma_0 \approx u$. Therefore, for this discussion, we will assume that K is large. We make the following remarks:

(a) Asymptotic analysis of M as $N \rightarrow \infty$: For the asymptotic analysis as $N \rightarrow \infty$, we consider two parameter regimes:

- Fixed K , $\frac{L}{N} = \alpha_0$, $0 < \alpha_0 < 1$: We will refer to this regime as “fixed- K ” regime.

Using Stirling’s formula, it can be easily shown that, for this regime

$$\lim_{N \rightarrow \infty} \frac{\log \binom{N-K}{L-1}}{(N-K)-(L-1)} \leq \frac{H_b(\alpha_0)}{1-\alpha_0} \quad (\text{see e.g., Lemma 3, Appendix A}),$$

where $H_b(\cdot)$ is the binary entropy function. Noting that $\gamma_0 \approx u$ and $(1 + \psi_0) > 1$, the sufficient number of tests M for the proposed algorithms, as in (3.2), (3.5) and (3.11) can be commonly represented as $M_{FK} \triangleq O\left(\frac{K}{(1-u)(1-q)} \frac{H_b(\alpha_0)}{1-\alpha_0}\right)$. Note that, in this regime, as $N \rightarrow \infty$ (and hence $L \rightarrow \infty$), the number of tests required to find L non-defective items stay constant.

- $\frac{K}{N} = \beta_0$, $\frac{L}{N} = \alpha_0$, $0 < \beta_0 < \alpha_0 < 1$: Note that $\alpha + \beta < 1$. We will refer to this

regime as “increasing- K ” regime. Define, $\phi_0 = \frac{\alpha_0}{1-\beta_0}$. The sufficient number of

tests for this scaling regime can be shown equal to $M_{VK} \triangleq O\left(\frac{K}{(1-u)(1-q)} \frac{H_b(\phi_0)}{1-\phi_0}\right)$.

In this regime, both K and L grow linearly with N and the number of tests

increase linearly with K , since $\frac{H_b(\phi_0)}{1-\phi_0}$ is constant.

- (b) Variation of M with L : Let $\zeta \triangleq \frac{L}{N-K}$, the fraction of non-defective items that need to be found. Let $g(\zeta) \triangleq \frac{H_b(\zeta)}{1-\zeta}$. We note that the parameter L impacts both M_{FK} and M_{VK} *only* via the function $g(\zeta)$, and this function characterizes the dependence of the sufficient number of tests on L .⁴ For small values (or even moderately high values) of ζ , it can be easily shown that $g(\zeta)$ is approximately linear in ζ ; this is also confirmed via simulation results in Section 3.4.
- (c) Comparison with the “indirect” approach: Note that one way to find L non-defective items, referred to as the “indirect” approach (see Chapter 2), is to first identify the defective set (using any defective set recovery algorithm), and subsequently choose the non-defective items randomly from the complement set. One such “indirect” decoding scheme, based on an exhaustive search over all possible defective sets, was investigated in Chapter 2. It can be easily shown that, for the fixed- K scaling regime, the sufficient number of tests for such a scheme scales as $\Omega(K \log(KL))$, i.e., the number of tests increase as $N \rightarrow \infty$. In contrast, the number of tests with the direct approach M_{FK} stays constant as $N \rightarrow \infty$, since it only depends upon $\frac{L}{N}$, which is a constant. Thus, the proposed algorithms perform significantly better than the “indirect” approach; this is also supported by numerical simulations in Section 3.4.
- (d) Order-wise comparison with the information theoretic lower bounds: We consider the fixed- K regime and compare M_{FK} with the lower bounds on the number of tests for non-defective subset recovery, as tabulated in Table 3.1. First, consider

⁴For a detailed characterization of the function $\frac{H_b(\zeta)}{1-\zeta}$, the reader is referred to Lemma 3, Appendix A (appendix for Chapter 2).

the noiseless case, i.e., $u = 0, q = 0$. For the ease of comparison we consider two parameter ranges for α_0 . When α_0 is small, e.g., close to zero, we note that $\frac{H_b(\alpha_0)}{1-\alpha_0}$ is dominated by the term $\frac{\alpha_0}{1-\alpha_0} \log(1/\alpha_0)$. Thus, the gap between upper and lower bounds scales as $O\left(\frac{\alpha_0}{1-\alpha_0} \frac{\log \alpha}{\log(1-\alpha_0)} \log K\right)$. When α_0 is close to 1, $\frac{H_b(\alpha_0)}{1-\alpha_0}$ is dominated by the term $\log \frac{1}{1-\alpha_0}$. Thus, M_{FK} is within a $O(\log K)$ factor of the lower bound. For the additive noise-only case, the algorithms incur a factor of $1/(1-q)$ increase in M . This is expected, since additive noise causes a fraction q of the tests with negative outcomes to be discarded. Also, in terms of the variation with q , we note that the factor $\frac{1}{\log(1/q)}$ in the lower bound behaves similar to $\frac{1}{(1-q)}$. For the dilution noise case, the algorithms exhibit an optimal dependence on u and incur a factor $\frac{1}{(1-u)}$ increase in M , which is the same as in the lower bound. Thus, the proposed algorithms for the non-defective subset recovery are nearly optimal, as the number of tests required for a guaranteed correct recovery fall within $O(\log K)$ of the information theoretic lower bounds. We have also compared the number of tests obtained via simulations with an exact computation of the lower bounds; we refer the reader to Figure 3.4, Section 3.4. Similar observations apply in the increasing- K regime also.

- (e) Defective set recovery via non-defective subset recovery: It is interesting to note that by substituting $L = N - K$ in (3.2) and (3.5), we get $M = O\left(\frac{K \log(N-K)}{(1-u)(1-q)}\right)$, which is order-wise similar to the number of tests required for defective set identification derived in the existing literature [15, 20, 36].
- (f) Robustness under uncertainty in the knowledge of K : The theoretical guarantees presented in the above theorems hold provided the design parameter p is chosen as

$O(\frac{1}{(1-u)K})$. This requires the knowledge of u , and, in particular, the number of defective items K .⁵ If we do not have perfect knowledge of u and K , similar guarantees can be easily derived, albeit with a penalty on the number of tests, as expected. For example, choosing p as $O(1/K)$, i.e., independent of u , results in a $\frac{1}{1-u}$ times increase in the number of tests. Similarly, the impact of using an imperfect value of K can also be quantified. Let \hat{K} be the value used and let $\Delta_k > 0$ be such that $\hat{K} = \Delta_k K$. That is, Δ_k parametrizes the estimation error in K . Using the fact that for large n , $(1 - \alpha/n)^n \approx \exp(-\alpha)$, it follows that with $p = O(\frac{1}{\Delta_k K})$, the number of tests increases approximately by a factor of $f_M(\Delta_k) \triangleq \Delta_k \exp\left(- (1-u)(\frac{1}{\Delta_k} - 1)\right)$ compared to the case with perfect knowledge of K , i.e., with $p = O(1/K)$. It is easy to see that the proposed algorithms are robust to the uncertainty in the knowledge of K . For example, with $u = 0$, $f_M(1.5) = 1.09$, i.e., a 50% error in the estimation of K leads to 9% increase in number of tests. Furthermore, the asymmetric nature of $f_M(\Delta_k)$ (e.g., $f_M(1.5) = 1.09$ and $f_M(0.5) = 1.3$) suggests that the algorithms are more robust when $\Delta_k > 1$ as compared to the case when $\Delta_k < 1$. We corroborate this behavior via numerical simulations also (see Table 3.2).

- (g) Operational complexity: It is easy to see that the execution of **RoAI** and **CoAI** requires $O(MN)$ operations, where M is the number of tests. The complexity of LP based algorithms **RoLpAI**, **RoLpAI++** and **CoLpAI** is implementation dependent, but is, in general, much higher than **RoAI** and **CoAI**. For example, an interior-point method based implementation will require $O(N^2(M+N)^{3/2})$ operations [52].

⁵Note that the implementation of the recovery algorithms do not require us to know the values of K or u . These system model parameters are only required to choose the value of p for constructing the test matrix.

Table 3.1: Finding a subset of L non-defective items: Order results for necessary number of group tests which hold asymptotically as $N \rightarrow \infty$ (see Theorem 3, Chapter 2). “Fixed- K ” regime: $\frac{L}{N} = \alpha_0$, $0 < \alpha_0 < 1$ and fixed K . “Increasing- K ” regime: $\frac{K}{N} = \beta_0$, $\frac{L}{N} = \alpha_0$, $\alpha_0 + \beta_0 < 1$, $0 < \beta_0 < \alpha_0 < 1$ and $\phi_0 \triangleq \frac{\alpha_0}{1-\beta_0}$.

	Fixed- K regime	Increasing- K regime
No Noise ($u = 0, q = 0$)	$\Omega \left(\frac{K}{\log K} \log \frac{1}{(1 - \alpha_0)} \right)$	$\Omega \left(\frac{K}{\log K} \log \frac{1}{(1 - \phi_0)} \right)$
Dilution Noise ($u > 0, q = 0$)	$\Omega \left(\frac{K}{(1 - u) \log K} \log \frac{1}{(1 - \alpha_0)} \right)$	$\Omega \left(\frac{K}{(1 - u) \log K} \log \frac{1}{(1 - \phi_0)} \right)$
Additive Noise ($u = 0, q > 0$)	$\Omega \left(\frac{K}{\log \frac{1}{q}} \log \frac{1}{(1 - \alpha_0)} \right)$	$\Omega \left(\frac{K}{\log \frac{1}{q}} \log \frac{1}{(1 - \phi_0)} \right)$

Although, this is higher compared to **RoAI** and **CoAI**, it is still attractive in comparison to the brute force search methods, due to its polynomial-time complexity.

3.3 Proofs of the Main Results

3.3.1 Proof of Theorem 4 and Corollary 2

The proof involves upper bounding the probability of non-defective subset recovery error of the decoding algorithm, **RoAI**, and deriving the parameter choices, i.e., M , p , that drive this probability arbitrarily close to 0. Let $S_d \subset [N]$ be the defective set and let $\hat{S}_L \subset [N]$ denote the set of L non-defective items output by the decoding algorithm. Let $T_n(i, l)$ denote the number of times an item i is tested in l tests with negative outcomes. Let $M_n(l)$ represent the number of negative outcomes in a group testing setup with l tests. For a given defective set S_d , let $\mathcal{E} \triangleq \left\{ \hat{S}_L \cap S_d \neq \{\emptyset\} \right\}$ denote the error event, i.e., the event that the above algorithm outputs an incorrect non-defective subset.

Clearly, **RoAI** succeeds when there exists a set of at least L non-defective items that

have been tested more number of times than any of the defective items, in the tests with negative outcomes. Let $S_z \subset [N] \setminus S_d$ be any set of non-defective items such that $|S_z| = N_0 \triangleq (N - K) - (L - 1)$. Further, let \mathcal{S}_z denote all such possible sets. Thus,

$$\mathcal{E} \subseteq \{\exists i \in S_d \text{ and } \exists S_z, \text{ such that } T_n(i, M_n(M)) \geq T_n(j, M_n(M)) \forall j \in S_z\} \quad (3.12)$$

$$\subseteq \bigcup_{i \in S_d} \bigcup_{S_z \in \mathcal{S}_z} \{T_n(i, M_n(M)) \geq T_n(j, M_n(M)), \forall j \in S_z\} \quad (3.13)$$

$$\subseteq \left[\bigcup_{i \in S_d} \bigcup_{S_z \in \mathcal{S}_z} \{T_n(i, M_n(M)) \geq T_n(j, M_n(M)), \forall j \in S_z\} \cap \overline{\mathcal{M}_0} \right] \cup \mathcal{M}_0, \quad (3.14)$$

where $\mathcal{M}_0 \triangleq \{M_n(M) < M_0\}$ for some $0 < M_0 < M$, and, will be chosen appropriately later. Define $\mathcal{E}_0(m) \triangleq \{T_n(i, m) \geq T_n(j, m), i \in S_d, j \in S_z\}$. Using the union bound and the independence assumptions on the test matrix \mathbf{X} , we get, from (3.14),

$$\mathbb{P}(\mathcal{E}) \leq \left[K \binom{N-K}{L-1} \sum_{m \geq M_0} \mathbb{P}(M_n(M) = m) \{\mathbb{P}(\mathcal{E}_0(m))\}^{N_0} \right] + \mathbb{P}(\mathcal{M}_0). \quad (3.15)$$

We now analyze $\mathbb{P}(\mathcal{E}_0(m))$. Let $Z_{ij} \triangleq T_n(j, m) - T_n(i, m)$ for any $i \in S_d$ and $j \in S_z$. Thus, $\mathbb{P}(\mathcal{E}_0(m)) = \mathbb{P}(Z_{ij} \leq 0)$. We note that Z_{ij} can be represented as a sum of m i.i.d. Bernoulli random variables. In particular, $Z_{ij} = \sum_{l=1}^m (B_l - A_l)$, where B_l, A_l are i.i.d. and, using results from Lemma 2, $A_l \sim \mathcal{B}(p\gamma_0)$ and $B_l \sim \mathcal{B}(p)$. Thus, $\mathbb{E}(Z_{ij}) \triangleq \mu_{ij} = mp(1 - \gamma_0)$. Further, $\text{Var}[Z_{ij}] \triangleq \sigma_{ij}^2 = mp(1 + \gamma_0 - p(1 + \gamma_0^2))$. Note that $\sigma_{ij}^2 < mp(1 + \gamma_0)$. As shown later, we will choose p such that $\gamma_0 < 1$ (in fact, for large values of K , $\gamma_0 \approx u$) and thus $\mu_{ij} > 0$. Intuitively, since Z_{ij} is a sum of i.i.d. RVs, it concentrates around μ_{ij} , and hence the probability that it is negative is small. We make this argument rigorous by using Bernstein's inequality [53].⁶ With $\epsilon \in (0, 1)$ small, e.g., $\epsilon = 0.01$, let $\delta = (1 - \epsilon)\mu_{ij}$. We note that $|B_l - A_l| \leq 1$. Further, since $\mathbb{P}(\mathcal{E}_0(m)) \leq \mathbb{P}(Z_{ij} < \epsilon\mu_{ij})$, using Bernstein's

⁶For ease of reference, we have stated it in Appendix B.5.

inequality, we get:

$$\mathbb{P}(\mathcal{E}_0(m)) \leq \exp\left(-\frac{(1-\epsilon)^2\mu_{ij}^2}{2\sigma_{ij}^2 + \frac{2}{3}(1-\epsilon)\mu_{ij}}\right) \quad (3.16)$$

$$\leq \exp\left(-mp(1-\gamma_0)^2\frac{(1-\epsilon)^2}{4}\right) \triangleq \bar{f}(m). \quad (3.17)$$

Using the fact that $\bar{f}(m)$ is monotonically decreasing in m and substituting the above into (3.15) we get

$$\mathbb{P}(\mathcal{E}) \leq \left[K \binom{N-K}{L-1} \bar{f}(M_0)^{N_0}\right] + \mathbb{P}(\mathcal{M}_0). \quad (3.18)$$

To analyze $\mathbb{P}(\mathcal{M}_0)$, let us define $Z_l \triangleq \mathbb{I}_{\{\underline{y}(l)=0\}}$ for all $l = 1, 2, \dots, M$, and note that $M_n(M) = \sum_{l=1}^M Z_l$. Since the entries of the test matrix are i.i.d., Z_l are also i.i.d. with $\mathbb{E}(Z_l) \triangleq \Gamma = (1-q)[1 - (1-u)p]^K$ (see Lemma 2). Thus, by choosing $M_0 = (1-\eta)M\Gamma$ for some $0 < \eta < 1$ and using the multiplicative form of Chernoff bound [53, 54] (also see Appendix B.5), we get

$$\mathbb{P}(\mathcal{M}_0) \leq \exp\left(-M\eta^2\frac{\Gamma}{2}\right). \quad (3.19)$$

Choosing $p = \frac{\alpha}{K}$ with $\alpha = \frac{1}{3(1-u)}$, we note that $\gamma_0 < 1$ for any $u \leq 0.5$ and for all $K \geq 1$. Further, using the fact that for $0 < b < 1$, $(1-b) \leq e^{-b} \leq 1 - \frac{b}{2}$, we note that, $\left[1 - \frac{(1-u)\alpha}{K}\right]^K \geq \exp(-2\alpha(1-u)) \geq e^{-2/3}$. Thus, $\Gamma \geq e^{-2/3}(1-q)$. Substituting the chosen value of M_0 and p in (3.17), we get

$$\bar{f}(M_0) \leq \exp\left(-Mp\Gamma(1-\gamma_0)^2\frac{(1-\eta)(1-\epsilon)^2}{4}\right) \quad (3.20)$$

$$\leq \exp\left(-\frac{M(1-q)(1-\gamma_0)^2}{K} \frac{1}{C_0}\right), \quad (3.21)$$

where $C_0 \triangleq \frac{12e^{2/3}}{(1-\eta)(1-\epsilon)^2}$.⁷ Finally, from (3.18), we get

$$\mathbb{P}(\mathcal{E}) \leq \exp \left[-M \left(\frac{(1-\gamma_0)^2(1-q)N_0}{C_0 K(1-u)} - \frac{\log \left(K \binom{N-K}{L-1} \right)}{M} \right) \right] + \exp \left(-M(1-q) \frac{\eta^2}{2e^{2/3}} \right). \quad (3.22)$$

Thus, if M is chosen as specified in (3.2), with the constants C_0 and C_1 chosen as above, then there exist positive constants $c_0, c_1 > 0$ such that the error probability remains smaller than $\exp(-Mc_0) + \exp(-Mc_1)$, i.e., the decoding algorithm succeeds with high probability.

For the uniform case, we use the union bound over all possible choices of the defective set. Recall that $\mathbb{P}(\mathcal{M}_0)$ in (3.19) was derived for a fixed defective set. With the same definitions as above, the probability that, for a given instance of test matrix, we get less than $M_0 \left(= (1-\eta)M(1-q)[1-(1-u)p]^K \right)$ negative outcomes for *any* defective set is upper bounded as

$$\mathbb{P}(\mathcal{M}_0) \leq \exp \left[-M \left((1-q) \frac{\eta^2}{2e^{2/3}} - \frac{\log \binom{N}{K}}{M} \right) \right]. \quad (3.23)$$

For the first term in (3.18), we note that the multiplicative factor of K , arising due to the union bound over the individual defective items of a given defective set gets replaced by a factor of N to take into account all possible index values for a defective item. Using the same choices as the above for α , C_0 , and with $C_1 = \frac{2e^{2/3}}{\eta^2}$, it is easy to show that when M satisfies the bound in (3.3), we can obtain the required forms for each of the probability terms involved.

⁷In this proof, we have made a conservative choice for α . The factor 3 in the denominator is chosen to ensure that γ_0 is less than 1 for any $K \geq 1$ and for any value of $u \leq 0.5$. Using similar arguments as above, it is easy to see that for all $K \geq 3$, α can be chosen as $\frac{1}{(1-u)}$ and all the conclusions above hold with slightly different choice of the constant C_0 , e.g., with $C_0 = \frac{4e^2}{(1-\eta)(1-\epsilon)^2}$.

3.3.2 Proof of Theorem 5

Recall, $\mathcal{T}(i) = \underline{x}_i^T \underline{y}^c - (\psi_{cb}) \underline{x}_i^T \underline{y}$. Let S_d denote the defective set, such that $|S_d| = K$. Intuitively, we expect that for all $i \in S_d$, $\mathcal{T}(i)$ to be lower than any item not belonging to the defective set. Our strategy will be to bound the probability that $\mathcal{T}(i)$ for an item belonging to a defective set is higher than that of an item from the non-defective set, and then use the union bound to obtain an upper bound on the total error probability. Define $S_z \subset [N] \setminus S_d$ such that $|S_z| = (N - K) - (L - 1)$. Define $N_0 \triangleq (N - K) - (L - 1)$. Let \mathcal{S}_z denote all such possible sets. Thus, $|\mathcal{S}_z| = \binom{N-K}{L-1}$. Let \hat{S}_L denote the set of L items declared non-defective by **CoAI** and let \mathcal{E} denote the event that the algorithm makes an error. For a given defective set S_d , we have

$$\mathcal{E} \subset \bigcup_{i \in S_d} \{i \in \hat{S}_L\} \subset \bigcup_{i \in S_d} \bigcup_{S_z \in \mathcal{S}_z} \{\mathcal{T}(j) \leq \mathcal{T}(i), \forall j \in S_z\}. \quad (3.24)$$

Thus, using the union bound and the independence assumptions on the test matrix \mathbf{X} , we get,

$$\mathbb{P}(\mathcal{E}) \leq K \binom{N-K}{L-1} \left[\mathbb{P}(\{\mathcal{T}(j) \leq \mathcal{T}(i), i \in S_d, j \in S_z\}) \right]^{(N-K)-(L-1)}. \quad (3.25)$$

Let $P_e \triangleq \mathbb{P}(\{\mathcal{T}(j) \leq \mathcal{T}(i), i \in S_d, j \in S_z\})$. For any $i \in S_d$ and $j \in S_z$, define $Z \triangleq \mathcal{T}(j) - \mathcal{T}(i)$. We note that $P_e = \mathbb{P}(Z \leq 0)$. Further, define $Z_0 = \sum_{l=1}^M A_l$, where $A_l \triangleq [\underline{x}_j(l) - \underline{x}_i(l)] \mathbb{I}_{\{y(l)=0\}}$ are i.i.d. random variables. Similarly, define $Z_1 = \sum_{l=1}^M B_l$, where $B_l \triangleq [\underline{x}_i(l) - \underline{x}_j(l)] \mathbb{I}_{\{y(l)=1\}}$, and B_l are i.i.d. random variables. We note that $Z = Z_0 + \psi_{cb} Z_1 = \sum_{l=1}^M (A_l + \psi_{cb} B_l)$. Using results from Lemma 2 to compute statistics of A_l and B_l it follows that $\mathbb{E}(Z_0) = \mathbb{E}(Z_1) = Mp\Gamma(1 - \gamma_0)$, where $\Gamma \triangleq (1 - q)(1 - (1 - u)p)^K$ and

$\gamma_0 \triangleq \frac{u}{(1-(1-u)p)}$, as before. Further,

$$\mathbb{E}(Z_0^2) = Mp\Gamma[(1 + \gamma_0) - 2p\gamma_0] \quad \text{and} \quad \mathbb{E}(Z_1^2) = Mp[(1 - p)(2 - \gamma_0\Gamma) - \Gamma(1 - p\gamma_0)]. \quad (3.26)$$

Using the above, $\mathbb{E}(Z) \triangleq \mu_Z$ and $\text{Var}(Z) \triangleq \sigma_Z^2$ can be easily computed. As mentioned in the proof of Theorem 4, we will choose a value of p to ensure that $\gamma_0 < 1$ and thus $\mathbb{E}(Z_0) > 0$ and $\mathbb{E}(Z_1) > 0$. Also, we note that an upper bound on σ_Z^2 is sufficient for the purpose of upper bounding the error probabilities. In particular, we have

$$\mu_Z = Mp\Gamma(1 - \gamma_0)(1 + \psi_{cb}) \quad \text{and} \quad (3.27)$$

$$\sigma_Z^2 = \text{Var}(Z_0) + \psi_{cb}^2 \text{Var}(Z_1) \leq Mp[\Gamma(1 + \gamma_0) + 2\psi_{cb}^2(1 - p)]. \quad (3.28)$$

Let $\delta = (1 - \epsilon)\mu_Z$ for any small $\epsilon \in (0, 1)$. We note that $P_e = \mathbb{P}(Z \leq 0) \leq \mathbb{P}(Z < \mu_Z - \delta)$. Further, $|A_l + \psi_{cb}B_l| \leq \max\{1, \psi_{cb}\}$. Now, we proceed by assuming that $\psi_{cb} < 1$, and we will later see that our choice of ψ_{cb} will indeed turn out to be less than 1. Thus, using Bernstein's inequality, and sequentially upper bounding the probability of error expression, we get,

$$P_e \leq \exp\left(-\frac{(1 - \epsilon)^2 \mu_Z^2}{2\sigma_Z^2 + \frac{2}{3}\mu_Z}\right) \quad (3.29)$$

$$\leq \exp\left(-Mp\Gamma^2(1 - \gamma_0)^2 \frac{(1 - \epsilon)^2}{2} g(\psi_{cb})\right), \quad \text{where} \quad (3.30)$$

$$g(\psi_{cb}) \triangleq \frac{(1 + \psi_{cb})^2}{\Gamma(1 + \gamma_0) + 2(1 - p)\psi_{cb}^2 + (1 + \psi_{cb})(1 - \gamma_0)\Gamma}. \quad (3.31)$$

Define, $\psi_0 \triangleq \frac{\Gamma(1 + \gamma_0)}{2(1 - p)}$. Note that, $g(\psi_{cb})|_{\psi_{cb}=\psi_0} = \frac{(1 + \psi_0)}{2\Gamma}$. As in the proof of previous theorem, we choose $p = \frac{\alpha}{K}$ with $\alpha = \frac{1}{3(1 - u)}$. Using the above values of p and ψ_{cb} , and

noting that $\Gamma \geq e^{-2/3}(1-q)$, we get

$$P_e \leq \exp\left(-\frac{M}{K(1-u)}(1-q)(1-\gamma_0)^2(1+\psi_0)\frac{1}{C_2}\right). \quad (3.32)$$

where $C_2 \triangleq \frac{12e^{2/3}}{(1-\epsilon)^2}$. Thus, from (3.25),

$$\mathbb{P}(\mathcal{E}) \leq \exp\left(-M\left(\frac{(1-q)(1+\psi_0)(1-\gamma_0)^2 N_0}{K(1-u)C_2} - \frac{\log[K\binom{N-K}{L-1}]}{M}\right)\right). \quad (3.33)$$

Thus, by choosing M as in (3.5), $\exists c_0 > 0$ such that $\mathbb{P}(\mathcal{E}) \leq \exp(-Mc_0)$. Hence the proof.

3.3.3 Proof of Theorem 6

Let $\mathbf{X} \in \{0, 1\}^{M \times N}$ denotes the random test matrix, \underline{y} the group test output vector, $Y_z \triangleq \{\underline{y} = 0\}$ with $M_z = |Y_z|$, and $Y_p \triangleq \{\underline{y} = 1\}$ with $M_p = |Y_p|$. Let $\mathbf{X}_0 \triangleq \mathbf{X}(Y_z, \cdot)$ and $\mathbf{X}_1 \triangleq \mathbf{X}(Y_p, \cdot)$. Note that $\mathbf{X}_0 \in \{0, 1\}^{M_z \times N}$ and $\mathbf{X}_1 \in \{0, 1\}^{M_p \times N}$. For the ease of performance analysis for the LP described in (3.6), we work with the following equivalent program:

$$\begin{aligned} & \text{minimize}_{\underline{z}} \quad \underline{\mathbf{1}}_{M_z}^T \mathbf{X}_0 \underline{z} & (3.34) \\ \text{(LP0a)} \quad & \text{subject to} \quad \underline{\mathbf{0}}_N \preceq \underline{z} \preceq \underline{\mathbf{1}}_N, \\ & \underline{\mathbf{1}}_N^T \underline{z} \geq (N-L). \end{aligned}$$

The above formulation has been arrived at by eliminating the equality constraints and replacing the optimization variable \underline{z} by $(\underline{\mathbf{1}}_N - \underline{z})$. Hence, the non-defective subset output by (3.34) is indexed by the *smallest* L entries in the solution for (LP0a) (as opposed to largest L entries in the solution for (LP0)). We know that strong duality holds for a linear program and that any pair of primal and dual optimal points satisfy the Karush-Kuhn-Tucker (KKT) conditions [55]. Hence, a characterization of the primal solution

can be obtained in terms of dual optimal points via the KKT conditions. Let $\underline{\lambda}_1, \underline{\lambda}_2 \in \mathbb{R}^N$ and $\nu \in \mathbb{R}$ denote the dual variables associated with the inequality constraints in (LP0a). The KKT conditions for any pair of primal and dual optimal points corresponding to (LP0a) can be written as follows:

$$\underline{1}_{M_z}^T \mathbf{X}_0 - \underline{\lambda}_1 + \underline{\lambda}_2 - \nu \underline{1}_N = \underline{0}_N \quad (3.35)$$

$$\underline{\lambda}_1 \circ \underline{z} = \underline{0}_N; \underline{\lambda}_2 \circ (\underline{z} - \underline{1}_N) = \underline{0}_N; \nu(\underline{1}_N^T \underline{z} - (N - L)) = 0; \quad (3.36)$$

$$\underline{0}_N \preceq \underline{z} \preceq \underline{1}_N; \underline{1}_N^T \underline{z} \geq (N - L); \underline{\lambda}_1 \succeq \underline{0}_N; \underline{\lambda}_2 \succeq \underline{0}_N; \nu \geq 0; \quad (3.37)$$

Let $(\underline{z}, \underline{\lambda}_1, \underline{\lambda}_2, \nu)$ be primal, dual optimal points, i.e., a point satisfying the set of equations (3.35)-(3.37). Let S_d denote the set of defective items. Further, let \hat{S}_L denote the index set corresponding to the smallest L entries, and hence the declared set of non-defective items, in the primal solution \underline{z} . We first derive a sufficient condition for successful non-defective subset recovery with **RoLpAI**.

Proposition 2. *If $\underline{\lambda}_2(i) > 0 \forall i \in S_d$, then $\hat{S}_L \cap S_d = \{\emptyset\}$.*

Proof: See Appendix B.2

Define $\mathcal{E} \triangleq \{\hat{S}_L \cap S_d \neq \{\emptyset\}\}$. Also, let $P_e \triangleq \mathbb{P}(\mathcal{E})$. Let $S_z \subset [N] \setminus S_d$ be any set of non-defective items such that $|S_z| = (N - K) - (L - 1)$. Let \mathcal{S}_z denote all such possible sets. The above sufficiency condition for successful non-defective subset recovery, in turn, leads to the following:

Proposition 3. *The error event associated with **RoLpAI** satisfies:*

$$\mathcal{E} \subseteq \bigcup_{i \in S_d} \bigcup_{S_z \in \mathcal{S}_z} \{ \underline{1}_{M_z}^T \mathbf{X}_0(:, i) \geq \underline{1}_{M_z}^T \mathbf{X}_0(:, j), \forall j \in S_z \}. \quad (3.38)$$

Proof. Define $\mathcal{E}_0(i) \triangleq \{\underline{\lambda}_2(i) = 0\}$. We first note, from (3.35), that for any $i \in [N]$

$$\underline{\lambda}_2(i) = 0 \implies \mathbf{1}_{M_z}^T \mathbf{X}_0(:, i) = \underline{\lambda}_1(i) + \nu \geq \nu. \quad (3.39)$$

Define $\theta_0 \triangleq \max_{\{i: \underline{\lambda}_1(i)=0\}} \mathbf{1}_{M_z}^T \mathbf{X}_0(:, i)$ and $\theta_1 \triangleq \min_{\{i: \underline{\lambda}_1(i)>0\}} \mathbf{1}_{M_z}^T \mathbf{X}_0(:, i)$. We relate θ_0 , θ_1 and ν as follows:

Proposition 4. *The dual optimal variable ν satisfies $\theta_0 \leq \nu < \theta_1$.*

Proof: See Appendix B.3.

From the above proposition and (3.39) it follows that

$$\mathcal{E}_0(i) \subseteq \{\mathbf{1}_{M_z}^T \mathbf{X}_0(:, i) \geq \theta_0\}. \quad (3.40)$$

We note that there exists at most L items for which $\underline{\lambda}_1(i) > 0$; otherwise the solution would violate the primal feasibility constraint: $\mathbf{1}_N^T \underline{\lambda}(i) \geq (N - L)$. Thus, it is easy to see that there exist at least $(N - K) - (L - 1)$ non-defective items in the set $\{i : \underline{\lambda}_1(i) = 0\}$.

Thus, from (3.40) we get

$$\mathcal{E}_0(i) \subseteq \bigcup_{S_z \in \mathcal{S}_z} \{\mathbf{1}_{M_z}^T \mathbf{X}_0(:, i) \geq \mathbf{1}_{M_z}^T \mathbf{X}_0(:, j), \forall j \in S_z\}, \quad (3.41)$$

and (3.38) now follows since using Proposition 2 we have, $\mathcal{E} \subseteq \bigcup_{i \in S_d} \mathcal{E}_0(i)$. \square

Note that, for a given i , the quantity $\mathbf{1}_{M_z}^T \mathbf{X}_0(:, i)$ is the same as the quantity $T_n(i, M_n(M))$ defined in the proof of Theorem 4 and (3.38) is the same as (3.12) (see Section 3.3.1). Thus, following the same analysis as in Section 3.3.1, it follows that, if M satisfies (3.2), the LP relaxation based algorithm **RoLPAl** succeeds in recovering L non-defective items with probability exceeding $1 - \exp(-Mc_0) - \exp(-Mc_1)$.

3.3.4 Proof Sketch for Theorem 7

We use the same notation as in Theorem 6 and analyze an equivalent program that is obtained by substituting $(1 - \underline{z})$ by \underline{z} . We note that **LP2** differs from **LP0** only in terms of the objective function, and the constraint set remains the same. And thus, the complimentary slackness and the primal dual feasibility conditions are the same as given in (3.36) and (3.37), respectively. The zero gradient condition for **LP2** is given by:

$$\underline{1}_{M_z}^T \mathbf{X}_0 - \psi_{lp} \underline{1}_{M_p}^T \mathbf{X}_1 - \underline{\lambda}_1 + \underline{\lambda}_2 - \nu \underline{1}_N = \underline{0}_N. \quad (3.42)$$

Let the error event associated with **CoLpAI** be denoted by \mathcal{E} . Let $i \in S_d$, and define $\mathcal{E}_i \triangleq \{i \in \hat{S}_L\}$. Note that $\mathcal{E} \subseteq \cup_{i \in S_d} \mathcal{E}_i$. Further, it follows that $\mathcal{E}_i \subseteq \mathcal{A}_i \cup \mathcal{B}_i$, where $\mathcal{A}_i \triangleq \{\lambda_2(i) = 0\}$ and $\mathcal{B}_i \triangleq \{\mathcal{E}_i \cap \{\lambda_2(i) > 0\}\}$. Let us first analyze \mathcal{B}_i . Using similar arguments as in Proposition 2 and 4, it is easy to see that,

$$\mathcal{B}_i \subseteq \{\nu = 0\} \subseteq \cup_{S_z \in \mathcal{S}_z} \left\{ \underline{1}_{M_z}^T \mathbf{X}_0(:, j) - \psi_{lp} \underline{1}_{M_p}^T \mathbf{X}_1(:, j) \leq 0, \forall j \in S_z \right\}, \quad (3.43)$$

where $S_z \subset [N] \setminus S_d$ is any set of non-defective items such that $|S_z| = (N - K) - (L - 1)$ and \mathcal{S}_z denotes all such possible sets. Further, using similar arguments as in the proof of Theorem 6, it can be shown that

$$\mathcal{A}_i \subseteq \cup_{S_z \in \mathcal{S}_z} \left\{ \underline{1}_{M_z}^T \mathbf{X}_0(:, i) - \psi_{lp} \underline{1}_{M_p}^T \mathbf{X}_1(:, i) \geq \underline{1}_{M_z}^T \mathbf{X}_0(:, j) - \psi_{lp} \underline{1}_{M_p}^T \mathbf{X}_1(:, j), \forall j \in S_z \right\}, \quad (3.44)$$

where S_z and \mathcal{S}_z are as defined above.

The subsequent analysis follows by using the Bernstein inequality to upper bound the probability of events \mathcal{A}_i and \mathcal{B}_i in a manner similar to previous proofs; we omit

the details for the sake of brevity. As before, let $N_0 \triangleq (N - K) - (L - 1)$. Define $\psi'_0 \triangleq \min\left(\frac{\Gamma(1+\gamma_0)}{2}, \frac{\Gamma}{2(1-\Gamma)}\right)$. Note that, with $\psi_{lp} = \psi'_0$, $\mathbb{E}([\mathbf{1}_{M_z}^T \mathbf{X}_0(:, j) - \psi_{lp} \mathbf{1}_{M_p}^T \mathbf{X}_1(:, j)]) \geq Mp\Gamma/2 > 0$ for any $j \in S_z$. Further, it can be shown that there exists absolute constant C_{4b} such that

$$\mathbb{P}\left(\bigcup_{i \in S_d} \mathcal{B}_i\right) \leq \exp\left(-M \left[\frac{p\Gamma N_0}{C_{4b}} - \frac{\log [K \binom{N-K}{L-1}]}{M}\right]\right). \quad (3.45)$$

Similarly, following the same steps as in the proof of Theorem 5, it can be shown that, for the chosen value of ψ_{lp} , there exists absolute constant C_{4a} such that

$$\mathbb{P}\left(\bigcup_{i \in S_d} \mathcal{A}_i\right) \leq \exp\left(-M \left[\frac{p\Gamma(1-\gamma_0)^2 N_0}{C_{4a}} - \frac{\log [K \binom{N-K}{L-1}]}{M}\right]\right). \quad (3.46)$$

The final result now follows easily by substituting $p = \frac{1}{3(1-u)K}$, since, by choosing M as in (3.11), the error terms in (3.45), (3.46) can be written as $\exp(-M\epsilon)$ for some $\epsilon > 0$.

3.4 Simulations

In this section, we investigate the empirical performance of the algorithms proposed in this chapter for non-defective subset recovery. In contrast to the previous section, where theoretical guarantees on the number of tests were derived based on the analysis of the upper bounds on probability of error of these algorithms, here we find the exact number of tests required to achieve a given performance level, thus highlighting the practical ability of the proposed algorithms to recover non-defective subset.

This, apart from validating the general theoretical trends, also facilitates a direct comparison of the presented algorithms. Our setup is as follows. For a given set of operating parameters, i.e., N , K , u , q and M , we choose a defective set $S_d \subset [N]$ randomly such that $|S_d| = K$ and generate the test output vector \underline{y} according to (3.1). We then recover a subset of L non-defective items using different recovery algorithms, i.e., **RoAl**, **CoAl**, **RoLpAl**, **RoLpAl++** and **CoLpAl**, and compare it with the defective set. The empirical probability of error is set equal to the fraction of the trials for which the recovery was not successful, i.e., the output non-defective subset contained at least one defective item. This experiment is repeated for different values of M and L . For each trial, the test matrix \mathbf{X} is generated with random Bernoulli i.i.d. entries, i.e., $\mathbf{X}_{ij} \sim \mathcal{B}(p)$, where p is a design parameter. As suggested by the theoretical analysis presented in the previous section we choose $p = \frac{1}{K}$. Also, for **CoAl** and **CoLpAl**, we set $\psi_{cb} = \frac{\Gamma(1+\gamma_0)}{2(1-p)}$ and $\psi_{lp} = \min\left(\frac{\Gamma(1+\gamma_0)}{2(1-p)}, \frac{\Gamma}{2(1-\Gamma)}\right)$ respectively, where $\Gamma \triangleq [1 - (1 - u)p]^K$ and $\gamma_0 = u/[1 - (1 - u)p]$. Unless otherwise stated, we set $N = 256$, $K = 16$, $u = 0.05$, $q = 0.1$ and we vary L and M .

Figure 3.2 shows the variation of the empirical probability of error with the number of tests, for $L = 64$ and $L = 128$. These curves demonstrate the theoretically expected exponential behavior of the average error rates, the similarity of the error rate performance of algorithms **RoAl** and **RoLpAl**, and the performance improvement offered by **RoLpAl++** at higher values of L . We also note that, as expected, the algorithms that use tests with both positive and negative outcomes perform better than the algorithms that use only tests with negative outcomes.

Figure 3.3 presents the number of tests M required to achieve a target error rate of

10% as a function of the *size* of non-defective subset, L . We note that for small values of L almost all algorithms perform similarly, but, in general, **CoAl** and **CoLpAl** are the best performing algorithms across all values of L . We also note that, as argued in Section 3.2.3, **RoLpAl++** performs similar to **RoLpAl** for small values of L and for large values of L the performance of the former is significantly better, and is the same as that of **CoLpAl**. Also, as mentioned in Section 3.2.4, we note the linear increase in M with L , especially for small values of L . We also compare the algorithms proposed in this work with an algorithm that identifies the non-defective items by first identifying the defective items, i.e., we compare the “direct” and “indirect” approach (see Chapter 2) of identifying a non-defective subset. We first employ a defective set recovery algorithm for identifying a defective set and then choose L items uniformly at random from the complement set. This algorithm is referred to as “InDirAl” algorithm in Figure 3.3. In particular, we have used “No-LiPo-” algorithm [20] for defective set identification. It can be easily seen that the “direct” approach significantly outperforms the “indirect” approach. We also compare against a non-adaptive scheme that tests items one-by-one. The item to be tested in each test is chosen uniformly at random from the population. We choose the top L items tested in all the tests with negative outcomes as the non-defective subset. This algorithm is referred to as “NA1by1” (Non-Adaptive 1-by-1) in Figure 3.3. Again, it is easy to see that the group testing based algorithms significantly outperform the NA1by1 strategy.

Figure 3.4 compares the number of tests required to achieve a target error rate of 10% for **CoLpAl** with the information theoretic lower bound for two different values of K .⁸

⁸We refer the reader to Theorem 3 in Chapter 2 for a detailed discussion on the information theoretic lower bound. Also, see equations (A.6) and (A.7) in Appendix A for the exact derivation of the mutual

It is easy to see that, as discussed in Section 3.2.4, the empirical performance of **CoLpAI** (and other algorithms) is within $O(\log K)$ of the lower bound.

As discussed in Section 3.2.4, the parameter settings require the knowledge of K . Here, we investigate the sensitivity of the algorithms on the test matrix designed assuming a nominal value of K to mismatches in its value. Let the true number of defective items be K_t . Let $M(\hat{K}, K_t)$ denote the number of tests required to achieve a given error rate when the test is designed with $K = \hat{K}$. Let $\Delta_M(\hat{K}, K_t) \triangleq \frac{M(\hat{K}, K_t)}{M(K_t, K_t)}$. Thus, $\Delta_M(\hat{K}, K_t)$ represents the penalty paid compared to the case when the test is designed knowing the number of defective items. Table 3.2 shows the empirically computed Δ_M for different values of uncertainty factor $\Delta_K \triangleq \frac{\hat{K}}{K_t}$ for different algorithms. We see that the algorithms exhibit robustness to the uncertainty in the knowledge of K . For example, even when $\hat{K} = 2K_t$, i.e., $\Delta_k = 200\%$, we only pay a penalty of approximately 17% for most of the algorithms. Also, as suggested by the analysis of upper bounds in Section 3.2.4, the algorithms exhibit asymmetric behavior in terms of robustness and are more robust for $\Delta_k > 1$ compared to when $\Delta_k < 1$.

Figure 3.5 shows the performance of different algorithms with the variations in the system noise parameters. Again, in agreement with the analysis of the probability of error, the algorithms perform similarly with respect to variations in both the additive and dilution noise.

information term that is required for computing the lower bound for the group testing signal model.

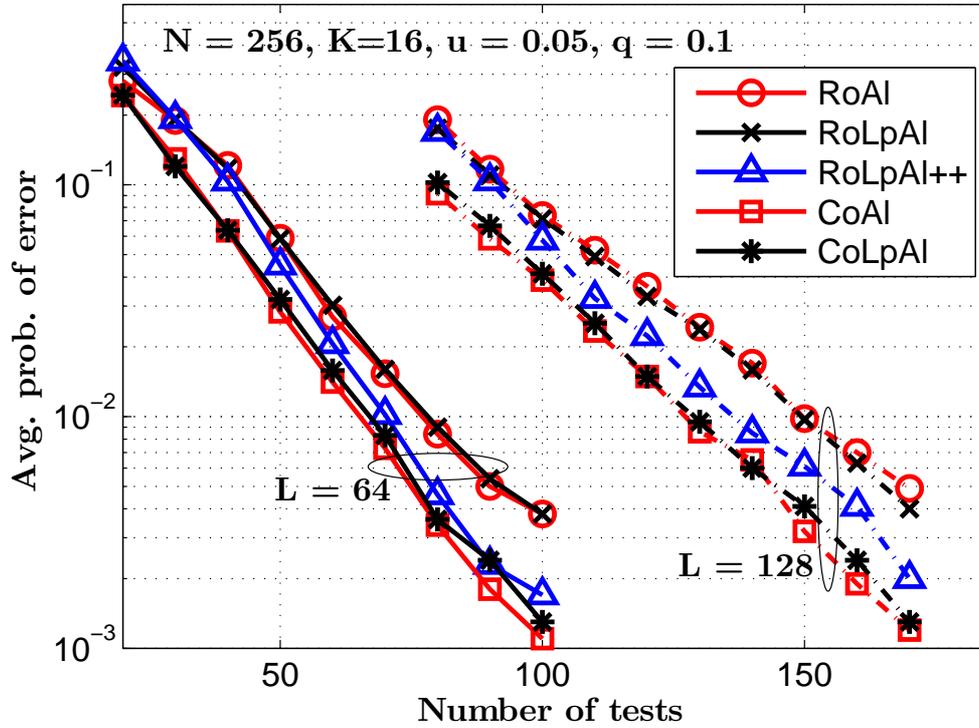


Figure 3.2: Average probability of error (APER) *vs.* number of tests M . The APER decays exponentially with M .

Table 3.2: Robustness of the non-defective subset identification algorithms in the presence of uncertainty in the knowledge of K . The numbers in the table are $M(\hat{K}, K_t)$, the relative increase in the number of tests due to the uncertainty in the knowledge of K .

$K_t = 16, N = 256, L = 128, q = 0.1, u = 0.05$			
	$\Delta_K = 75\%$	$\Delta_K = 150\%$	$\Delta_K = 200\%$
RoAl	1.13	1.06	1.20
CoAl	1.13	1.02	1.13
RoLpAl	1.09	1.04	1.17
RoLpAl++	1.04	1.00	1.17
CoLpAl	1.13	1.00	1.18

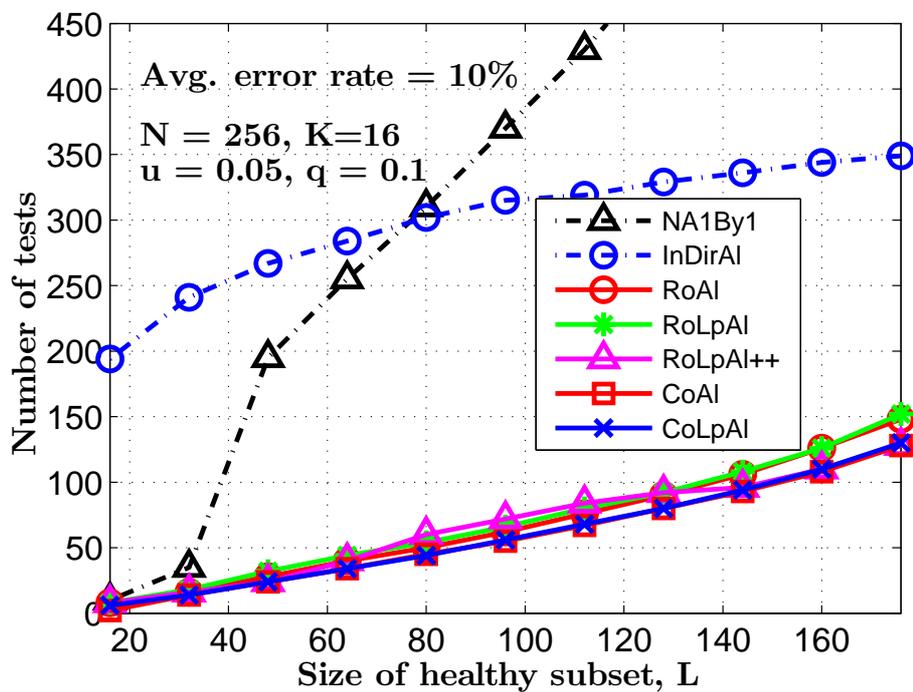


Figure 3.3: Number of tests *vs.* size of non-defective subset. Algorithm **CoLpAI** performs the best among the ones considered. The direct approach for finding non-defective items significantly outperforms both the indirect approach (“InDirAI”), where defective items are identified first and the non-defective items are subsequently chosen from the complement set (see Chapter 2), as well as the item-by-item testing approach (“NA1By1”).

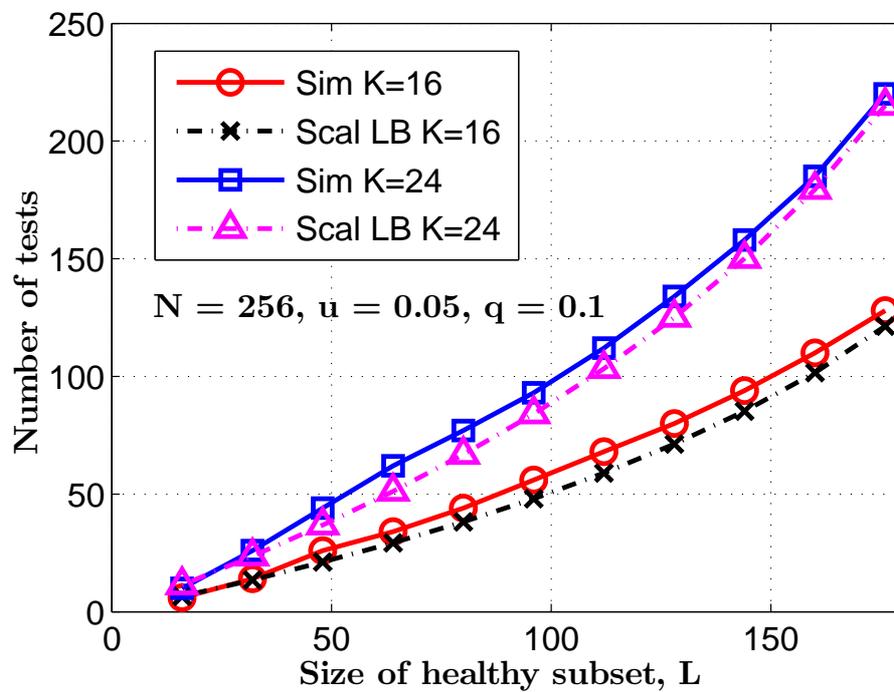
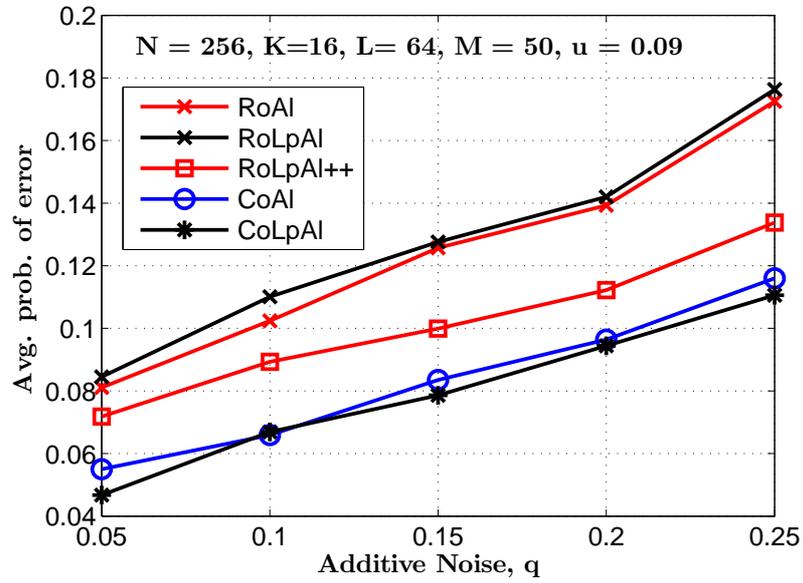
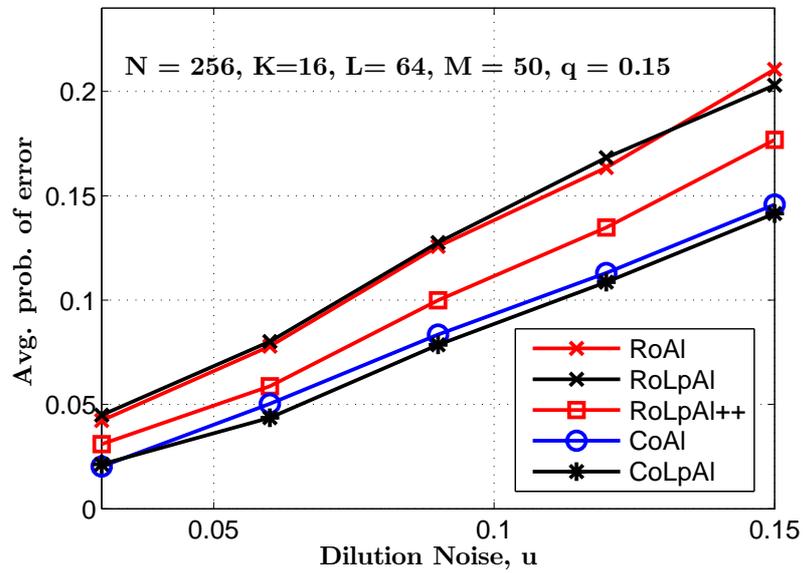


Figure 3.4: Comparison of CoLpAI with the *scaled* information theoretic lower bounds. A factor of $\log(K)$ is used for scaling the lower bounds. The algorithm performance is within $\log(K)$ factor of the lower bounds.



(a)



(b)

Figure 3.5: Average probability of error variation with noise. Panel (a) Variation with the additive noise (q); Panel (b) Variation with dilution noise (u). The algorithms exhibit similar performance with variations in additive and dilution noise.

3.5 Conclusions

In this chapter, we proposed analytically tractable and computationally efficient algorithms for identifying a non-defective subset of a given size in a noisy non-adaptive group testing setup. We derived upper bounds on the number of tests for guaranteed correct identification and showed that the algorithms are nearly optimal, as the upper bounds and information theoretic lower bounds are order-wise tight up to a log factor. We showed that the algorithms are robust to the uncertainty in the knowledge of system parameters. Also, it was found that the algorithms that use both positive and negative outcomes, namely **CoAI** and the LP relaxation based **CoLPAl**, gave the best performance for a wide range of values of L , the size of non-defective subset to be identified. In this work, we considered the randomized pooling strategy. It will be interesting to study deterministic constructions for the purpose of non-defective subset identification; this could be considered in a future extension of this work.

Chapter 4

Group Testing Based Spectrum Hole Search for Cognitive Radios

Group testing is a natural framework for efficiently identifying the defective items in a large population containing a small fraction of defective items [6]. It is applicable in scenarios where multiple items can be tested together in a single test; the group test returns positive if at least one item in the group is defective, and returns negative otherwise. Group tests are particularly useful when individually testing each item is prohibitively time-consuming, since testing multiple items in a single test leads to time savings when test outcomes are negative. One area where group tests could potentially offer significant benefits is that of spectrum hole search for Cognitive Radio (CR) [56–58]. The CR paradigm is based on the fact that, at any given time, the spectral occupancy by the primary users is sparse over a wideband of interest [21, 22]. For efficient functioning, CR networks need accurate and up-to-date information about the availability of spectrum holes, i.e., frequency bands where the primary users are inactive.

In this chapter, we focus on the application of *adaptive* group testing to the task of

finding a spectrum hole of a pre-specified bandwidth within a given wideband of interest for use by the CR network. This problem is relevant in many CR scenarios. For example, in the IEEE 802.22 standard for cognitive radio, the primary users occupy a bandwidth of 6 MHz each. A secondary network that requires 40 MHz of bandwidth for its operation will need to find a spectrum hole consisting of 7 contiguous unoccupied bands. In practice, it is desirable to have the CR network operate in a contiguous frequency band, as this simplifies transceiver hardware design and helps improve the energy efficiency of the CR network compared to using non-contiguous frequency bins. Other considerations for preferring contiguous frequency bin allocation are the physical layer access mechanism (e.g., code division multiple access), network quality of service requirements, spectral mask constraints, etc.

We consider a setup where a CR wishes to identify a given number, say N_e , contiguous unoccupied sub-bands over a given wide bandwidth. A straightforward approach to this problem would be to test each sub-band sequentially, one at a time, till the required N_e contiguous bins are found. On the other hand, group testing can be used to reduce the search time in such a problem, if a set of adjacent sub-bands can be tested at one shot. One way to accomplish this without increasing the sampling rate and processing requirements at the CR node is to acquire the analog signal corresponding to $M(\geq 1)$ sub-bands using a wide front-end anti-aliasing filter, followed by sampling at a rate corresponding to the Nyquist rate for a *single* sub-band. Although sampling at the Nyquist rate of a single sub-band results in aliasing, it provides the receiver with a signal that is the *sum of the signals in all the acquired sub-bands*. Based on the energy of the aliased signal, in this work, we develop an energy-based detector, referred to as *an*

M-bin group test, to provide a joint occupancy decision on the group of *M* adjacent sub-bands over which the signal is acquired. We consider the popular energy-based detection (see [59–61] for an excellent survey of spectrum sensing), as it is easy to implement and is optimal when the CR has no prior information about the primary signal [62].

In the literature, the idea of sampling the signal over multiple sub-bands and make joint occupancy decisions has been explored, but with sampling at the higher Nyquist rate corresponding to the multiple sub-bands. For example, FFT-based architectures that collect samples at a Nyquist rate corresponding to C_N narrowbands and can provide simultaneous decisions for all the C_N bins have been considered [63–65]. A two-stage sensing architecture that reduces search times by extending the narrowband energy detector to wider bands has been proposed in [66]. Another way to reduce the search time is by employing parallel data chains, e.g., using multiple antenna receivers [67, 68]. However, the effective sampling rate requirements of these architectures are higher than the narrowband detector, since data is acquired over multiple narrowbands at Nyquist rates. Increasing the sampling rate entails higher power consumption and processing requirements, which is undesirable in tasks such as spectrum sensing, which are frequently performed at the CR nodes. A wideband sensing framework is proposed in [69,70], where a bank of multiple narrowband energy detectors operating at Nyquist sampling rate are jointly optimized by choosing different thresholds, to maximize the total opportunistic CR throughput, while constraining the interference to the primary users. The above framework requires the knowledge of primary-to-secondary channel coefficients and secondary throughput values for each narrowband, and is therefore limited to scenarios where such information is available. In practice, it may be hard for

the CR to obtain or estimate these parameters.

Group testing based signal acquisition leads to noise enhancement (folding), due to the aliasing introduced by the sub-Nyquist sampling. This phenomenon is also seen in other wideband acquisition systems operating at sub-Nyquist rates, such as compressive sensing based methods [71]. This raises important questions about the efficacy of group testing in the face of noise folding, and the dependence of the optimal test parameters on the operating SNR and the sparsity in the frequency occupancy of primary users. In this context, our contributions in this chapter are as follows:

1. We introduce a signal acquisition scheme that enables the use of group testing based spectrum hole search by acquiring wideband signals at a fixed sampling rate. The acquisition scheme entails only a minimal hardware change, compared to the narrowband energy detector, in the form of a programmable anti-aliasing filter (See Sec. 4.1).
2. We present a search algorithm that minimizes the time to detect a spectrum hole of a specified bandwidth while satisfying an upper bound on the probability of incorrectly identifying the hole (See Sec. 4.2).
3. We theoretically analyze the detection delay behavior of the algorithm, and use it to optimize the parameters (group size, samples per test, and detection thresholds) of the search algorithm. We also identify the regimes of the sparsity and detection SNR where group tests offer performance benefits over the conventional bin-by-bin search scheme. In particular, our analytical characterization of the detection delay of the bin-by-bin detector is also new (See Sec. 4.3).

4. We present a multi-stage detection algorithm that combines multiple group sizes to identify a large fraction of the available spectrum holes as fast as possible (See Sec. 4.4).

Extensive simulation results corroborate our theoretical analysis and illustrate the performance benefits obtainable from the group testing approach under favorable conditions on sparsity and SNR (See Sec. 4.5). The use of group tests with the optimal group size leads to a faster acquisition of the desired spectrum hole. This, in turn, leads to a better utilization of the available spectrum, since a shorter sensing duration leaves more time for data transmission. Reducing the sensing duration is also power efficient, since spectrum sensing is a frequently-running task on CR devices. Moreover, group test based schemes significantly reduce the total number of tests that need to be set up while searching over a given wideband, thereby reducing the test setup overheads [66]. Finally, we note that the spectrum hole search algorithm always selects the optimal group size M . In particular, in scenarios where $M = 1$ is optimal, the group testing scheme defaults to, and hence performs at least as well as, the conventional bin-by-bin search scheme.

4.1 Signal Acquisition Scheme

We model the wideband as a set of consecutive non-overlapping frequency narrowbands. Let f_b denote the bandwidth of a narrowband channel, also referred to as a bin or a sub-band. Let the wideband being searched consist of N contiguous subbands. Let $S_l(j\Omega)$ and $s_l(t)$ represent the frequency domain and time domain signal components in the l^{th} bin, respectively, down converted to the baseband (denoted

$s_l(t) \xleftrightarrow{\mathcal{F}} S_l(j\Omega)$). Let $\Omega \triangleq 2\pi f$. By our assumption on the bandwidth of each bin, we have $S_l(j\Omega) = 0, |\Omega| > 2\pi f_b/2$. The frequency domain signal for the wideband channel, down converted to the baseband, can be represented as

$$X(j\Omega) = \sum_{l=-(N-1)/2}^{(N-1)/2} S_l(j\Omega - j\Omega_l) \quad (4.1)$$

where $\Omega_l \triangleq 2\pi l f_b$ represents the center frequency of the l^{th} bin. In the above, N is assumed to be odd, but the extension to even N is straightforward.

We now describe a signal acquisition scheme that enables group testing of multiple adjacent bins without increasing the sampling rate requirements at the CR node. Let M denote the number of adjacent bins over which the signal is acquired. The incoming down-converted signal is first passed through an anti-aliasing filter of bandwidth $M f_b$ to eliminate the out-of-band signals and noise. The frequency domain signal at the output of the anti-aliasing filter can be represented as

$$X_a(j\Omega) = \sum_{l=-(M-1)/2}^{(M-1)/2} S_l(j\Omega - j\Omega_l). \quad (4.2)$$

The signal $x_a(t) \xleftrightarrow{\mathcal{F}} X_a(j\Omega)$ is sampled at a rate $f_s = f_b$. Since the anti-aliasing filter band-limits the signal to $[-M f_b/2, M f_b/2]$, sampling at a rate f_b introduces *aliasing*. Let $T_b = \frac{1}{f_b}$. Let $x_d[n] = x_a(nT_b)$ represent the sampled signal and $x_d[n] \xleftrightarrow{\mathcal{F}} X_d(e^{j\omega})$, where $\omega = 2\pi \frac{f}{f_b}$ [72]. With the above notation, for odd M , $X_d(e^{j\omega}) = \frac{1}{T_b} \sum_{l=-(M-1)/2}^{(M-1)/2} S_l\left(\frac{j\omega}{T_b}\right)$. Again, the extension to even M is straightforward. The received signal, $x_d[n]$, is thus the sum of the signal components in the individual bins. Now, the received signal $y_g[n]$

can be described by

$$y_g[n] = x_d[n] + v_g[n] \quad (4.3)$$

where $v_g[n]$ is the white noise component after aliasing.

Now, due to possibly independent fading across the bins and mismatches between transmit and receive pulse shaping filters, timing and frequency offsets, etc between the primary transmitter and the CR receivers, the signals from different bins are effectively the result of passing a random signal through orthogonal filters, since the different frequency bins are non-overlapping. Hence, it is reasonable to model the signal contributions from different bins as being mutually independent. Since we assume no knowledge about the primary signal characteristics at the secondary node, we model the signal contributions from l^{th} bin as Gaussian distributed with zero mean and variance P_l , as in [65, 73, 74]. We assume that the baseband signal is real-valued for simplicity of exposition; the extension to complex signals is immediate. We note that, $v_g[n] \sim \mathcal{N}(0, M\sigma^2)$, where $\sigma^2 = N_o f_b$, and N_o is the white noise power spectral density. The factor M in the noise variance is due to the aliasing introduced by sampling at rate f_b . We accumulate the energy from K samples at the output of the filter, and compute the following test statistic:

$$T(y) = \sqrt{\sum_{n=1}^K |y_g[n]|^2}. \quad (4.4)$$

Note that, the conventional narrowband signal acquisition is a special case of the above signal model with $M = 1$. Figure 4.1 shows the block diagram for the proposed wide-band signal acquisition scheme. Let the bandwidth of the spectrum holes that need to be found be denoted by W_h , such that $W_h = N_e f_b$. With this setting, the task of the

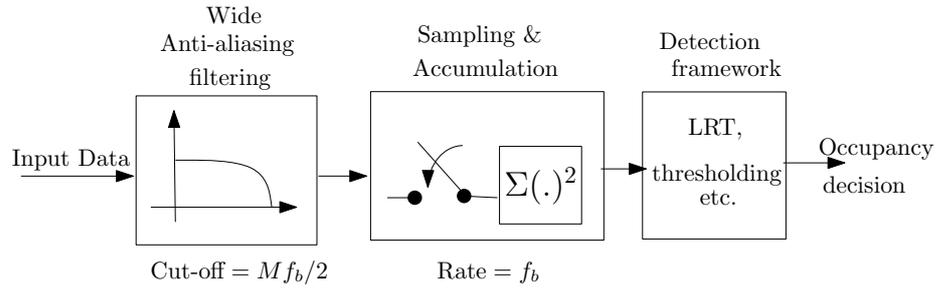


Figure 4.1: Block diagram of the wideband signal acquisition system.

sensing algorithm is to find a set of $N_e (\ll N)$ *consecutive* unoccupied bins in the given wideband. We discuss this in the next section.

4.2 The M -Bin Group Test Design

Using the above data acquisition scheme, we first describe an algorithm to find a contiguous spectrum hole of the specified bandwidth of N_e bins. Let the parameters N and M be as defined in the previous section. Let N_e be an integer multiple of M and define¹ $b = \frac{N_e}{M}$. To find N_e contiguous unoccupied bins, the proposed algorithm makes an occupancy decision on a group of $M \leq N_e$ contiguous bins in a single test. Such group tests, referred to as M -bin tests, are conducted sequentially on multiple adjacent groups until b consecutive M -bin tests declare the set of bins being tested as unoccupied. Also, for simplicity, we assume that the required N_e empty bins can be found using one pass of the algorithm over the given wideband consisting of N bins. This holds true when the occupancy of the primary is sparse in the frequency domain and $N_e \ll N$, which is typically the case in scenarios relevant for CR deployment.

An M -bin group test forms the basic building block of the above algorithm to find N_e

¹We consider such a combination of N_e and M for the simplicity of exposition. The algorithm can be easily adapted to non-integer multiples also.

unoccupied bins. Let \mathcal{H}_l denote the hypothesis that l out of M bins are occupied in the group under test and let $\{\mathcal{H}_l\}_{l=1}^M$ denote the composite alternate hypothesis, i.e., \mathcal{H}_l is true for some $l = 1, 2, \dots, M$. An M -bin group test distinguishes between the following hypotheses:

$$\begin{aligned} \mathcal{H}_0 &: \text{No primary signal on any of the } M\text{-bins} \\ \{\mathcal{H}_l\}_{l=1}^M &: \text{Primary signal present on at least one bin} \end{aligned} \quad (4.5)$$

To find N_e empty bins, the algorithm can use different values of M (and hence b) and K . We first describe the algorithm with a fixed value of M and K , and later present a way of choosing the best M and K . The following optimization problem arises naturally in the context of the M -bin test:

$$\text{minimize } \bar{N}_t \quad \text{subject to } P_e \leq P_0, \quad (4.6)$$

where P_e denotes the probability that the overall search algorithm makes an error, and \bar{N}_t denotes the average number of tests required to find N_e consecutive bins. The minimization in (4.6) is over the parameters M , b , K and the detection thresholds used in the M -bin tests. Since the algorithm terminates once it has declared a set, say \mathcal{A} , of N_e consecutive bins as unoccupied, we say that an error has occurred if the primary signal is present in one or more of the bins in \mathcal{A} . Mathematically,

$$P_e \triangleq \Pr(\text{Primary present in a set } \mathcal{A} \text{ of } N_e \text{ bins} \mid b \text{ consecutive } M\text{-bin tests succeed for the first time}). \quad (4.7)$$

Note that, P_e is related to the miss detection probability, i.e., the probability that the group \mathcal{A} is declared as empty given that at least one bin in \mathcal{A} is actually occupied, through Bayes' rule. Also, a false alarm event, i.e., the event that \mathcal{A} is declared occupied given that it is actually empty, leads to an increased detection delay; its effect is captured in \bar{N}_t . See Propositions 5 and 5 and Appendix C.2.

Let occupancy across the bins be i.i.d. and distributed as binary Bernoulli random variables (denoted $\sim \mathcal{B}(\rho)$), where ρ is the fraction of bins occupied on average, over the long term. In the current work, we focus on the i.i.d. occupancy model [69], that, apart from being analytically tractable, might also be of independent interest in the area of adaptive group testing where the items being defective is independent of each other. We further assume that the occupancy pattern stays fixed over the search duration. Let H_{0d} denote the event that \mathcal{H}_0 was declared by a single group test and define $p \triangleq \Pr\{H_{0d}\}$. Let Π_0 and Π_1 represent the prior probabilities for the null and alternate hypotheses for the M -bin group test. With our assumptions, $\Pi_0 = (1 - \rho)^M$ and $\Pi_1 = 1 - \Pi_0$. Define $P_{me} \triangleq \Pr\{\{\mathcal{H}_l\}_{l=1}^M | H_{0d}\}$ as the probability that the single group test makes an incorrect decision, given that the group test has declared \mathcal{H}_0 . The following proposition connects P_e and \bar{N}_t to P_{me} and p , the parameters of the M -bin group test.

Proposition 5. *Let \bar{N}_t , M , b , P_e , P_{me} , and p be as defined above. Then the following hold:*

$$P_e = 1 - (1 - P_{me})^b \quad \text{and} \quad \bar{N}_t = \sum_{i=1}^b (p)^{-i}. \quad (4.8)$$

Proof: See Appendix C.1.

From Proposition 5, \bar{N}_t depends solely on p and b , and is monotonically decreasing

with p . Thus, the design goal for the single M -bin test can be stated as

$$\text{maximize } p \quad \text{subject to } P_{me} \leq P'_0, \quad (4.9)$$

where $P'_0 = 1 - (1 - P_0)^{\frac{1}{b}}$. The maximization above is over all possible decision rules, denoted by $\delta(T)$. Our next proposition establishes that a likelihood ratio test (LRT) [75] is optimal for the design criterion specified in (4.9). Let $\delta_L(T)$ be a decision rule based on the LRT with threshold $\eta_{gt} > 0$, defined as $\delta_L(T) = 1, Pr\{T|\{\mathcal{H}_l\}_{l=1}^M\} \geq \eta_{gt} Pr\{T|\mathcal{H}_0\}$, and $= 0$ otherwise, where T is given by (4.4). We state:

Proposition 6. *Let $\delta_L(T)$ be the LRT decision rule defined above, with η_{gt} chosen such that $P_{me}(\delta_L) = P'_0$. Let $\delta'(T)$ be any other decision rule such that $P_{me}(\delta') \leq P'_0$. If $\eta_{gt} > \frac{\Pi_0}{\Pi_1} \frac{P'_0}{1-P'_0}$, then $\bar{N}_t(\delta_L) \leq \bar{N}_t(\delta')$, where $\bar{N}_t(\delta_L)$ (or $\bar{N}_t(\delta')$) represents the average number of tests required to find a consecutive set of N_e vacant bins using the test δ_L (or δ').*

Proof: See Appendix C.2.

To compute likelihood ratios, we need the probability distributions of the test statistic defined in (4.4) under the two hypotheses. To this end, we need to know the variances under the primary signal present hypothesis $\{P_l\}_{l=1}^M$, but these are unknown and in general hard to estimate. To get around the problem of unknown $\{P_l\}$, we define a bin as being occupied if the received primary signal power in the bin is at least P_s . Further, we *design* the test conservatively by assuming that the received primary signal power in any occupied bin *equals* P_s . This is in line with the approach recommended in emerging CR standards such as the IEEE 802.22, where the CR is required to reliably sense the primary signal whenever the received signal power exceeds -116dBm [76]. With these

assumptions, it can be shown that

$$\begin{aligned} \mathcal{H}_0 : T &\sim \mathcal{N}(m_0, \sigma_0^2) \\ \{\mathcal{H}_l\}_{l=1}^M : T &\sim \sum_{l=1}^M \theta_l \mathcal{N}(m_l, \sigma_l^2), \end{aligned} \quad (4.10)$$

where $\sigma_l^2 \triangleq (M\sigma^2 + lP_s)/2$, $m_l \triangleq \sqrt{(2K-1)\sigma_l^2}$ and $\theta_l \triangleq \binom{M}{l} \rho^l (1-\rho)^{M-l} / \Pi_1$, such that $\sum_{l=1}^M \theta_l = 1$. In deriving the above distributions, we have used the approximation that if $X \sim \chi^2(K)$, then $\sqrt{2X} \sim \mathcal{N}(\sqrt{2K-1}, 1)$ ([77], Ch. 26).

The log-likelihood function corresponding to the test in (4.10) is analytically intractable, making it hard to obtain the detection threshold in closed form. However, it can be easily shown that it is approximately quadratic in T (when $M = 1$, it is exactly quadratic in T . For $M > 1$, when one of the terms in the mixture density dominates the other terms for T close to the threshold, the error in the approximation is small.) Due to this, the critical region is of the form $\{T \leq \eta'_{gt0}\} \cup \{T \geq \eta'_{gt1}\}$, where η'_{gt0} and η'_{gt1} are lower and upper thresholds. For most scenarios of interest, the contribution to P_{me} from $\{T \leq \eta'_{gt0}\}$ is small, since it represents the unlikely event that, due to its larger variance, the instantiation of the received signal power estimate under $\{\mathcal{H}_l\}_{l=1}^M$ is unusually small. This allows us to replace the LRT test by a simple, albeit sub-optimal, one-sided threshold test on T :

$$T \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \eta'_{gt1}. \quad (4.11)$$

The threshold η'_{gt1} is chosen to satisfy the constraint on P_{me} in (4.9). For notational simplicity, let η be the threshold used in the test. Define the false alarm and miss detection rate of a single M -bin test as $\alpha(\eta) \triangleq \Pr\{\{\mathcal{H}_l\}_{l=1}^M \text{ declared} | \mathcal{H}_0\} = \Pr\{T \geq \eta | \mathcal{H}_0\}$ and $\beta(\eta) \triangleq \Pr\{\mathcal{H}_0 \text{ declared} | \{\mathcal{H}_l\}_{l=1}^M\} = \Pr\{T < \eta | \{\mathcal{H}_l\}_{l=1}^M\}$, respectively. Also, define

$\beta_l \triangleq \Pr\{\mathcal{H}_0 \text{ declared} | \mathcal{H}_l\}$, i.e., the miss detection rate when exactly l bins are occupied.

These can be computed as

$$\begin{aligned}\alpha(\eta) &= 1 - Q\left(\frac{m_0 - \eta}{\sigma_0}\right), \\ \beta(\eta) &= \sum_{l=1}^M \theta_l \beta_l \text{ where } \beta_l = Q\left(\frac{m_l - \eta}{\sigma_l}\right),\end{aligned}\quad (4.12)$$

where m_l and θ_l are as defined in (4.10). The constraint on P_{me} in (4.9) leads to the following nonlinear equation, the solution to which yields η'_{gt1} :

$$\frac{\beta(\eta) \Pi_1}{p(\eta)} = P'_0, \quad (4.13)$$

where $p(\eta)$ can be computed as

$$p(\eta) = \Pi_0 (1 - \alpha(\eta)) + \Pi_1 \beta(\eta). \quad (4.14)$$

Numerical techniques such as the bisection method have to be used to solve (4.13) to obtain η'_{gt1} .

This completes the design for an M -bin group test using given values of the number of samples, K , and the group size, M . In the next section, we show how to find an optimal value for K and M .

4.3 Optimal Parameters for the M -Bin Test

We start by describing a procedure for finding the optimal K . Note that, (4.13) can be written as

$$\Gamma(\eta, K) = C_1, \quad (4.15)$$

where $C_1 = \frac{\Pi_0 P'_0}{\Pi_1 1-P'_0} > 0$, and $\Gamma(\eta, K) \triangleq \frac{\sum_{l=1}^M \theta_l Q(z_l)}{Q(z_0)}$, with z_l , for $l = 0, 1, \dots, M$, defined as

$$z_l \triangleq \frac{(m_l - \eta)}{\sigma_l} = \sqrt{2K - 1} - \frac{\eta}{\sigma_l}. \quad (4.16)$$

Since σ_l increases with l , as seen from (4.10), z_l also increases with l for a fixed K , i.e., $z_0 < z_1 < \dots < z_M$. We first study the variation of Γ with η and K . Although K is an integer valued variable, for the purpose of analysis, we treat K as a positive real number. Due to the continuity of the Q -function, $\Gamma(\eta, K)$ is continuous with respect to η and K , and since $z_l > z_0, 0 < \Gamma(\eta, K) < 1$. Further, we have:

Proposition 7. (a) For fixed K , $\Gamma(\eta, K)$ is a quasi-convex function of η and attains a minimum value, $\Gamma_{min}(K) = \Gamma(\eta_0, K)$, where η_0 satisfies $\left. \frac{\partial \Gamma(\eta, K)}{\partial \eta} \right|_{\eta=\eta_0} = 0$, i.e.,

$$\sum_{l=1}^M \theta_l \left[\frac{Q(z_0) e^{-z_l^2/2}}{\sigma_l} - \frac{Q(z_l) e^{-z_0^2/2}}{\sigma_0} \right] = 0, \quad (4.17)$$

where $z_l, l = 0, 1, \dots, M$ are evaluated at η_0 .

(b) For fixed η , $\Gamma(\eta, K)$ is a decreasing function of K .

Proof: See Appendix C.3.

Figure 4.2 depicts the behavior of $\Gamma(\eta, K)$ as implied by the above proposition. We make the following remarks: (a) For a given probability of incorrect detection, P'_0 , there is a

certain minimum number of samples, $K_{min}(\geq 1)$, that are required to set up the M -bin group test to satisfy the performance requirement P'_0 . In fact, if $K_0(\geq 1)$ is such that $\Gamma_{min}(K_0) > C_1$ (e.g., with $K = 20$ in Figure 4.2), a test cannot be designed with K_0 samples. However, since $\Gamma_{min}(K)$ is a decreasing function of K , we can find the smallest integer, denoted K_{min} , such that $\Gamma_{min}(K_{min}) \leq C_1$. For any $K \geq K_{min}$, an M -bin test can be designed and each will result in a different average number of tests required to find N_e unoccupied bins. (b) For each $K \geq K_{min}$, due to the quasi-convex nature of $\Gamma(\eta, K)$, there exist exactly two solutions of (4.15), and since the M -bin test is a threshold test, we pick the larger of the two as the threshold to be used. Define $\eta_s(K)$, the computed threshold for a given value of K , as

$$\eta_s(K) = \max \{ \eta : \Gamma(\eta, K) = C_1 \}. \quad (4.18)$$

Note that, due to the way η_s is chosen, $\left. \frac{\partial \Gamma(\eta, K)}{\partial \eta} \right|_{\eta=\eta_s} > 0$. Also, since $\Gamma(\eta, K)$ decreases with K , it is easy to see that $\eta_s(K)$ is an increasing function of K .

For each $K \geq K_{min}$, a test that satisfies the constraint on the probability of incorrect decision can be designed; and our next task is to decide which K to use. Note that, since $M \leq N_e$, multiple M -bin tests are required to find N_e consecutive unoccupied bins. To run the test on a different set of M bins, we need to move to a different center frequency. Due to the time required by various RF-components such as the PLL, oscillators, etc, to reach a steady state after the change in center frequency, there is a setup delay (also referred to as settling time) of N_S samples associated with each M -bin test [66]. Thus,

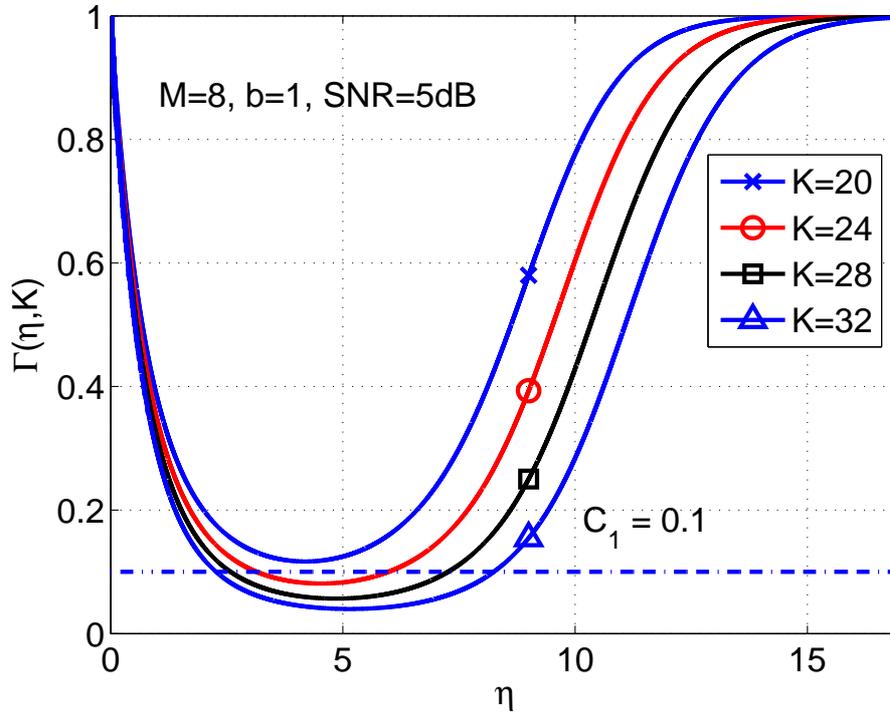


Figure 4.2: Family of $\Gamma(\eta, K)$ functions: increasing K helps in meeting the minimum performance target.

we define the following objective function:

$$D_{avg}(K) = \bar{N}_t(K) (K + N_S). \quad (4.19)$$

$D_{avg}(K)$ can be viewed as the average search duration for finding N_e unoccupied bins with an M -bin test designed with K samples. The value of N_S is known, as it depends on the front-end RF chain at the CR, and is part of its technical specifications. From (4.19), N_S plays significant role in the time taken to find N_e bins when it is comparable to K or when $\bar{N}_t(K)$ is high. We state the following:

Proposition 8. For $K \geq K_{min}$, $D_{avg}(K)$ is a convex function of K .

Proof: See Appendix C.4.

Thus, for a given M , the optimum K can be computed as the solution to:

$$K_{opt} = \underset{K \geq K_{min}}{\text{minimize}} \quad D_{avg}(K). \quad (4.20)$$

With regards to minimization, $D_{avg}(K)$ is a well-behaved function and simple convex optimization techniques, e.g., the Newton method [55], can be used to find the optimum K . Note that the computational complexity of evaluating K_{opt} (and η'_{gt} in (4.13)) is not of major concern here, as these will be computed offline and remain unchanged as long as the primary usage statistics remain the same. In the above analysis, we have assumed K to be a real number. In practice, we compute $D_{avg}(K)$ at the two integers nearest to the optimum real value and pick K_{opt} to be the one with smaller $D_{avg}(K)$. Figure 4.3 illustrates the convex behavior of $D_{avg}(K)$ with K for a particular set of operating parameters.

The last step in the design of the detector is to find the optimum combination (M, b) that minimizes the average detection delay. To this end, for a given M , let $K_{opt}(M)$ be the number of samples that minimize the average search duration to find $N_e = Mb$ consecutive unoccupied bins. We use $D_{avg}(K_{opt}(M))$ as the metric to compare the performance with different values of M . Let $\mathcal{M} = \{(M, b) : b = \lceil \frac{N_e}{M} \rceil\}$ be the set of all combinations of (M, b) that can be used to find N_e consecutive empty bins. To find the optimal value of M for a given test scenario, we minimize $D_{avg}(K_{opt})$ over \mathcal{M} :

$$(M_{opt}, b_{opt}) = \underset{(M,b) \in \mathcal{M}}{\text{minimize}} \quad D_{avg}(K_{opt}(M)). \quad (4.21)$$

We solve the above optimization problem by simply searching over the set \mathcal{M} , since there are only a finite number of combinations, and computing $D_{avg}(K_{opt}(M))$ is not

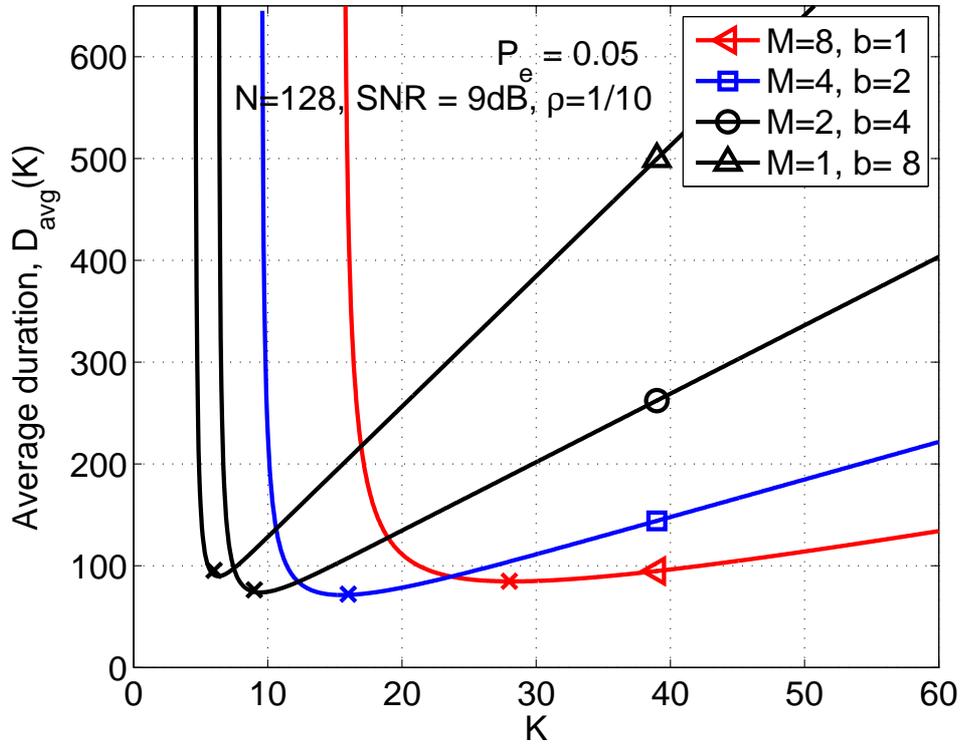


Figure 4.3: $D_{avg}(K)$ is convex in K : Minimizing $D_{avg}(K)$ leads to an optimal value of K ; 'x' markers show the experimental $D_{avg}(K)$ at computed K_{opt} .

computationally demanding. Thus, given the operating parameters: (σ^2, ρ, P_s) , we have fully specified an M -bin test, and a method to choose the (M, K) that minimizes the average delay in searching for a spectrum hole consisting of N_e unoccupied bins.

4.4 Multi-Stage Sensing Algorithm

In this section, we describe an M -bin test based multi-stage sensing algorithm, to find the available spectrum holes in a given wideband of interest. The basic idea, as shown in Figure 4.4, is to search for spectrum holes of different bandwidths by making multiple passes of search on a given wideband. With each successive pass, the width of the hole (specified by the value of M used in the M -bin test) being searched for is halved,

and only the parts of the wideband that have been declared occupied in the previous stages are considered in the search. A multi-stage algorithm can be used, for example, in spectrum hole search with a frequency hopping primary. Here, the occupancy pattern keeps changing periodically, and we refer to this time interval as hopping interval, denoted by N_h .² In terms of CR usage, each hopping interval is split into two phases: *sensing phase*, to find the unoccupied spectrum and *usage phase*, to use the spectrum found. Let n represent the duration of sensing phase. Thus, the goal for sensing algorithms in a frequency hopping scenario is to maximize $N_{TB} \triangleq M_h(n)(N_h - n)$, where $M_h(n)$ is the number of unoccupied bins found during the sensing phase. In other words, we want as much usable spectrum for as much time as possible in a given hopping interval.

Let M , ρ , K_{opt} , η_s and p be as defined earlier. Let $M^{(l)}$, $\rho^{(l)}$, $K_o^{(l)}$, $\eta_s^{(l)}$ and $p^{(l)}$ denote the values of above parameters at the l^{th} stage. Let L be the number of stages. Since, at each stage, the value of M is halved, $M^{(1)} = 2^{L-1}$. Let P_{ems} denote the bin-level error probability in the sense of (4.7), i.e., the probability that an individual bin is actually occupied given that it has been declared unoccupied by the algorithm. An overall bin-level error probability constraint can be satisfied if the M -bin test at every stage satisfies the same bin-level error probability constraint. For an M -bin test, define $P_{me}^{(i)} \triangleq \Pr(i^{\text{th}} \text{ bin in } \mathcal{A}_M \text{ is occupied} \mid \mathcal{A}_M \text{ is declared unoccupied})$, where \mathcal{A}_M is the set of M bins being tested. That is, $P_{me}^{(i)}$ quantifies the bin-level error probability of an M -bin test. It can be shown that $P_{me}^{(i)} = P_0$, i.e., it meets the target error probability constraint, if the threshold used in the M -bin test, η , is chosen as the solution to the

²For example, in a Bluetooth network, the hopping time is 1/1600 s. With a sampling rate of 2 MHz, this implies $N_h = 1250$.

following equation:

$$\sum_{k=1}^M \binom{M-1}{k-1} \rho^k (1-\rho)^{M-k} \frac{\beta_k(\eta)}{p(\eta)} = P_0. \quad (4.22)$$

Here, ρ denotes the occupancy rate, $\beta_k(\eta)$ and $p(\eta)$ are as defined in (4.12) and (4.14), respectively. Thus, depending upon P_0 , M and ρ , an M -bin test satisfying a specified bin-level error probability constraint can be designed by choosing the detection threshold according to the above equation. We also note that since each stage removes a part of the unoccupied spectrum from the given band of operation, the occupancy rate, $\rho^{(l)}$, for each stage needs to be updated accordingly.

We now describe the multi-stage algorithm. Let an estimate of the number of bins found at the end of the l^{th} stage be denoted by $T_f(l)$.³ Let ρ_0 be the occupancy rate for the wideband over which multi-stage algorithm is operating. Let $P_0 \in (0, 1)$ denote the target bin-level error probability.

1. Initialize: $\rho^{(1)} = \rho_0$; $T_f(0) = 0$; and set all the bins in the wideband as occupied.
2. for $l = 1, 2, \dots, L$
 - (a) Using $M = M^{(l)}$, $\rho = \rho^{(l)}$ and P_0 , find the detection threshold, $\eta_s^{(l)}$, using (4.22).
 - (b) Find the optimum number of samples, $K_o^{(l)}$, using (4.20) with $b = 1$. That is, find the number of samples required to minimize the detection delay in finding a spectrum hole of size M . Also, find the corresponding $p^{(l)}$ using (4.14).

³Note that, this can also be replaced by the number of bins that are declared as unoccupied in the actual running of the algorithm. Here, we use an estimate of the number of bins declared as unoccupied in order to facilitate an offline calculation of the thresholds to be used at each stage, and to analytically compute the detection delay.

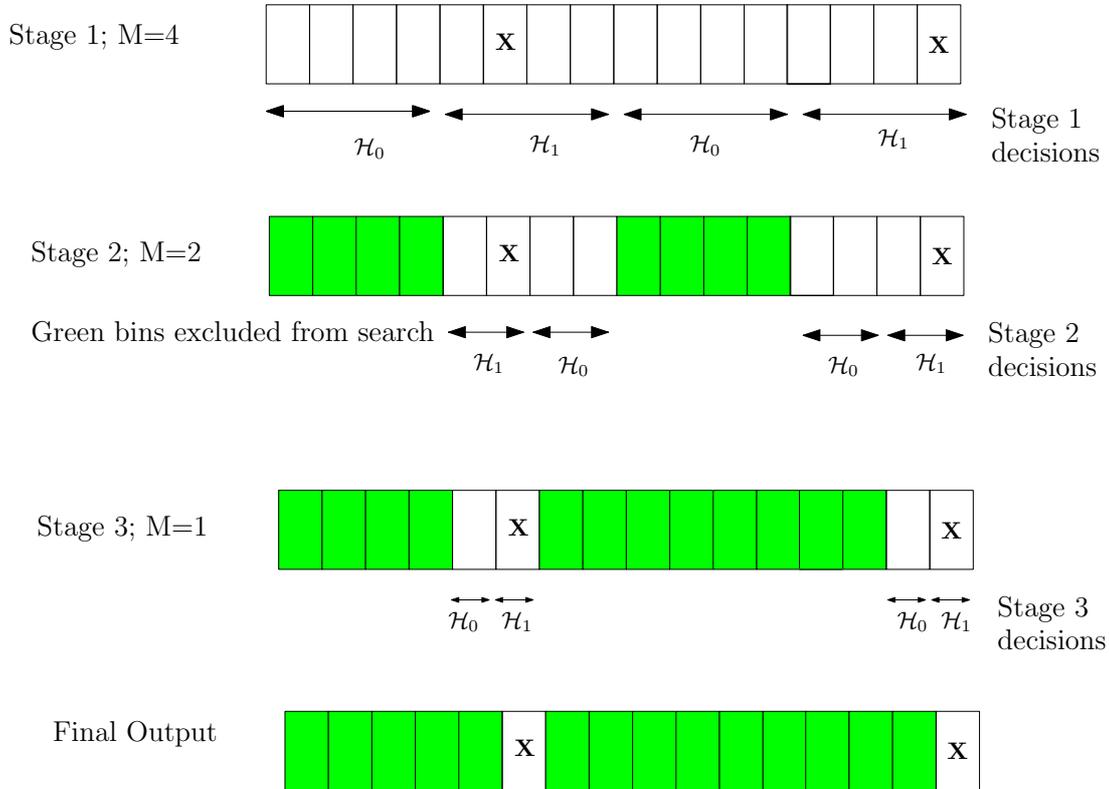
- (c) Make a pass over the full wideband, i.e., perform a series of M -bin tests for the bins that are set as occupied, with $M = M^{(l)}$, $K = K_o^{(l)}$ and $\eta = \eta_s^{(l)}$. If a test declares \mathcal{H}_0 , then set the corresponding $M^{(l)}$ bins as unoccupied.
- (d) Update $T_f(l)$: $T_f(l) = T_f(l-1) + [N - T_f(l-1)]p^{(l)}$.
- (e) Update the occupancy rate for the next stage: $\rho^{(l+1)} = \frac{N\rho - T_f(l)P_0}{N - T_f(l)}$.
- (f) Update $M^{(l+1)} = M^{(l)}/2$.

Note that, the above algorithm ensures that the overall bin-level error probability constraint is met, since each stage is designed such that the bin-level error probability in that stage is P_0 . Also, the above update for the occupancy rate works well when the primary powers in the different bins are approximately equal and known. In the unequal or unknown transmit power case, the M -bin test ensures that the empirical values of $P_{me}^{(i)}$ remain below the specified target. We have observed, via simulations, that the algorithm works well in terms of the detection delay, and also satisfies the specified target error rates. When $M^{(1)} = 1$, we get the conventional bin-by-bin detection scheme as a special case, as before.

4.5 Simulations and Results

We now present simulation results to illustrate the performance of the proposed signal acquisition scheme and spectrum hole search algorithms. We first present the results for the contiguous hole search algorithm in the AWGN case. We consider a test setup with $N = 128$, $N_e = 8$ and $\rho \in \{\frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}, \frac{1}{16}, \frac{1}{20}\}$. The signal samples used for computing the test statistic are generated according to (4.3). The signal powers for bins with

16 bins to be searched; 3 stages; Test are assumed to be error-free;
 Bins marked 'X' indicate primary occupancy



Group of bins declared unoccupied are marked green

Figure 4.4: A pictorial description for the multi-stage algorithm.

occupancy, P_l , are chosen uniformly at random from the set $\{P_s, P_s+2, P_s+4, P_s+6\}$ dB, with $P_s = 0$ dB. We refer to $\frac{P_s}{\sigma^2}$ as the detection SNR, and vary it from 0 to 13 dB. The test outputs a sequence of N_e bins that are declared unoccupied. An error occurs if this declared set contains any occupied bin. The target error rate constraint is set as $P_0 = 0.1$. For different combinations of (M, b) , the value of K_{opt} is numerically computed, and the corresponding detection delay ($D_{avg}(K_{opt})$, henceforth denoted D_{avg} for short) performance is evaluated. For calculating D_{avg} , we used $N_S = 0$.

Figures 4.5 and 4.6 show the variation of D_{avg} with sparsity and SNR, respectively. As expected, with increasing sparsity, the group tests with higher M perform better. At

low SNR, $M = 1$ is optimal. As the SNR increases, $M > 1$ outperforms $M = 1$, and interestingly, the relative reduction in D_{avg} is higher for higher M . Note that, the bin-by-bin ($M = 1$) test can be considered to be the result of using the framework in [70] when the primary transmitter to secondary receiver channel gain information is absent and the test is designed to minimize the average test duration. We have also empirically verified that the observed P_e is below the target $P_0 = 0.1$. We also evaluate the performance of the group tests with the optimal values of M , K and η_s computed as described in Section 4.2. We see that the tests with optimal parameter values perform the best in all the simulated scenarios. In Figure 4.7, we show the excellent match between the simulation and analytical results for D_{avg} and P_e at different sparsity values, when $P_l = P_s$.

Next, we consider the scenario where the primary signals undergo Rayleigh multipath fading and lognormal shadowing with variance 4 dB. Figure 4.8 shows the variation of D_{avg} with detection SNR. We see the same behavior as in the AWGN case, albeit at roughly 7dB higher SNR values. This is expected, as the M -bin detector is an energy based detector and its performance degrades in the presence of fading.

Table 4.1 shows the reduction in D_{avg} compared to tests with $M = 1$ at an SNR of 9 dB for the AWGN case and 16 dB for the fading case, and with different sparsity values. With $N_S = 0$, the reduction in D_{avg} of the proposed group test compared to the $M = 1$ case is 20 to 30%, depending on the sparsity level. For a conservative value⁴ of $N_S = 5$, the percentage reduction in D_{avg} is significantly higher, and varies between 35 and 60%. This is because tests with higher M result in a significantly lower average number of

⁴At a sampling rate of 1 MHz and with a low-power phase-locked loop (see [66], Table I), N_S can be as high as 120 samples.

tests (see Table 4.2) and thus save on the test setup overheads, compared to the $M = 1$ case.

Figure 4.9 shows the variation of $D_{avg}(M_{opt}, K_{opt})$ with primary SNR for different values of target P_0 in the AWGN case. It is interesting to note that the curves are approximately linear, even though different values of M are optimal for different primary SNRs. Also, larger sensing times are required to satisfy smaller values of the target probability of error, as expected.

We now compare the performance of the M -bin detector with the energy-based Single Slot Detector (SSD) proposed in [1]. We use the same setup as in [1], and study the achievable opportunistic secondary throughput, $R(\tau)$, defined as, $R(\tau) \triangleq C_0(1 - \frac{\tau}{T_F})(1 - P_f)(1 - \rho)B$ bits/s, where τ is the sensing (search) duration, T_F is the frame duration and C_0 is the secondary throughput when the primary is absent, ρ is the prior probability of the primary being present and B is bandwidth of a single slot. We assume that the secondary network does not obtain any throughput if it transmits data in the presence of the primary (that is, C_1 , as defined in [1], is zero). Note that, P_f is the probability of false alarm obtained by designing the detector to ensure that the probability of miss stays below a specified target. The M -bin detector searches M consecutive adjacent slots simultaneously in each test, and the corresponding secondary throughput is given by: $R(\tau) \triangleq MC_0(1 - \frac{\tau}{T_F})(1 - P_{fM})(1 - \rho)^M B$ bits/s, where P_{fM} is the probability of false alarm for an M -bin detector with the given target probability of miss detection. We assume BPSK signaling, a sampling rate of 6 MHz, $C_0 = 6.6582$ (which corresponds to a secondary-to-secondary SNR of 20 dB), and a target probability of miss of 0.1, as in [1]. Figure 4.10 compares the normalized throughputs $R(\tau)/B$ for different values

of τ for the scenario where $\rho = 0.1$, $T_F = 100$ ms and a primary SNR value of -13 dB. Also, Table 4.3 compares the achievable throughput for different operating scenarios. As the SNR and sparsity increase, M -bin tests with larger M provide the best secondary throughput, and the group test significantly outperforms the SSD.

We now present a few representative results that illustrate the key aspects of multi-stage algorithm discussed in Section 4.4. Here, we consider the AWGN case and an i.i.d. occupancy model with $N = 256$. We first investigate the application of the multi-stage algorithm for finding non-contiguous spectrum holes in the case with equal power primary signals. Figure 4.11 shows the results of running the multi-stage algorithm with different values of $M^{(1)}$, i.e., with different number of stages. It can be observed that, depending upon the number of non-contiguous unoccupied bins we wish to find, different multi-stage instantiations lead to faster search times. For example, $M^{(1)} = 8$ (i.e., $L = 4$) is optimal if we wish to find about 100 unoccupied bins, whereas $M^{(1)} = 4$ (i.e., $L = 3$) is optimal if we wish to find about 200 unoccupied bins. Figure 4.12 shows the results for the spectrum hole search in the presence of frequency hopping. For the multi-stage algorithm with different initial group sizes and different search durations, we need to pick the one that maximizes N_{TB} , as defined in Section 4.4. It can be seen that, under sparse spectrum occupancy by the primary, the group testing based sensing schemes are more efficient in harvesting the available spectrum in a given hopping interval. For example, at an SNR= 7dB, multi-stage instantiations with $M^{(1)} = 2, 4$ and 8 all have higher maximum N_{TB} compared to the bin-by-bin test.

Table 4.1: Percentage D_{avg} reduction compared to $M = 1$. The quantities in the parenthesis represent the percentage reduction compared to $M = 1$ and the value of M that achieves it, respectively.

Sparsity	1/8		1/12		1/16	
	$N_S = 0$	$N_S = 5$	$N_S = 0$	$N_S = 5$	$N_S = 0$	$N_S = 5$
AWGN, 9dB	(19%, 4)	(43%, 4)	(26%, 4)	(52%, 8)	(29%, 4)	(61%, 8)
Fading, 16dB	(21%, 4)	(44%, 4)	(26%, 4)	(52%, 8)	(30%, 4)	(62%, 8)

Table 4.2: Average number of tests for varying sparsity at SNR = 9dB.

Sparsity	1/6	1/8	1/12	1/16
$M = 8, b = 1$	5.6	3.8	2.6	2.1
$M = 4, b = 2$	7.9	5.9	4.4	3.9
$M = 2, b = 4$	12.6	9.0	7.1	6.2
$M = 1, b = 8$	21.8	17.5	13.5	12.8

4.6 Conclusions

In this chapter, we investigated the use of adaptive group testing based techniques for spectrum hole search in cognitive radios. To enable this, we proposed a signal acquisition scheme that deliberately introduces aliasing by sampling a wideband signal at a sub-Nyquist rate. We developed spectrum hole search algorithms based on the energy of the aliased signal. The algorithms exploit the sparsity in the primary spectral occupancy by making a joint occupancy decision on the group of narrowband bins over which the signal is acquired. We first designed the group testing based algorithm to search contiguous spectrum holes while guaranteeing a given level of protection to the primary network. We extended the group tests to a multi-stage sensing algorithm that looks for contiguous holes of different widths at each stage. Based on the theoretical analysis of the group tests, we provided a computational procedure to obtain the optimal group size, number of samples, and the detection thresholds, that minimize the

Table 4.3: Achievable throughput, R/B bits/s: Comparison of the M -bin test with the detector in [1].

	SNR=-15 dB, $\rho = 1/10$	SNR=-12 dB, $\rho = 1/10$	SNR=-10 dB, $\rho = 1/16$
SSD	5.66	5.88	6.19
M -bin	9.09 ($M = 2$)	14.81 ($M = 4$)	25.13 ($M = 8$)

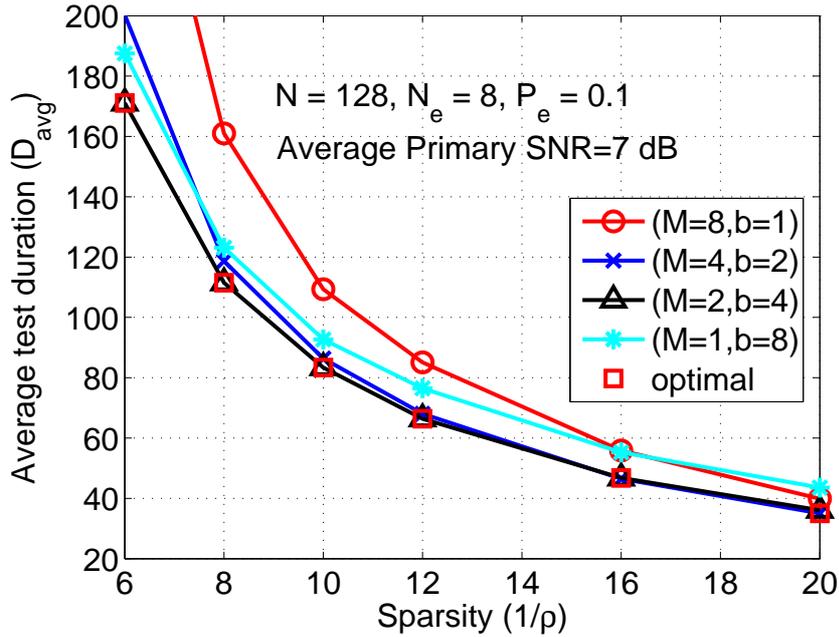


Figure 4.5: D_{avg} Vs. Sparsity in the AWGN case. At higher sparsity, group tests outperform the bin-by-bin test.

average search duration given a target error rate constraint. This enabled us to identify the operating parameter regimes where group testing based algorithms outperform their narrowband counterparts. The performance gains are achieved at a minimal additional hardware cost, which makes the group testing based sensing schemes attractive for practical implementation.

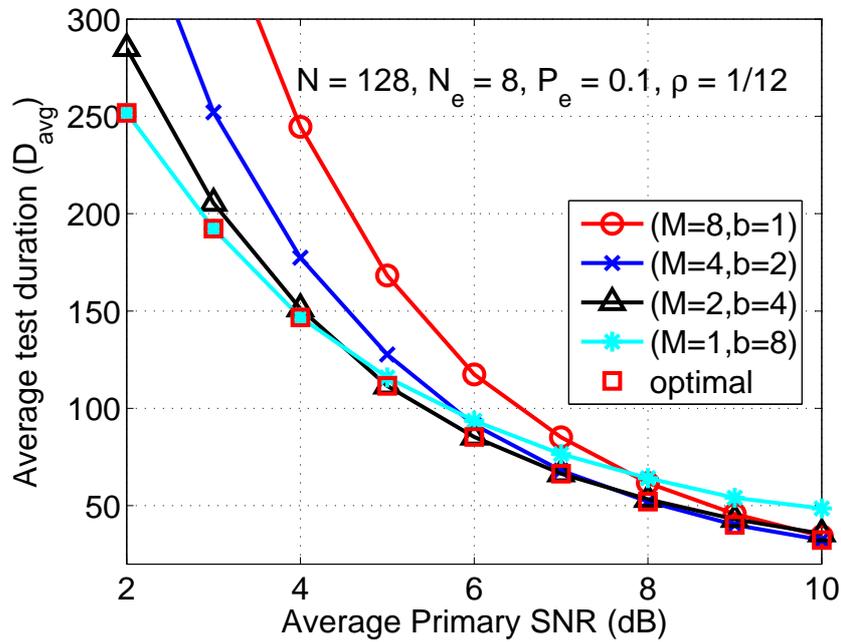


Figure 4.6: D_{avg} Vs. SNR in the AWGN case. At higher SNRs, group tests outperform the bin-by-bin test.

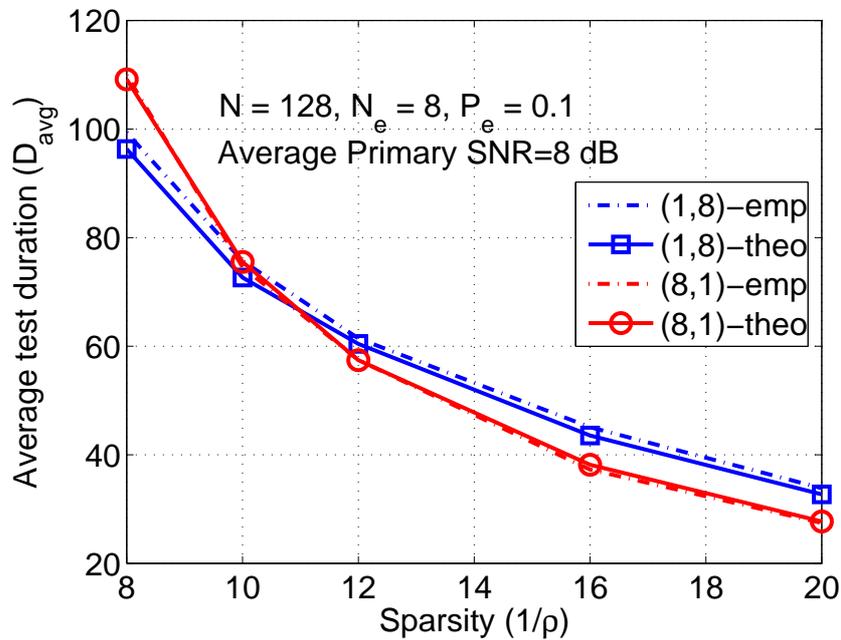


Figure 4.7: Good match with theoretical results, D_{avg} Vs. Sparsity in the AWGN case. Empirical P_e for the above data points: Case ($M = 8, b = 1$): [0.092 0.093 0.095 0.095 0.097] and Case ($M = 1, b = 8$): [0.084 0.090 0.091 0.100 0.094] at sparsity values of $1/\rho = [8 10 12 16 20]$, respectively. The empirical P_e always remains below the target value of 0.1.

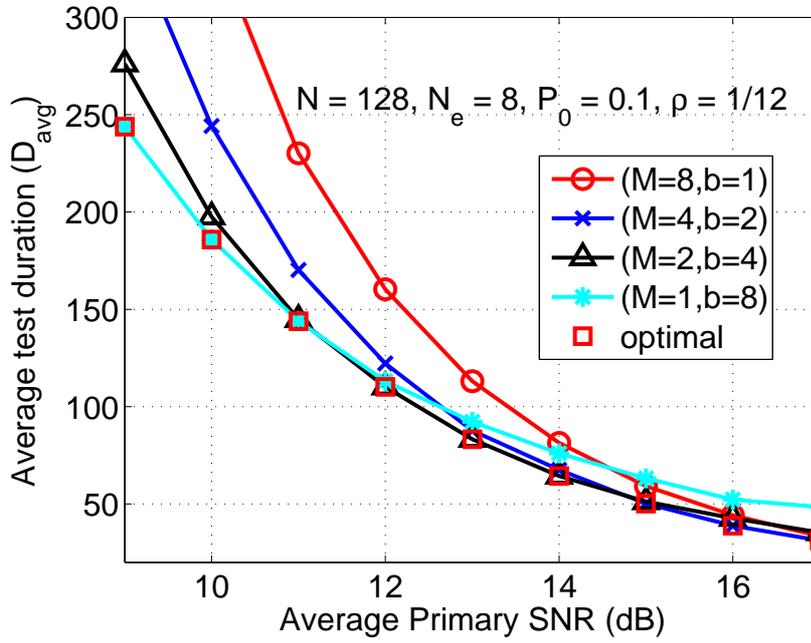


Figure 4.8: Contiguous hole search, with fading and shadowing, D_{avg} Vs. SNR: Similar trends as compared to AWGN case, but at higher SNRs.

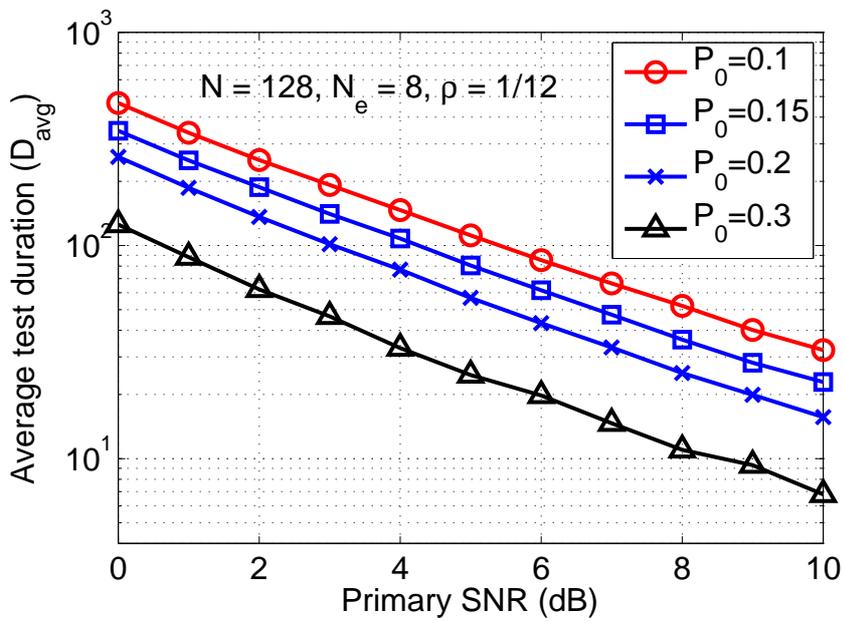


Figure 4.9: Contiguous hole search, AWGN, D_{avg} Vs. SNR: Performance at different target error probabilities.

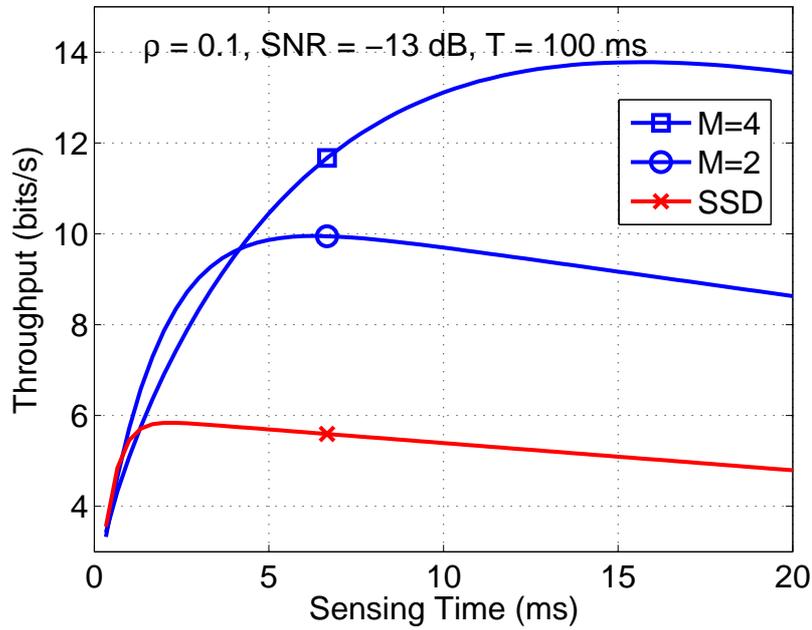


Figure 4.10: Normalized throughput vs. the sensing time: M -bin detector significantly improves secondary throughput. “SSD” refers to the energy-based single slot detection scheme in [1].

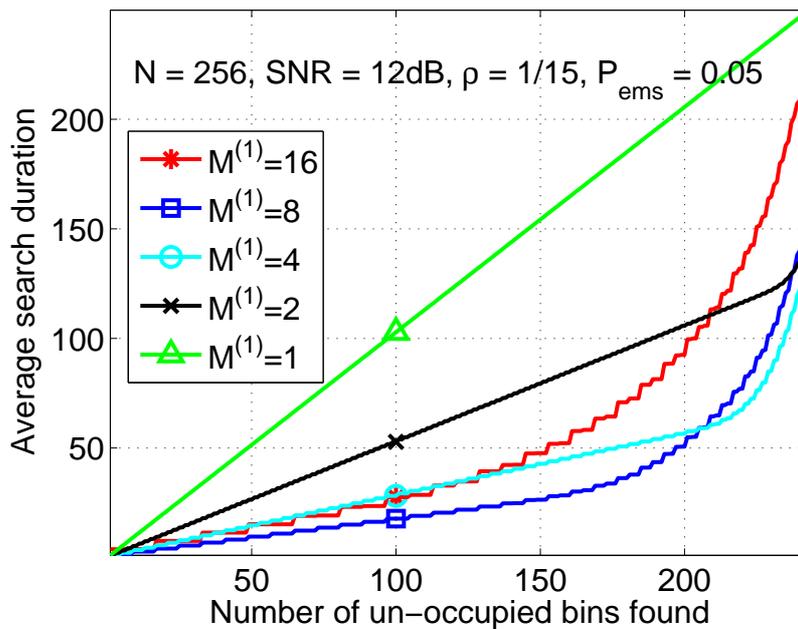


Figure 4.11: Search durations for different multi-stage instantiations.

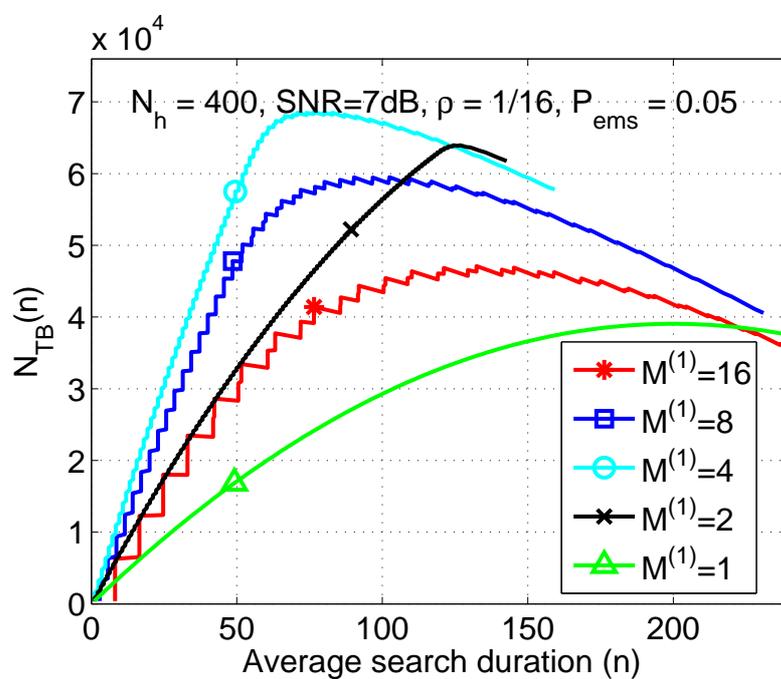


Figure 4.12: Spectrum hole search and frequency hopping. Group testing based sensing exploits the available spectrum more efficiently, compared to the bin-by-bin case.

Chapter 5

Conclusions and Future Work

In this thesis, we studied the problem of identifying a given number of “non-defective” items from a large population containing a small number of “defective” items, in a non-adaptive group testing framework. The contributions of this thesis are summarized below.

5.1 Summary of Contributions

In Chapter 2, we studied the information theoretic aspects of the problem of identifying L non-defective items out of a large population of N items containing K defective items. We studied the problem in a general sparse signal modeling framework where we compared the following two approaches: identifying the defective items using the observations followed by picking L items from the complement set, and directly identifying non-defective items from the observations. We derived upper and lower bounds on the number of observations required for identifying the L non-defective items. We showed that an impressive gain in the number of observations is obtainable by directly identifying the non-defective items. We applied the results to the noisy nonadaptive

group testing setup, where we accounted for the impact of both dilution and additive noise in the tests. We characterized the number of tests that are sufficient to identify a subset of non-defective items of a large population and also showed that the upper and lower bounds are order-wise tight.

In Chapter 3, we proposed analytically tractable and computationally efficient algorithms for identifying a non-defective subset of a given size in a noisy non-adaptive group testing setup. We derived upper bounds on the number of tests for guaranteed correct non-defective subset identification and we showed that the algorithms are nearly optimal, as the upper bounds and information theoretic lower bounds are order-wise tight up to a $\log K$ factor, where K is the number of defective items. While the algorithms did not require knowledge of system parameters such as K , the statistics of the random test matrix did depend on K . We showed that the algorithms and results were robust to uncertainty in the knowledge of the system parameters. Also, it was found that the algorithms that use both positive and negative outcomes, namely **CoAI** and the LP relaxation based **CoLpAI**, gave the best performance for a wide range of values of L , the size of non-defective subset to be identified.

Finally, in Chapter 4 we investigated the use of adaptive group testing for spectrum hole search in cognitive radios. To enable this, we proposed a signal acquisition scheme that deliberately introduces aliasing by sampling a wideband signal at a sub-Nyquist rate. The energy in the aliased signal is used as the basis for detection. We proposed a group testing based algorithm to search for contiguous spectrum holes while guaranteeing a given level of protection to the primary network. We also extended the

group tests to a multi-stage sensing algorithm that looks for contiguous holes of different widths at each stage. We provided a computational procedure to obtain the optimal algorithm parameters that minimize the average search duration given a target error rate constraint. Further, our theoretical analysis enabled us to identify the operating parameter regimes where group testing based algorithms outperform their narrowband counterparts.

5.2 Future Work

Future work could include the extension of the non-defective subset identification problem to more complex sparse signal models. In Chapters 2 and 3, we considered test/measurement matrix ensembles that were generated in i.i.d. fashion. It will be interesting to extend the results on non-defective subset identification to non-i.i.d. test matrices. For example, Gaussian ensembles with independent rows and correlated columns have been studied in the context of sparse recovery in [35, 78]. In Chapter 3, we considered a randomized pooling strategy. An interesting extension could be to study deterministic or structured random constructions, see e.g., [17, 18], for the purpose of non-defective subset identification. Another possible direction is to study the non-defective subset identification when structured pooling strategies are used. For example, it would be interesting to derive bounds on the number of tests for non-defective subset recovery, when the items to be pooled are constrained to be nodes that form a path obtained by taking a random walk over a given graph [16].

Appendix A

Appendix for Chapter 2

A.1 Asymptotic Properties of the Upper Bound for Decoding Scheme 2

In this subsection, we analyze the behavior of Γ_{u2} , the key quantity used in deriving the upper bound on the number of observations under decoding schemes discussed in Sections 2.2.2 and 2.2.3. Let N, K and L be as defined before. Let $\Gamma_{u2} \triangleq \max_{1 \leq j \leq K} \frac{\log C_2(L, N, K, j)}{L_j}$, where C_2 and L_j are as defined in Theorem 2. We have the following lemma regarding the asymptotic behavior of Γ_{u2} .

Lemma 3. Let $\alpha \triangleq \frac{L-1}{N-K}$. Define $N_b \triangleq \frac{N-K}{K-1}$. Let L be an integer multiple of K . Let $(N-K) > 4$. Then for all L such that $L \geq K$ and $2 \leq (L-1) \leq (N-K) - 2$,

$$\Gamma_{u2} \leq \Gamma_{mu} \triangleq \frac{H_b(\alpha)}{1-\alpha} + \frac{1}{1-\alpha} \left[\frac{\log(e\alpha N_b)}{N_b} + \frac{\log(Ke^K)}{N-K} \right], \quad (\text{A.1})$$

where $H_b(\cdot)$ represents the binary entropy function. In particular, for the scaling regime: $L, N \rightarrow \infty, \frac{L}{N} = \alpha_0$ for a fixed $0 \leq \alpha_0 < 1$ and fixed K , we have $\lim_{N \rightarrow \infty} \Gamma_{u2} \leq \frac{H_b(\alpha_0)}{1-\alpha_0}$.

Furthermore, for the same scaling regime as above, and for any given $\alpha_0 \leq \frac{1}{2}$, there exist constants c_0, c_1 such that

$$\lim_{N \rightarrow \infty} \Gamma_{u2} \leq c_0 \alpha_0 + c_1. \quad (\text{A.2})$$

Proof. We note the following: (a) Since L is an integer multiple of K , $L_j = (N - K) - (L - j)$ and $L_1 < L_j \forall j > 1$. Also, $\frac{L-1}{L_1} = \frac{\alpha}{1-\alpha}$ and $\frac{N-K}{L_1} = \frac{1}{1-\alpha}$. (b) $\binom{L-1}{K-1} > \binom{L-j}{K-j} \forall j > 1$ and $L \geq K$. (c) $\binom{K-1}{j-1} \leq e^K$; and finally (d) $\frac{\log \binom{N-K}{L_j}}{L_j}$ is a decreasing function of j . Thus,

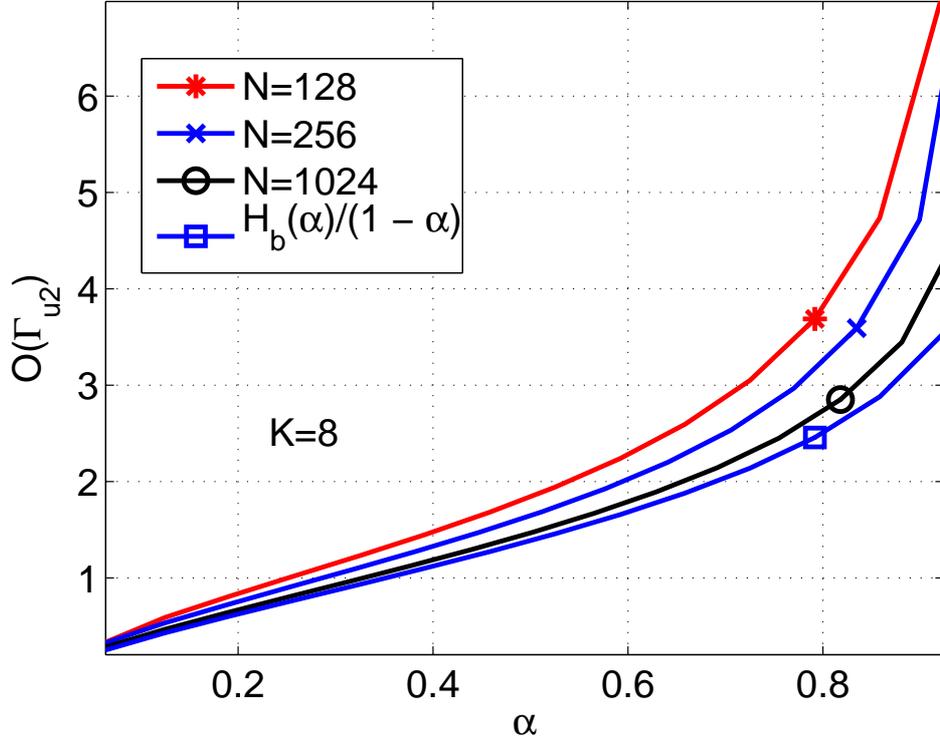
$$\begin{aligned} \Gamma_{u2} &\leq \frac{\log \binom{N-K}{L_1}}{L_1} + \frac{\log(Ke^K)}{L_1} + \frac{(K-1) \log \frac{e(L-1)}{K-1}}{L_1} \\ &\leq \frac{\log \binom{N-K}{L_1}}{L_1} + \frac{\frac{\log(Ke^K)}{(N-K)}}{1-\alpha} + \frac{\frac{1}{N_b} \log(e\alpha N_b)}{1-\alpha}. \end{aligned} \quad (\text{A.3})$$

Using Stirling's formula, for any $n \in \mathbb{Z}_+$: $\sqrt{2\pi n} n^{n+1/2} e^{-n} \leq n! \leq e n^{n+1/2} e^{-n}$, we get

$$\log \binom{N-K}{L_1} \leq L_1 \log \left(\frac{N-K}{L_1} \right) + (L-1) \log \left(\frac{N-K}{L-1} \right) + \frac{1}{2} \log \frac{N-K}{L_1(L-1)} \quad (\text{A.4})$$

$$\leq L_1 \log \left(\frac{N-K}{L_1} \right) + (L-1) \log \left(\frac{N-K}{L-1} \right). \quad (\text{A.5})$$

The second inequality follows since under the assumptions on the range of L , $\frac{N-K}{L_1(L-1)} < 1$. Thus, $\frac{\log \binom{N-K}{L_1}}{L_1} \leq \frac{H_b(\alpha)}{1-\alpha}$, from which we arrive at (A.1). We note that, for the given scaling regime $\lim_{N \rightarrow \infty} \alpha = \alpha_0$, and, for a fixed α_0 , the second and third term in (A.3) vanish as $N \rightarrow \infty$. Thus, $\lim_{N \rightarrow \infty} \Gamma_{u2} \leq \frac{H_b(\alpha_0)}{1-\alpha_0}$. Figure A.1 shows the variation of upper bound of Γ_{u2} with N , with $K = 8$.

Figure A.1: Behavior of $\Gamma_{u2}(\alpha)$ with N .

To establish (A.2), we note that

$$\begin{aligned}
 \frac{H_b(\alpha)}{1-\alpha} &= -\frac{\alpha}{1-\alpha} \log(\alpha) - \log(1-\alpha) = \frac{\alpha}{1-\alpha} \sum_{i=1}^{\infty} \frac{(1-\alpha)^i}{i} + \sum_{i=1}^{\infty} \frac{\alpha^i}{i} \\
 &\leq \alpha \left(1 + \frac{(1-\alpha)}{2} + \frac{(1-\alpha)^2}{3} \right) + \frac{\alpha(1-\alpha)^3}{4} \left(\sum_{i=1}^{\infty} (1-\alpha)^{i-1} \right) \\
 &\quad + \alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{3} + \frac{\alpha^4}{4} \left(\sum_{i=1}^{\infty} \alpha^{i-1} \right) \\
 &\leq \frac{17}{6}\alpha + \frac{1}{4} \left[(1-\alpha)^3 + \frac{\alpha^4}{1-\alpha} \right] \leq c_0\alpha + c_1,
 \end{aligned}$$

where c_1 is obtained by appropriately bounding the second term by a constant when $\alpha \leq \frac{1}{2}$. Thus, (A.2) follows since $\alpha \rightarrow \alpha_0 \leq \frac{1}{2}$ as $N \rightarrow \infty$. \square

A.2 Order-Tight Results for Necessary and Sufficient Number of Tests with Group Testing

In this section, we present a brief sketch of the derivation of the order results for the sufficient and necessary number of tests presented in Tables 2.1 and 2.2, respectively.

We first note that $I^{(j)} = H(Y|X_{S^{(K-j)}}) - H(Y|X_{S^{(K-j)}}, X_{S^{(j)}})$ [30], where $H(\cdot|\cdot)$ represents the entropy function [44]. From (2.2), we have

$$H(Y|X_{S^{(K-j)}}) = \sum_{l=0}^{K-j} \left[\binom{K-j}{l} p^l (1-p)^{K-j-l} H_b((1-q)u^l(1-p(1-u))^j) \right] \quad (\text{A.6})$$

$$H(Y|X_{S^{(K-j)}}, X_{S^{(j)}}) = \sum_{i=0}^K \left[\binom{K}{i} p^i (1-p)^{K-i} H_b((1-q)u^i) \right]. \quad (\text{A.7})$$

We use the results from [47] for bounding the mutual information term. We collect the required results from [47] in the following lemma.

Lemma 4. *Bounds on $I^{(j)}$ [47]: Let $p = \frac{\delta}{K}$. $I^{(j)}$ can be expressed as $I_1^{(j)} + I_2^{(j)}$, where*

$$I_1^{(j)} = \delta e^{-\delta(1-u)} (1-q) (u \log u + 1-u) \frac{j}{K} + O\left(\frac{1}{K^2}\right). \quad (\text{A.8})$$

For the case with $u = 0$ and $q > 0$ we have:

$$I_2^{(j)} = \delta e^{-\delta} \left(\log\left(\frac{1}{q}\right) - (1-q) \right) \frac{j}{K} + O\left(\frac{1}{K^2}\right), \quad (\text{A.9})$$

and for $q = 0, u \geq 0$ we have:

$$\begin{aligned} \delta e^{-\delta} \left((1-u) \left[\log \frac{K}{j\delta(1-u)} \right] - u \right) \frac{j}{K} + O\left(\frac{1}{K^2}\right) &\leq I_2^{(j)} \\ &\leq \delta e^{-\delta(1-u^2)} \left((1-u) \left[\log \frac{K}{j\delta(1-u)} \right] - u + u^2 \right) \frac{j}{K} + O\left(\frac{1}{K^2}\right). \end{aligned} \quad (\text{A.10})$$

Thus, with $\delta = 1$ and neglecting the $O(1/K^2)$ term compared to the first term, we get:

(a) For $u = 0, q > 0$ case, $I^{(j)} \approx \frac{j}{eK} \log(\frac{1}{q})$. (b) For $q = 0, 0 \leq u \leq 0.5$ case, simplifying further, we get

$$\frac{j}{eK}(1-u) \log \frac{K}{j} \lesssim I^{(j)} \lesssim \frac{j}{e^{1/2}K}(1-u) \left(\log \frac{K}{j} + 1 \right). \quad (\text{A.11})$$

The order results now follow by using Lemma 3 to asymptotically upper bound the combinatorial term, Γ_{u2} , for the upper bounds and by noting that, for the scaling regimes under consideration, the combinatorial term for the lower bounds, Γ_l , can be asymptotically bounded as $\lim_{N \rightarrow \infty} \Gamma_l(L, N, K, j) \geq j \log \frac{1}{1-\alpha_0}$.

A.3 Proof of Lemma 1

Let $E_0(\rho, j, n)$ be as defined in (2.3). In contrast to the fixed K case, where analysis of $E_0(\rho, j, n)$ at $\rho = 0$ was sufficient, we need control over the behavior of $E_0(\rho, j, n)$ at other values of ρ and thus we need to consider the exact expression for $E_0(\rho, j, n)$. For the noiseless group testing case (see (2.2)), from the definition (see (2.3)), $E_0(\rho, 1, L_1)$ can be shown to be given by

$$E_0(\rho, 1, L_1) = -\log \left[(1-p)^{(K-1)} \left((1-p)^{(1+\rho L_1)} + p^{(1+\rho L_1)} \right) + 1 - (1-p)^{(K-1)} \right]. \quad (\text{A.12})$$

It is easy to verify that $E_0(\rho, 1, L_j)$ is an increasing function of j , where $L_j = (N - K) - (L - j)$. Hence, from (2.10), it follows that if

$$ME_0(\rho, 1, L_1) - \rho \max_{1 \leq j \leq K} \log C_2(L, N, K, j) - \log K > 0 \quad (\text{A.13})$$

for some $\rho < 1$, then there exists some $\epsilon > 0$ such that $P_e \leq \exp(-M\epsilon)$. To find a sufficient condition on M , we first lower bound E_0 . Note that, $E_0(0, 1, n) = 0$, is positive and is concave in ρ [43]. Define $\rho_0 = \frac{K}{L_1}$. Note that $\rho_0 < 1$. We first compute E_0 at ρ_0 . Let $p = \frac{1}{K}$. For large K , we can neglect the term involving powers of p . Thus $E_0(\rho_0, 1, L_1) \geq -\log [1 - ((1 - \frac{1}{K})^{K-1} + \exp(-2))] \geq C_3$ for all $K \geq K_0$. Using the concavity of E_0 , we get $E_0(\rho, 1, L_1) \geq \rho \frac{C_3 L_1}{K}$ for $0 \leq \rho \leq \rho_0$. Using this bound in (A.13), for $\rho = \frac{K}{L_1}$, the main result follows. We further note that, from Lemma 3, $\max_{1 \leq j \leq K} \frac{\log C_2(L, N, K, j)}{L_1} \leq \Gamma_{mu}$, where Γ_{mu} is given by (A.1).

Appendix B

Appendix for Chapter 3

B.1 Proof of Lemma 2

We note that a test outcome is 0 only if none of the K defective items participate in the test and the output is not corrupted by the additive noise. (a) now follows easily by noting that the probability that an item does not participate in the group test is given by $(1 - p) + pu$. (b) follows easily from (3.1). For (c) we note that, given that $X_{li} = 1$ for any $i \in S_d$, the outcome is 0 only if the i^{th} item does not participate in the test (despite $X_{li} = 1$) and none of the remaining $K - 1$ defective items participate in the test and the test outcome is not corrupted by the additive noise. That is, $\mathbb{P}(Y_l = 0 | X_{li} = 1) = u(1 - (1 - p)u)^{K-1}(1 - q) = \gamma_0\Gamma$. The other part follows similarly. (d) follows by noting that for any $i \in S_d$ and $j \notin S_d$, $\mathbb{P}(Y_l | X_{li}, X_{lj}) = \mathbb{P}(Y_l | X_{li})$. By Bayes rule and part (b) in this lemma, we get: $\mathbb{P}(X_{li}, X_{lj} | Y_l) = \frac{\mathbb{P}(Y_l | X_{li}, X_{lj})\mathbb{P}(X_{li})\mathbb{P}(X_{lj})}{\mathbb{P}(Y_l)} = \mathbb{P}(X_{li} | Y_l)\mathbb{P}(X_{lj} | Y_l)$. Hence the proof.

B.2 Proof of Proposition 2

We first prove that, for all $i \in \hat{S}_L$, $\underline{\lambda}_2(i) = 0$. The proof is based on contradiction. Suppose $\exists j \in \hat{S}_L$ such that $\underline{\lambda}_2(j) > 0$. This implies, from the complimentary slackness conditions (3.36), $\underline{z}(j) = 1$ and thus, $\underline{\lambda}_1(j) = 0$. Since j^{th} item is amongst the smallest L entries, this implies that $\underline{\mathbf{1}}_N^T \underline{\mathbf{z}} > (N - L)$. Hence, $\nu = 0$. From the zero gradient condition in (3.35), it follows that $\underline{\mathbf{1}}_{M_z}^T \mathbf{X}_0(:, j) = -\underline{\lambda}_2(j) < 0$, which is not possible, as all entries in \mathbf{X} are nonnegative. It then follows that $\forall j \in \hat{S}_L$ $\underline{\lambda}_2(j) = 0$. Thus, if $\underline{\lambda}_2(i) > 0$, $\forall i \in S_d$, then these items cannot belong to the first L entries in the primal solution $\underline{\mathbf{z}}$, i.e., $S_d \cap \hat{S}_L = \{\emptyset\}$.

B.3 Proof of Proposition 4

Suppose $\nu < \theta_0$. Then $\exists i$ such that $\underline{\lambda}_1(i) = 0$ and $\nu < \underline{\mathbf{1}}_{M_z}^T \mathbf{X}_0(:, i)$. Thus, from (3.35), $\underline{\lambda}_2(i) = \nu - \underline{\mathbf{1}}_{M_z}^T \mathbf{X}_0(:, i) < 0$, which violates the dual feasibility conditions (3.37). Thus, $\nu \geq \theta_0$. Similarly, let $\nu \geq \theta_1$. Then $\exists i$ such that $\underline{\lambda}_1(i) = 1$ and $\nu \geq \underline{\mathbf{1}}_{M_z}^T \mathbf{X}_0(:, i)$. Thus, from (3.35), $\underline{\lambda}_2(i) = \underline{\lambda}_1(i) + \nu - \underline{\mathbf{1}}_{M_z}^T \mathbf{X}_0(:, i) \geq 1$, which is a contradiction since $\underline{\lambda}_1(i) > 0$ implies $\underline{\lambda}_2(i) = 0$. Thus, $\nu \geq \theta_1$ is not possible.

B.4 Discussion on the theoretical guarantees for RoLpAl++

The discussion for RoLpAl++ proceeds on similar lines as RoLpAl. We use the same notation as in Section 3.3.3, and, as above, we analyze an equivalent LP obtained by eliminating the equality constraints and substituting $(1 - \underline{\mathbf{z}})$ by $\underline{\mathbf{z}}$. The corresponding

KKT conditions for a pair of primal and dual optimal points are as follows:

$$\underline{\mathbf{1}}_{M_z}^T \mathbf{X}_0 - \underline{\mu}^T \mathbf{X}_1 - \underline{\lambda}_1 + \underline{\lambda}_2 - \nu \underline{\mathbf{1}}_N = \underline{\mathbf{0}}_N \quad (\text{B.1})$$

$$\underline{\mu} \circ (\mathbf{X}_1 \underline{z} - (1 - \epsilon_0) \underline{\mathbf{1}}_{M_p}) = \underline{\mathbf{0}}_{M_p}; \quad \underline{\lambda}_1 \circ \underline{z} = \underline{\mathbf{0}}_N; \quad \underline{\lambda}_2 \circ (\underline{z} - \underline{\mathbf{1}}_N) = \underline{\mathbf{0}}_N; \quad \nu (\underline{\mathbf{1}}_N^T \underline{z} - (N - L)) = 0; \quad (\text{B.2})$$

$$\underline{\mathbf{0}}_N \preceq \underline{z} \preceq \underline{\mathbf{1}}_N; \quad \underline{\mathbf{1}}_N^T \underline{z} \geq (N - L); \quad \underline{\mu} \succeq \underline{\mathbf{0}}_{M_p}; \quad \underline{\lambda}_1 \succeq \underline{\mathbf{0}}_N; \quad \underline{\lambda}_2 \succeq \underline{\mathbf{0}}_N; \quad \nu \geq 0; \quad (\text{B.3})$$

Note that in the above $\underline{\mu} \in \mathbb{R}^{M_p}$ is the dual variable associated with constraint (3.9) of

LP1. Let $(\underline{z}, \underline{\mu}, \underline{\lambda}_1, \underline{\lambda}_2, \nu)$ be a primal, dual optimal point satisfying the above equations.

We first prove the following:

Proposition 9. *If $\lambda_2(i) > 0$, then $\underline{\mu}^T \mathbf{X}_1(:, i) = 0$.*

Proof. For any $l \in [M_p]$, if $\mathbf{X}_1(l, i) = 0$ then $\underline{\mu}(l) \mathbf{X}_1(l, i) = 0$. If $\mathbf{X}_1(l, i) = 1$, then for the l^{th} test $\mathbf{X}_1(l, :)\underline{z} \geq 1 > (1 - \epsilon_0)$, since $\lambda_2(i) > 0$ implies $z(i) = 1$. This implies $\underline{\mu}(l) = 0$, and thus $\underline{\mu}(l) \mathbf{X}_1(l, i) = 0$, for this case also. \square

Using the above, it is easy to see that Proposition 2 holds in this case also. Furthermore, using the same arguments as in Section 3.3.3, it can be shown that the error event associated with **RoLpAl++**, \mathcal{E} , satisfies $\mathcal{E} \subseteq \cup_{i \in S_d} \cup_{S_z \in \mathcal{S}_z} \{\mathcal{E}_0(i, j), \forall j \in S_z\}$, where

$$\mathcal{E}_0(i, j) = \{\underline{\mathbf{1}}_{M_z}^T \mathbf{X}_0(:, i) - \underline{\mu}^T \mathbf{X}_1(:, i) - \lambda_1(i) \geq \underline{\mathbf{1}}_{M_z}^T \mathbf{X}_0(:, j) - \underline{\mu}^T \mathbf{X}_1(:, j)\}, \quad (\text{B.4})$$

and, $S_z \subset [N] \setminus S_d$ is any set of non-defective items such that $|S_z| = (N - K) - (L - 1)$ and \mathcal{S}_z denotes all such sets possible. In the following discussion, since i is fixed, for notational simplicity we will use $\mathcal{E}_0(j) \triangleq \mathcal{E}_0(i, j)$. Note that, for **RoLpAl**, the error event is upper bounded by a similar expression as above but with $\mathcal{E}_0(j)$ replaced by $\mathcal{E}_1(j)$

where $\mathcal{E}_1(j) \triangleq \{\underline{\mathbf{1}}_{M_z}^T \mathbf{X}_0(:, i) \geq \underline{\mathbf{1}}_{M_z}^T \mathbf{X}_0(:, j)\}$. In order to analytically compare the performances of **RoLpAl** and **RoLpAl++**, we try to relate the events $\mathcal{E}_0(j)$ and $\mathcal{E}_1(j)$. Note that if $\mathcal{E}_0(j) \subseteq \mathcal{E}_1(j)$, then $\mathbb{P}(\mathcal{E}_0(j)) \leq \mathbb{P}(\mathcal{E}_1(j))$, and hence, **RoLpAl++** would outperform **RoLpAl**. Now, when $\underline{\mu} = \underline{\mathbf{0}}_{M_z}$, $\mathcal{E}_0(j) \subseteq \mathcal{E}_1(j)$, $\forall j \in S_z$. For $\underline{\mu} \neq \underline{\mathbf{0}}$, we divide the items in S_z into two disjoint groups:

(a) $\lambda_2(j) > 0$: Since $\underline{\mu}^T \mathbf{X}_1(:, j) = 0$, $\underline{\mu}^T \mathbf{X}_1(:, i) \geq 0$ and $\lambda_1(i) \geq 0$, it follows that $\mathcal{E}_0 \subseteq \mathcal{E}_1$.

(b) $\lambda_2(j) = 0$: We note that $\mathcal{E}_0(j) \subseteq \mathcal{E}_1(j) \cup \mathcal{E}'_1(j)$ where

$$\mathcal{E}'_1(j) = \{\underline{\mu}^T [\mathbf{X}_1(:, j) - \mathbf{X}_1(:, i)] \geq \kappa + \lambda_1(i)\}, \text{ where } \kappa + \lambda_1(i) > 0.$$

A technical problem, which does not allows us to state the categorical performance result, arises now. It is difficult to obtain the estimates for the dual variables $\underline{\mu}$ and hence of $\mathbb{P}(\mathcal{E}'_1(j))$. Therefore, we offer two intuitive arguments that provide insight into the relative performance of **RoLpAl++** and **RoLpAl**. The first argument is that the *majority* of the items in S_z will have $\lambda_2(j) > 0$ and thus, for a majority items in S_z , it follows that $\mathbb{P}(\mathcal{E}_0(j)) \leq \mathbb{P}(\mathcal{E}_1(j))$. This is because the set $\{\lambda_2(j) = 0\}$ is given by,

$$\{\lambda_2(j) = 0\} = \left\{ j : (\underline{\mathbf{1}}_{M_z}^T \mathbf{X}_0(:, j) - \underline{\mu}^T \mathbf{X}_1(:, j)) = \max_{\{l: \lambda_1(l)=0\}} (\underline{\mathbf{1}}_{M_z}^T \mathbf{X}_0(:, l) - \underline{\mu}^T \mathbf{X}_1(:, l)) \right\}, \quad (\text{B.5})$$

and, as the number of tests increase and the number of non-zero components of $\underline{\mu}$ increase, the probability that above equality holds becomes smaller and smaller. Furthermore, for a small number of items $j \in S_z$ with $\lambda_2(j) = 0$, it is reasonable to expect that $\mathbb{P}(\mathcal{E}'_1(j))$ will be small. This is because the probability that a defective item is tested in a pool with positive outcome is higher than the probability that a non-defective item is tested in a pool with positive outcome. Thus, the expected value of $\underline{\mu}^T [\mathbf{X}(:, j) - \mathbf{X}(:, i)]$

will be negative for a non-negative $\underline{\mu}$ and, thus using concentration of measure arguments, we can expect $\mathbb{P}(\mathcal{E}'_1(j))$ to be small. Thus, we expect that **RoLpAl++** to perform similar (or even better) than **RoLpAl**.

B.5 Chernoff Bounds

Theorem 8. (*[54], Ch. 4*) Let X_1, X_2, \dots, X_n be independent $\mathcal{B}(p)$ random variables. Let $X = \sum_{i=1}^n X_i$ and let $\mu = \mathbb{E}(X)$. Then, for any $0 < \delta < 1$, the following Chernoff bounds hold:

$$\mathbb{P}(X \geq (1 + \delta)\mu) \leq \exp\left(-\frac{\delta^2\mu}{3}\right) \quad (\text{B.6})$$

$$\mathbb{P}(X \leq (1 - \delta)\mu) \leq \exp\left(-\frac{\delta^2\mu}{2}\right) \quad (\text{B.7})$$

Theorem 9. (*Bernstein Inequality [53]*) Let X_1, X_2, \dots, X_n be independent real valued random variables, and assume that $|X_i| < c$ with probability one. Let $X = \sum_{i=1}^n X_i$, $\mu = \mathbb{E}(X)$ and $\sigma = \text{Var}(X)$. Then, for any $\delta > 0$, the following hold:

$$\mathbb{P}(X > \mu + \delta) \leq \exp\left(-\frac{\delta^2}{2\sigma^2 + \frac{2}{3}c\delta}\right) \quad (\text{B.8})$$

$$\mathbb{P}(X < \mu - \delta) \leq \exp\left(-\frac{\delta^2}{2\sigma^2 + \frac{2}{3}c\delta}\right) \quad (\text{B.9})$$

Appendix C

Appendix for Chapter 4

C.1 Proof of Proposition 5

From the definition of P_e in (4.7), it follows that the algorithm makes an error if any of the last b M -bins tests, that have all declared \mathcal{H}_0 , make an error. Thus,

$$\begin{aligned} P_e &= 1 - \Pr\left(\text{None of the last } b \text{ } M\text{-bin tests is in error} \mid \text{All} \right. \\ &\quad \left. \text{the last } b \text{ tests have declared } \mathcal{H}_0\right) \\ &= 1 - \left(1 - \Pr(\{\mathcal{H}_l\}_{l=1}^M | H_{0d})\right)^b = 1 - (1 - P_{me})^b \end{aligned} \quad (\text{C.1})$$

Now, let \bar{N}_t and p be as defined before. \bar{N}_t can be found by setting up a recursive equation using the following arguments: (i) If we get alternate hypothesis declaration on the k^{th} attempt with $k = 1, 2, \dots, b$, then the search process restarts since each test is independent. (ii) If we get b successive null hypothesis outputs then our search terminates. Note that declaring the alternate hypothesis on the k^{th} attempt leads to an increase in the number of tests by k . The probability that we get the first alternate hypothesis on

the k^{th} attempt is given by $p^{(k-1)}(1-p)$. Thus,

$$\bar{N}_t = (1-p) \sum_{k=1}^b p^{k-1} (\bar{N}_t + k) + bp^b = (1-p^b) \bar{N}_t + (1-p) \sum_{k=1}^b kp^{k-1} + bp^b.$$

Using $\sum_{k=1}^b kp^{k-1} = \left[\frac{1-p^{b+1}}{(1-p)^2} - \frac{(b+1)p^b}{1-p} \right]$ and simplifying, the result follows. ■

C.2 Proof of Proposition 6

Let t be the random variable describing the test statistic defined in (4.4). Let \mathcal{G} be the observation set. Let $f_0(t)$ and $f_1(t)$ denote probability distributions of t under the null and alternate hypothesis, respectively, and these are as defined in (4.10). Let $\alpha(\delta)$ and $\beta(\delta)$ represent the false alarm and miss detection rate, respectively, for a decision rule δ . Thus, $\alpha(\delta) = \int_{\mathcal{G}} (1 - \delta(t)) f_0(t) dt$ and $\beta(\delta) = \int_{\mathcal{G}} (1 - \delta(t)) f_1(t) dt$. From Bayes' rule, $P_{me}(\delta) = \frac{\beta(\delta) \Pi_1}{p(\delta)}$. Since $P_{me}(\delta') \leq P'_0$, (4.14) imply that

$$\beta(\delta') \leq \frac{P'_0}{\Pi_1} p(\delta') \quad \text{and} \quad 1 - \alpha(\delta') \geq \frac{1 - P'_0}{\Pi_0} p(\delta'), \quad (\text{C.2})$$

where Π_0 and Π_1 represent the prior probabilities for the null and alternate hypotheses for the M -bin group test. From the definition of $\delta_L(t)$, we have $[(1 - \delta(t)) - (1 - \delta(t'))][f_1(t) - \eta_{gt} f_0(t)] \leq 0$ for any $t \in \mathcal{G}$. Integrating over the entire observation space, we get

$$\beta(\delta) - \beta(\delta') \leq \eta_{gt} [(1 - \alpha(\delta)) (1 - \alpha(\delta'))].$$

Using (4.14) and (C.2), we can further simplify the above to

$\left(\frac{1 - P'_0}{\Pi_0} \eta_{gt} - \frac{P'_0}{\Pi_1} \right) [p(\delta) - p(\delta')] \geq 0$. For $\eta_{gt} > \frac{\Pi_0}{\Pi_1} \frac{P'_0}{1 - P'_0}$, $p(\delta) \geq p(\delta')$, and the assertion follows by noting that \bar{N}_t is monotonic in p . ■

C.3 Proof of Proposition 7

Part (a)

For a fixed K , $z_l = \sqrt{2K-1} - \eta/\sigma_l$, $l = 0, 1, \dots, M$ are functions of η . Hence, $\Gamma(\eta, K)$ is a real valued, continuously differentiable function of η , denoted $\Gamma(\eta)$ for short, with $\eta > 0$. Let $\Gamma'(\eta) \triangleq \frac{d\Gamma(\eta)}{d\eta}$ and $\Gamma''(\eta) \triangleq \frac{d^2\Gamma(\eta)}{d\eta^2}$. We use the second order conditions to prove quasi-convexity ([55], Section 3.4.3): $\Gamma(\eta)$ is quasi-convex in η , if, for all η_0 such that $\Gamma'(\eta_0) = 0$, we have $\Gamma''(\eta_0) > 0$. Since $\frac{d}{dx}(Q(x)) = -\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, and $\frac{d}{d\eta}(z_l) = -\frac{1}{\sigma_l}$ for $l = 0, 1, \dots, M$, we have

$$\Gamma'(\eta) = \frac{1}{\sqrt{2\pi}Q^2(z_0)} \left[Q(z_0) \left[\sum_{l=1}^M \frac{\theta_l}{\sigma_l} e^{-z_l^2/2} \right] - \frac{e^{-z_0^2/2}}{\sigma_0} \left[\sum_{l=1}^M \theta_l Q(z_l) \right] \right]. \quad (\text{C.3})$$

Setting $\Gamma'(\eta) = 0$, (4.17) follows. We now evaluate $\Gamma''(\eta)$ at $\eta = \eta_0$ such that η_0 satisfies (4.17). Differentiating (C.3) and substituting (4.17), we get

$$\Gamma''(\eta_0) = \frac{1}{\sqrt{2\pi}Q(z_0)} \left[\sum_{l=1}^M \frac{\theta_l}{\sigma_l} e^{-z_l^2/2} \left[\frac{z_l}{\sigma_l} - \frac{z_0}{\sigma_0} \right] \right]. \quad (\text{C.4})$$

Note that z_l and z_0 are evaluated at η_0 in the above equation. Moreover, $z_0 < z_1 < \dots < z_M$. We consider following two scenarios:

- (i) $z_0 > 0$: This implies that $z_l > 0$ for all $l = 1, \dots, M$. It is easy to show that $g(z) \triangleq zQ(z)e^{z^2/2}$ is an increasing function of $z > 0$. For any $l = 1, \dots, M$, since $z_0 < z_l$, we get $g(z_0) < g(z_l)$, i.e., $z_0Q(z_0)e^{-z_0^2/2} < z_lQ(z_l)e^{-z_l^2/2}$. Also, $\frac{z_0}{z_l}Q(z_0)e^{-z_l^2/2} < Q(z_l)e^{-z_0^2/2}$, since $z_l > 0$. It follows that

$$\sum_{l=1}^M \theta_l Q(z_l) e^{-z_0^2/2} > Q(z_0) \sum_{l=1}^M \theta_l \frac{z_0}{z_l} e^{-z_l^2/2}. \quad (\text{C.5})$$

Using (4.17) and re-arranging, we get

$$\sum_{l=1}^M \frac{\theta_l e^{-z_l^2/2}}{\sigma_l} \frac{1}{(z_l/\sigma_l)} \left[\frac{z_l}{\sigma_l} - \frac{z_0}{\sigma_0} \right] > 0. \quad (\text{C.6})$$

We claim that, $\sum_{l=1}^M \frac{\theta_l e^{-z_l^2/2}}{\sigma_l} \frac{1}{(z_0/\sigma_0)} \left[\frac{z_l}{\sigma_l} - \frac{z_0}{\sigma_0} \right] > \sum_{l=1}^M \frac{\theta_l e^{-z_l^2/2}}{\sigma_l} \frac{1}{(z_l/\sigma_l)} \left[\frac{z_l}{\sigma_l} - \frac{z_0}{\sigma_0} \right]$ and thus from the above equation and (C.4), it follows that $\Gamma''(\eta) > 0$. To prove the above claim, let us denote $g_l \triangleq \frac{\theta_l e^{-z_l^2/2}}{\sigma_l} \frac{1}{(z_l/\sigma_l)} \left[\frac{z_l}{\sigma_l} - \frac{z_0}{\sigma_0} \right]$ and $h_l \triangleq \frac{\theta_l e^{-z_l^2/2}}{\sigma_l} \frac{1}{(z_0/\sigma_0)} \left[\frac{z_l}{\sigma_l} - \frac{z_0}{\sigma_0} \right]$. Consider the following two scenarios: (a) $g_l \geq 0$: In this case, $(z_l/\sigma_l) \geq (z_0/\sigma_0)$ and thus $h_l \geq g_l$. Thus, replacing g_l by h_l makes (C.6) more positive. (b) $g_l < 0$: In this case, $(z_l/\sigma_l) < (z_0/\sigma_0)$ and thus h_l becomes less negative compared to g_l and thus replacing g_l by h_l still makes (C.6) more positive.

- (ii) $z_0 \leq 0$: For this case, we prove that $\frac{z_l}{\sigma_l} - \frac{z_0}{\sigma_0} > 0$ for each $l = 1, \dots, M$ and thus from (C.4) it follows that $\Gamma''(\eta) > 0$. Let l_0 ($1 \leq l_0 \leq M$) be such that $z_l < 0$ for $l \leq l_0$ and $z_l \geq 0$ for $l > l_0$. For $l > l_0$, it is straightforward to see that $\frac{z_l}{\sigma_l} - \frac{z_0}{\sigma_0} > 0$. For $l \leq l_0$, since $z_0 < z_l < 0$, it implies $|z_l| < |z_0|$. This implies $\frac{|z_l|}{\sigma_l} < \frac{|z_0|}{\sigma_0}$ since $\sigma_l > \sigma_0$. Hence, $-\frac{|z_l|}{\sigma_l} > -\frac{|z_0|}{\sigma_0}$, thereby implying $\frac{z_l}{\sigma_l} - \frac{z_0}{\sigma_0} > 0$, since $z_0 < z_l < 0$.

Part (b)

For a fixed η , $\{z_0, z_1, \Gamma\}$ are functions of K . Let $\Gamma_l(K) \triangleq \frac{Q(z_l)}{Q(z_0)}$ and thus

$\Gamma(K) = \sum_{l=1}^M \theta_l \Gamma_l(K)$. We prove that each Γ_l is a decreasing function of K , and thereby prove that Γ is a decreasing function of K . Let $\Gamma'_l(K) \triangleq \frac{\partial \Gamma_l(K)}{\partial K}$. Using the derivative of

Q -function and $\frac{dz_0}{dK} = \frac{dz_1}{dK} = \frac{1}{\sqrt{2K-1}}$, we have

$$\Gamma'_l(K) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2K-1} Q^2(z_0)} \left[Q(z_l) e^{-z_0^2/2} - Q(z_0) e^{-z_l^2/2} \right]. \quad (\text{C.7})$$

Proving $\Gamma'_l(K) < 0$ is equivalent to proving $Q(z_l)e^{-z_0^2/2} < Q(z_0)e^{-z_l^2/2}$. We first consider the case:

$z_l > 0$ and $z_0 \geq 0$: We first prove that for $z \geq 0$, $g(z) \triangleq Q(z)e^{z^2/2}$ is a decreasing function of z . Note that $g'(z) \triangleq \frac{d}{dz}g(z) = e^{z^2/2}Q(z)z - \frac{1}{\sqrt{2\pi}}$. We use the following upper bound for Q -function [79], $Q(y) < \frac{1}{(1-a)y+a\sqrt{y^2+b}} \frac{e^{-y^2/2}}{\sqrt{2\pi}}$ for $y > 0$ and $a = 0.344$ and $b = 5.334$, in the above expression, to get $g'(z) < \frac{1}{(1-a)z+a\sqrt{z^2+b}} \frac{1}{\sqrt{2\pi}} [az - a\sqrt{z^2+b}] < 0$, since, $a > 0$ and $b > 0$. And since $z_l > z_0$, this implies $Q(z_l)e^{-z_0^2/2} < Q(z_0)e^{-z_l^2/2}$. The above can be easily shown for the other cases, i.e., $\{z_l \leq 0 \text{ and } z_0 < 0\}$, $\{z_l > 0 \text{ and } z_0 < 0\}$, and $\{z_l > 0 \text{ and } z_0 < 0\}$, using similar arguments.

C.4 Proof of Proposition 8

Define $Z_0(K) \triangleq z_0(\eta_s(K), K) = \sqrt{2K-1} - \frac{\eta_s(K)}{\sigma_0}$ and $Z_1(K) \triangleq z_1(\eta_s(K), K) = \sqrt{2K-1} - \frac{\eta_s(K)}{\sigma_1}$. From (4.8), we see that $\bar{N}_t(K)$ depends only on $p(\eta_s(K))$ and using (4.14), we get $p(\eta_s(K)) = C_2Q(Z_0(K))$, where $C_2 = \frac{\Pi_0}{1-P'_0}$. Note that $D_{avg}(K)$ is a continuous function of K (for real K). We first prove that for all $K \geq K_{min}$, $K/Q(Z_0(K))$ is a convex function of K . Let $\Gamma(\eta, K)$, C_1 be as defined before. Let $F(K) \triangleq \frac{K}{Q(Z_0(K))}$, where $Z_0(K) \triangleq \sqrt{(2K-1)} - \frac{\eta(K)}{\sigma_0}$. Also let $z_0 \triangleq \sqrt{(2K-1)} - \frac{\eta}{\sigma_0}$. For a given K , $\eta(K)$ is chosen as the threshold value that satisfies $\Gamma(\eta, K) = C_1$, as given in (4.18). Using the quasi-convexity properties of $\Gamma(\eta, K)$ we can write $\eta(K)$ as:

$$\eta(K) = \inf_{\{\eta: \Gamma(\eta, K) \geq C_1\}} \eta. \quad (\text{C.8})$$

For a fixed K , $Q(z_0)$ is a monotonically increasing function of η . Combining this with (C.8), we get

$$Q(Z_0(K)) = \inf_{\{\eta: \Gamma(\eta, K) \geq C_1\}} Q(z_0), \text{ and } F(K) = \sup_{\{\eta: \Gamma(\eta, K) \geq C_1\}} \frac{K}{Q(z_0)} = \sup_{\{\eta \geq \eta_{min}\}} L(K, \eta), \quad (\text{C.9})$$

where $L(K, \eta) \triangleq \frac{K}{Q(z_0)}$ is a two dimensional function of K and η with $\text{dom } L = \{(K, \eta) : K \geq K_{min}; \eta \geq \eta(K)\}$. In the above equation, η_{min} is the threshold corresponding to $K = K_{min}$. Thus, $F(K)$ can be represented as point-wise supremum of a family of functions and the convexity follows by proving that for each $\eta \geq \eta_{min}$, $L(K, \eta)$ is a convex function of K , $K \in \text{dom } L$ [55]. Differentiating $L(K, \eta)$ twice w.r.t. K , we get

$$\frac{\partial^2 L}{\partial K^2} = \frac{e^{-z_0^2/2}}{\sqrt{2\pi}Q(z_0)^2} \left[\frac{3K-2}{(2K-1)^{3/2}} + \frac{K}{2K-1} \left(\frac{2e^{-z_0^2/2}}{\sqrt{2\pi}Q(z_0)} - z_0 \right) \right]. \quad (\text{C.10})$$

For practical values of $K (\geq 1)$, the first term is always positive. The second term is always positive for $z_0 < 0$. For $z_0 \geq 0$, we use the upper bound for Q -function [79], with $a = 0.344, b = 5.334$ and get,

$$\frac{2e^{-z_0^2/2}}{\sqrt{2\pi}Q(z_0)} - z_0 \geq 2 \left[(1-a)z_0 + a\sqrt{z_0^2 + b} \right] - z_0 \geq (1-a)z_0 \geq 0. \quad (\text{C.11})$$

Thus, $L(K, \eta)$ is convex in K and hence $F(K)$ is convex in K .

We now prove the main proposition. From (4.14), we see that $p(K) = C_2 Q(Z_0(K))$, where $C_2 = \frac{\Pi_0}{1-p_0}$. Using (4.8), $D_{avg}(K)$ can be written as $D_{avg}(K) = \sum_{n=1}^M \frac{1}{C_2^n} \frac{(K+N_S)}{Q^n(Z_0)}$. Let $G_n(K) \triangleq \frac{K}{Q^n(Z_0)}$ and $H_n(K) \triangleq \frac{1}{Q^n(Z_0)}$, with $n \geq 1$. The arguments that were used to prove convexity of $\frac{K}{Q(Z_0)}$ hold for $G_n(K)$ and $H_n(K)$ as well, and it can be easily

verified that $G_n(K)$ and $H_n(K)$ are convex in K . The convexity of $D_{avg}(K)$ follows from this, since it can be written as a non-negative weighted sum of convex functions and is therefore convex [55]. ■

Bibliography

- [1] Y. C. Liang, Y. Zeng, E. C. Y. Peh, and A. T. Hoang, "Sensing-throughput tradeoff for cognitive radio networks," *IEEE Trans. Wireless Commun.*, vol. 7, no. 4, pp. 1326–1337, 2008.
- [2] R. Dorfman, "The Detection of Defective Members of Large Populations," *The Annals of Mathematical Statistics*, vol. 14, no. 4, Dec. 1943. [Online]. Available: <http://projecteuclid.org/euclid.aoms/1177731363>
- [3] H. Q. Ngo and D.-Z. Du, "A survey on combinatorial group testing algorithms with applications to DNA library screening," in *Discrete mathematical problems with medical applications*, ser. DIMACS Ser. Discrete Math. Theoret. Comput. Sci. Amer. Math. Soc., 2000, vol. 55, pp. 171–182.
- [4] D. Du and F. Hwang, *Pooling designs and non-adaptive group testing: Important tools for DNA sequencing*. World Scientific, 2006.
- [5] M. Sobel and P. A. Groll, "Group testing to eliminate efficiently all defectives in a binomial sample," *Bell System Technical Journal*, vol. 38, no. 5, pp. 1179–1252, 1959.
- [6] D. Du and F. Hwang, *Combinatorial group testing and its applications*. World Scientific, 1993.

- [7] C. Chen and F. Hwang, "Detecting and locating electrical shorts using group testing," *IEEE Trans. Circuits Syst.*, vol. 36, no. 8, pp. 1113–1116, Aug 1989.
- [8] A. J. Macula and L. J. Popyack, "A group testing method for finding patterns in data," *Discrete Appl. Math.*, vol. 144, no. 1-2, pp. 149–157, 2004. [Online]. Available: <http://dx.doi.org/10.1016/j.dam.2003.07.009>
- [9] J. Macula, A., "Probabilistic nonadaptive group testing in the presence of errors and dna library screening," *Annals of Combinatorics*, vol. 3, no. 1, pp. 61–69, 1999. [Online]. Available: <http://dx.doi.org/10.1007/BF01609876>
- [10] C. R. Bilder and J. M. Tebbs, "Pooled testing procedures for screening high volume clinical specimens in heterogeneous populations," *Statistics in medicine*, vol. 31(27), pp. 3261–3268, 2012.
- [11] J. Wolf, "Born again group testing: Multiaccess communications," *IEEE Trans. Inf. Theory*, vol. 31, no. 2, pp. 185–191, Mar 1985.
- [12] G. Cormode and S. Muthukrishnan, "What's hot and what's not: Tracking most frequent items dynamically," *ACM Trans. Database Syst.*, vol. 30, no. 1, pp. 249–278, Mar. 2005. [Online]. Available: <http://doi.acm.org/10.1145/1061318.1061325>
- [13] A. C. Gilbert, M. A. Iwen, and M. J. Strauss, "Group testing and sparse signal recovery," in *Signals, Systems and Computers, 2008 42nd Asilomar Conference on*, Oct. 2008, pp. 1059–1063.
- [14] J. W. Fahey, P. J. Ourisson, and F. H. Degnan, "Pathogen detection, testing, and control in fresh broccoli sprouts," *Nutrition Journal*, vol. 5, no. 1, 2006.

- [15] M. Cheraghchi, A. Hormati, A. Karbasi, and M. Vetterli, "Group testing with probabilistic tests: Theory, design and application," *IEEE Trans. Inf. Theory*, vol. 57, no. 10, pp. 7057–7067, Oct. 2011.
- [16] M. Cheraghchi, A. Karbasi, S. Mohajer, and V. Saligrama, "Graph-constrained group testing," *IEEE Trans. Inf. Theor.*, vol. 58, no. 1, pp. 248–262, Jan. 2012.
- [17] W. Kautz and R. Singleton, "Nonrandom binary superimposed codes," *Information Theory, IEEE Transactions on*, vol. 10, no. 4, pp. 363–377, 1964.
- [18] P. Erdos, P. Frankl, and Z. Furedi, "Families of finite sets in which no set is covered by the union of r others," *Israel Journal of Mathematics*, vol. 51, no. 1-2, pp. 79–89, 1985. [Online]. Available: <http://dx.doi.org/10.1007/BF02772959>
- [19] M. Ruszinkó, "On the upper bound of the size of the r -cover-free families," *J. Comb. Theory, Ser. A*, vol. 66, no. 2, pp. 302–310, 1994.
- [20] C. L. Chan, S. Jaggi, V. Saligrama, and S. Agnihotri, "Non-adaptive group testing: Explicit bounds and novel algorithms," *CoRR*, vol. abs/1202.0206, 2012.
- [21] D. Cabric, S. M. Mishra, D. Willkomm, R. Brodersen, and A. Wolisz, "A cognitive radio approach for usage of virtual unlicensed spectrum," in *Proc. of 14th IST Mobile Wireless Communications Summit*, 2005.
- [22] FCC, "Et docket no. 02-155," *Spectrum policy task force report*, Nov. 2002.
- [23] A. Sharma and C. R. Murthy, "Group testing based spectrum hole search for cognitive radios (accepted)," *IEEE Trans. Veh. Technol.*, 2014.

- [24] A. C. Gilbert and M. J. Strauss, "Analysis of data streams: Computational and algorithmic challenges," *Technometrics*, vol. 49, no. 3, pp. 346–356, 2007.
- [25] A. G. Dyachkov and V. V. Rykov, "Bounds on the length of disjunctive codes," *Problems of Information Transmission*, vol. 18, no. 3, pp. 7–13, 1982.
- [26] A. Sebo, "On two random search problems," *Journal of Statistical Planning and Inference*, vol. 11, no. 1, pp. 23–31, Jan. 1985.
- [27] M. B. Malyutov, "The separating property of random matrices," *Mat. Zametki*, vol. 23, no. 1, pp. 155–167, 1978. [Online]. Available: <http://dx.doi.org/10.1007/BF01104893>
- [28] —, "On the maximal rate of screening designs," *Theory Probab. and Appl.*, vol. 24, pp. 655–667, 1979. [Online]. Available: <http://dx.doi.org/10.1007/BF01104893>
- [29] M. B. Malyutov and P. S. Mateev, "Planning of screening experiments for a nonsymmetric response function," *Mat. Zametki*, vol. 27, no. 1, pp. 109–127, 1980. [Online]. Available: <http://dx.doi.org/10.1007/BF01149816>
- [30] G. Atia and V. Saligrama, "Boolean compressed sensing and noisy group testing," *IEEE Trans. Inf. Theory*, vol. 58, no. 3, pp. 1880–1901, 2012.
- [31] E. J. Candés and T. Tao, "Decoding by linear programming," *IEEE Trans. Inf. Theory*, vol. 51, no. 12, pp. 4203–4215, Dec. 2005.
- [32] A. M. Bruckstein, D. L. Donoho, and M. Elad, "From sparse solutions of systems of equations to sparse modeling of signals and images," *SIAM Rev.*, vol. 51, no. 1, pp. 34–81, Feb. 2009. [Online]. Available: <http://dx.doi.org/10.1137/060657704>

- [33] J. A. Tropp, "Just relax: convex programming methods for identifying sparse signals in noise," *IEEE Trans. Inf. Theory*, vol. 52, no. 3, pp. 1030–1051, Mar. 2006.
- [34] E. J. Candés, J. K. Romberg, and T. Tao, "Stable signal recovery from incomplete and inaccurate measurements," *Communications on Pure and Applied Mathematics*, vol. 59, no. 8, pp. 1207–1223, 2006. [Online]. Available: <http://dx.doi.org/10.1002/cpa.20124>
- [35] M. J. Wainwright, "Sharp thresholds for high-dimensional and noisy sparsity recovery using ℓ_1 -constrained quadratic programming (LASSO)," *IEEE Trans. Inf. Theory*, vol. 55, no. 5, pp. 2183–2202, May 2009.
- [36] S. Aeron, V. Saligrama, and M. Zhao, "Information theoretic bounds for compressed sensing," *IEEE Trans. Inf. Theory*, vol. 56, no. 10, pp. 5111–5130, Oct. 2010.
- [37] A. K. Fletcher, S. Rangan, and V. K. Goyal, "Necessary and sufficient conditions for sparsity pattern recovery," *IEEE Trans. Inf. Theory*, vol. 55, no. 12, pp. 5758–5772, 2009.
- [38] G. Reeves and M. Gastpar, "A note on optimal support recovery in compressed sensing," in *Proc. Asilomar Conf. on Signals, Syst., and Comput.*, Nov. 2009, pp. 1576–1580.
- [39] Y. Jin, Y. Kim, and B. D. Rao, "Limits on support recovery of sparse signals via multiple-access communication techniques," *IEEE Trans. Inf. Theory*, vol. 57, no. 12, pp. 7877–7892, 2011.

- [40] M. J. Wainwright, "Information-theoretic limits on sparsity recovery in the high-dimensional and noisy setting," *IEEE Trans. Inf. Theory*, vol. 55, no. 12, pp. 5728–5741, 2009.
- [41] J. Yoo, Y. Xie, A. Harms, W. U. Bajwa, and R. A. Calderbank, "Finding zeros: Greedy detection of holes," *CoRR*, vol. abs/1303.2048, 2013.
- [42] A. Sharma and C. R. Murthy, "On finding a set of healthy individuals from a large population," in *Information Theory and Applications Workshop, San Diego, CA, USA*, 2013, pp. 1–5.
- [43] R. G. Gallager, *Information Theory and Reliable Communication*. New York, NY, USA: John Wiley & Sons, Inc., 1968.
- [44] T. M. Cover and J. A. Thomas, *Elements of information theory*. New York, NY, USA: Wiley-Interscience, 1991.
- [45] G. Atia and V. Saligrama, "A mutual information characterization for sparse signal processing," in *The 38th International Colloquium on Automata, Languages and Programming (ICALP), Switzerland*, 2011.
- [46] C. Aksoylar, G. Atia, and V. Saligrama, "Sparse signal processing with linear and non-linear observations: A unified shannon theoretic approach," *ArXiv e-prints*, Apr. 2013.
- [47] D. Sejdinovic and O. Johnson, "Note on noisy group testing: Asymptotic bounds and belief propagation reconstruction," in *Proc. Allerton Conf. on Commun., Control and Comput.*, 2010, pp. 998–1003.

- [48] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed. McGraw-Hill, 1976.
- [49] H.-B. Chen and F. K. Hwang, "A survey on nonadaptive group testing algorithms through the angle of decoding," *Journal of Combinatorial Optimization*, vol. 15, no. 1, pp. 49–59, 2008.
- [50] M. Aldridge, L. Baldassini, and O. Johnson, "Group testing algorithms: Bounds and simulations," *IEEE Trans. Inf. Theory*, vol. 60, no. 6, pp. 3671–3687, June 2014.
- [51] D. M. Malioutov and M. B. Malyutov, "Boolean compressed sensing: Lp relaxation for group testing." in *Proc. ICASSP*. IEEE, 2012, pp. 3305–3308.
- [52] Y. Nesterov, *Introductory lectures on convex optimization : A basic course*, ser. Applied optimization. Kluwer Academic Publ., 2004.
- [53] G. Lugosi, *Concentration-of-Measure Inequalities*, ser. Lecture Notes. Available at <http://www.econ.upf.edu/~lugosi/anu.pdf>, 2006. [Online]. Available: www.econ.upf.edu/~lugosi/anu.pdf
- [54] M. Mitzenmacher and E. Upfal, *Probability and Computing: Randomized Algorithms and Probabilistic Analysis*. New York, NY, USA: Cambridge University Press, 2005.
- [55] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, Mar. 2004. [Online]. Available: <http://www.amazon.com/exec/obidos/redirect?tag=citeulike07-20&path=ASIN/0521833787>
- [56] S. Haykin, "Cognitive radio: brain-empowered wireless communications," *IEEE J. Sel. Areas Commun.*, vol. 23, no. 2, pp. 201–220, Feb. 2005.

- [57] J. Mitola and G. Q. Maguire, Jr., "Cognitive radio: making software radios more personal," *IEEE Personal Commun. Mag.*, vol. 6, no. 4, pp. 13–18, Aug. 1999.
- [58] Y. C. Liang, K. C. Chen, G. Y. Li, and P. Mahonen, "Cognitive radio networking and communications: An overview," *IEEE Trans. Veh. Technol.*, vol. 60, no. 7, pp. 3386–3407, Sep. 2011.
- [59] T. Yucek and H. Arslan, "A survey of spectrum sensing algorithms for cognitive radio applications," *IEEE Commun. Surveys Tuts.*, vol. 11, no. 1, pp. 116–130, 2009.
- [60] Y. Zeng, Y. C. Liang, A. T. Hoang, and R. Zhang, "A review on spectrum sensing for cognitive radio: challenges and solutions," *EURASIP J. Adv. Signal Process.*, vol. 2010, Jan. 2010. [Online]. Available: <http://dx.doi.org/10.1155/2010/381465>
- [61] E. Axell, G. Leus, E. Larsson, and H. Poor, "Spectrum sensing for cognitive radio : State-of-the-art and recent advances," *IEEE Signal Process. Mag.*, vol. 29, no. 3, pp. 101–116, 2012.
- [62] H. Urkowitz, "Energy detection of unknown deterministic signals," *Proc. IEEE*, vol. 55, no. 4, pp. 523–531, Apr. 1967.
- [63] N. C. Beaulieu, W. L. Hopkins, and P. J. McLane, "Interception of frequency-hopped spread-spectrum signals," *IEEE J. Sel. Areas Commun.*, vol. 8, no. 5, pp. 853–870, Jun. 1990.
- [64] B. K. Levitt, U. Cheng, A. Polydoros, and M. K. Simon, "Optimum detection of slow frequency-hopped signals," *IEEE Trans. Commun.*, vol. 42, no. 2/3/4, pp. 1990–2000, Feb./Mar./Apr. 1994.

- [65] D. Cabric, A. Tkachenko, and R. W. Brodersen, "Experimental study of spectrum sensing based on energy detection and network cooperation," in *ACM TAPAS*, New York, NY, USA, 2006. [Online]. Available: <http://doi.acm.org/10.1145/1234388.1234400>
- [66] L. Luo, N. M. Neihart, S. Roy, and D. J. Allstot, "A two-stage sensing technique for dynamic spectrum access," *IEEE Trans. Wireless Commun.*, vol. 8, no. 6, pp. 3028–3037, 2009.
- [67] N. M. Neihart, S. Roy, and D. J. Allstot, "A parallel, multi-resolution sensing technique for multiple antenna cognitive radios," in *ISCAS*, 2007, pp. 2530–2533.
- [68] M. Hanif, P. Smith, D. Taylor, and P. Martin, "Mimo cognitive radios with antenna selection," *IEEE Trans. Wireless Commun.*, vol. 10, no. 11, pp. 3688–3699, 2011.
- [69] Z. Quan, S. Cui, A. H. Sayed, and H. V. Poor, "Optimal multiband joint detection for spectrum sensing in cognitive radio networks," *IEEE Trans. Signal Process.*, vol. 57, no. 3, pp. 1128–1140, Mar. 2009.
- [70] P. Paysarvi-Hoseini and N. C. Beaulieu, "Optimal wideband spectrum sensing framework for cognitive radio systems," *IEEE Trans. Signal Process.*, vol. 59, no. 3, pp. 1170–1182, Mar. 2011.
- [71] J. Treichler, M. Davenport, and R. Baraniuk, "Application of compressive sensing to the design of wideband signal acquisition receivers," in *Proc. U.S./Australia Joint Work. DASP*, Sep. 2009.

- [72] A. V. Oppenheim and R. W. Schaffer, *Discrete-time signal processing*. Prentice Hall, 1989.
- [73] Z. Quan, S. Cui, H. V. Poor, and A. Sayed, "Collaborative wideband sensing for cognitive radios," *IEEE Signal Process. Mag.*, vol. 25, no. 6, pp. 60–73, Nov. 2008.
- [74] P. Wang, J. Fang, N. Han, and H. Li, "Multiantenna-assisted spectrum sensing for cognitive radio," *IEEE Trans. Veh. Technol.*, vol. 59, no. 4, pp. 1791–1800, May 2010.
- [75] H. V. Poor, *An introduction to signal detection and estimation*. Springer-Verlag, 1994. [Online]. Available: <http://books.google.com/books?id=eI1DBIUbzAQC>
- [76] C. Cordeiro, K. Challapali, D. Birru, and S. S. N., "IEEE 802.22: the first worldwide wireless standard based on cognitive radios," in *Proc. DySPAN*, 2005, pp. 328–337.
- [77] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. Dover Publications, 1964. [Online]. Available: <http://books.google.co.in/books?id=MtU8uP7XMvoC>
- [78] C. Aksoylar, G. Atia, and V. Saligrama, "Sparse recovery with linear and nonlinear observations: Dependent and noisy data," *CoRR*, vol. abs/1403.3109, 2014. [Online]. Available: <http://arxiv.org/abs/1403.3109>
- [79] P. Borjesson and C.-E. Sundberg, "Simple approximations of the error function $q(x)$ for communications applications," *IEEE Trans. Commun.*, vol. 27, no. 3, pp. 639–643, Mar. 1979.