Multiple Transmitter Localization and Whitespace Identification using Randomly Deployed Binary Sensors

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***Abstract*—This work analyzes the limiting behavior of the uncertainty in localizing an unknown number of transmitters within a given geographical area. The set-up consists of n binary sensors that are deployed uniformly at random locations within the area. These sensors detect for the presence of a transmitter within their radio range, and their individual decisions are combined to estimate the number of transmitters as well as their locations. With the mean sum absolute error in transmitter localization as the metric, the optimal scaling of the radio range and the necessary minimum transmitter separation is determined, as n gets large. It is shown that both the localization error and the radio range optimally scale as $\log(n)/n$. The analysis is extended to the case of unreliable sensors, where, surprisingly, the optimal scaling is found to still be $\log(n)/n$. The cognitive radio problem of identifying the available whitespace, i.e., the regions that do not contain any transmitter, emerges as a special case. Finally, the optimal distribution of sensor deployment is determined, given the distribution of the transmitters. Simulation results illustrate the significant performance benefit that can be obtained by optimally scaling the radio range, compared to existing fixed sensing range based designs.**

***Index Terms*—Multiple transmitter localization, whitespace identification, k-coverage, cognitive radio.**

I. Introduction

Determining the number of transmitters, their locations, and communication footprints within a given geographical area of interest is useful in several applications [2]–[5]. A related problem in cognitive radios (CRs) is of whitespace identification, where whitespace is defined as the regions that are not covered by any transmitter. This information is also useful to wireless service providers, for finding *dead-zones* or the coverage holes in their service area. In cognitive radio (CR) networks, knowledge of the available whitespace is crucial for effective spatial spectral reuse by CRs and in order to ensure that the CR nodes do not cause harmful interference to the licensed/primary receivers.

This paper addresses the problem of multiple transmitter localization and whitespace detection, using an approach where n binary sensors are deployed in the geographical area of interest. Their collective observations regarding the presence or absence of a transmitter in their *radio range* r_s are used to localize the transmitters and estimate the available whitespace. A challenge here is to determine the optimal scaling of the radio range of the sensors that minimizes the error in estimating the transmitter locations and the available whitespace, as a function of the number of sensors deployed.

In the literature, various approaches for transmitter localization and whitespace identification have been considered. One approach is to use the received signal strength (RSS) measurements obtained from the sensors, and, to estimate the number of transmitters and their powers such that the sum of the MSE in the location and power estimates is minimized [6]. Other range-based approaches include localization based on the time difference of arrival, angle of arrival, etc [7], [8]. Alternatively, a small, fixed number of sensors are used, and deterministic results for the MSE in transmitter localization are derived [9]–[12]. Decentralized estimation of transmitter locations using messages exchanged between neighboring nodes was considered in [13]–[17]. More recently, this was extended to include the possibility of node failures in [18].

Another popular technique for estimating the locations of transmitters is by using binary sensors instead of the analog RSS measurements [19]–[23]. These binary sensors detect the presence or absence of a transmitter within a *sensing radius* r_s from their location. In practice, binary sensing is accomplished by comparing the RSS observed at the sensor with a predetermined threshold. The sensors return a reading of 1 if the RSS is above the threshold, meaning that there is a transmitter in their vicinity, and return 0 otherwise. In the sequel, we will consider a model where r_s is allowed to decrease with the number of sensors n. This can be accomplished, for example, by appropriately increasing the threshold signal level for declaring the presence of a primary transmitter in the vicinity of the sensor, when the primary transmitters emit their signals at the same power. The latter assumption is reasonable, for example, in a factory floor where the primary transmitters are access points or two-way walkie-talkie radios operating at a fixed power, and our goal is to determine the available whitespace in the factory floor in order to set up a secondary network in that area. The binary measurement model, although idealized, is attractive because of its simplicity and analytical tractability, well as because it is a promising option for use in inexpensive, energy-starved motes that may be deployed purely for monitoring the spatial spectral usage. Moreover, even though binary sensing model

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requires low-complexity sensors, it is a robust and effective approach, as demonstrated in [24]. Hence, this paper focuses on the binary sensing model for developing the results.

Tracking one or more targets with the help of multiple sensors is another related problem that has received considerable research attention in recent years (see [25] for a comprehensive review of literature). More specifically, the problem of target tracking under the binary sensing model has been studied both theoretically [10], [25]–[29] and experimentally [30]. In [25], [31]–[33], one or more targets that are located arbitrarily in the field of interest are tracked using multiple sensors, while in [26], [29], binary sensor measurements are used to decide on the presence or absence of targets at given locations. However, to the best of our knowledge, there have been no studies in the literature on the fundamental limits of localization accuracy and whitespace detection as a function of the number of deployed sensors. Of particular interest are the optimal scaling of the radio range of the sensors to achieve the minimum transmitter localization and whitespace recovery error, the resulting recovery performance, the optimum spatial distribution of sensors, etc., and this forms the focus of this work.

In this paper, we consider a scenario where n binary sensors are deployed uniformly at random locations in a given geographical area for transmitter localization and whitespace identification. Random deployment of sensors is often desirable when the transmitters that need to be localized can be arbitrarily distributed in the area of interest. It is also used in military applications, where a systematic deployment is generally not feasible [34], [35]. We also note that, given the locations of transmitters, one can compute their communication footprints, and thereby determine the available whitespace. However, the opposite is not true; i.e., given the available whitespace, one cannot directly find the locations of the transmitters. Thus, in this paper, we first address the *harder* problem of transmitter localization, and then use the results to determine the available whitespace in Secs. III and IV. Our main contributions are:

1) We start with the problem of determining the number of active transmitters and their locations, with the sum absolute error in transmitter localization as the metric. We show that the optimum minimum localization error scales as $\log(n)/n$, and derive the optimum radio range as well as the minimum separation between transmitters that guarantees that the localization error scales as $\log(n)/n$ with high probability. (see Sec. II.)
2) Next, we focus on the problem of whitespace recovery error, where the total whitespace recovered is determined as the union of the sensing regions around the sensors that return 0. We show that the whitespace recovery error (loss), i.e., the fraction of the available whitespace that is not recovered by the n sensors, and the radio range r_s, both optimally scale as $\log(n)/n$ as n gets large.[1] (see Sec. III.)
3) We extend the analysis to a more practical case, where the sensors report possibly erroneous measurements. We show that, surprisingly, the optimal scaling of the whitespace recovery error and the optimal radio range is still $\log(n)/n$. (see Sec. IV.)
4) Finally, for a given spatial distribution of the transmitter locations, we derive closed-form expressions for the optimal spatial distribution of sensor locations that minimizes the probability of not detecting a transmitter and the resulting minimum miss-detection probability. (see Sec. V.)

Through our derived results, we obtain insights into the number of sensors to be deployed, and their radio range, for accurately localizing the transmitters and for maximizing the recovered whitespace within a given geographical area. We validate our analytical results through Monte Carlo simulations. (see Sec. VI.) The simulation results also illustrate the significant performance improvement that is obtainable by using the optimal scaling for r_s, as the number of sensors n is increased, compared to using a slower or faster decrease of r_s with n (see Fig. 1). Moreover, even though the results are true for large n, the scaling of $r_s = \log(n)/n$ is *optimal* even at moderate or low values of n.

We note that, under a similar binary observation model in a 2-dimensional region with fixed sensor placements, the *expected* whitespace identification error is known to scale as $\frac{1}{\rho r_s}$, where ρ is the density of sensors and r_s is the radio range [25]. However, in [25], r_s is assumed to remain fixed as ρ is increased, i.e., the results do not hold if r_s is allowed to vary as ρ increases. Intuitively, the optimal r_s should decrease with increasing n, since, otherwise, increasing n does not improve the transmitter localization error. In this paper, with $\rho = n$, we are interested in optimal scaling of r_s with n, as ρ is increased. We show that we can achieve a whitespace identification error scaling of $\Theta\left(\frac{\log n}{n}\right)$, *with high probability*.

The organization of the rest of this paper is as follows. In Sec. II, we derive fundamental limits on the sum absolute error in jointly identifying the number of transmitters and their locations, as the number of sensors gets large. In Sec. III, we derive optimal scaling results on the whitespace identification when the sensors are reliable. Section IV extends the results to the case of unreliable sensors. Section V presents the optimum distribution of sensors that minimizes the probability of missing a transmitter. Simulation results are presented in Sec. VI, and concluding remarks are offered in Sec. VII.

II. Transmitter Localization

We let $\mathcal{S}$ denote the region of interest within which M arbitrarily distributed transmitters need to be localized. Without loss of generality, we consider the unit-square $[0, 1]^2$, i.e., the square formed by the points $(0, 0), (0, 1), (1, 0), (1, 1)$, as the area of interest. In the sequel, for simplicity of exposition while developing the fundamental ideas, we will first discuss the results in the 1-dimensional (1-d) case, with the unit length segment $\mathcal{L} \triangleq [0, 1]$ as our region of interest. We assume that n binary sensors are deployed uniformly at random locations on $\mathcal{S}$. Physically, this corresponds to an unplanned deployment of sensors, or the use of existing sensors (e.g., cell-phone users) whose locations can be modeled as being uniformly

[1] All logarithms in the sequel are to the base e.

distributed over the area of interest. Each sensor returns one of two possible readings $b \in \{0, 1\}$; $b = 1$ if there is at least one transmitter within a distance of $r_s(n)$ from it, and $b = 0$ otherwise [10], [25], [26], [29]–[33]. This corresponds to the sensors thresholding their observed RSS value to determine the presence or absence of a primary transmitter. It ignores the effects of path loss and shadowing of the signal, but we include these effects in terms of the analysis of sensing errors in Sec. IV and while presenting simulation results in Sec. VI. We find that the scaling laws derived do carry over to the more practical scenarios. Note that, the sensing radius is allowed to scale with n. Also, M is fixed but possibly unknown, and *does not depend on* n. The sensor readings are combined at a fusion center to localize the transmitters and find the region $\mathcal{A}_{\text{void}}$ of $\mathcal{S}$ that is guaranteed to not contain any transmitter. For example, in the absence of sensing errors, the whitespace is identified as the union of circular regions of radius $r_s(n)$ around the sensors that return a reading of 0.

In this section, we are interested in finding how many transmitters are present and estimating their locations in $\mathcal{S}$, using binary readings from n sensors deployed uniformly at random over $\mathcal{S}$. To this end, we note that each disjoint region containing sensors that returned the value 1 contains at least one transmitter. Hence, we estimate the number of transmitters to be equal to the number of disjoint regions containing sensors that returned the value 1, and we estimate the transmitter locations $\hat{x}_i$ to be the geometric centroid of each such region. Note that, any contiguous region containing sensors that returned the value of 1 could potentially have more than 1 transmitter.[2] This could lead to errors in estimating the number of transmitters and/or their locations, as there is no way of identifying the number of transmitters within regions containing sensors that measured a 1. To overcome this, in this section, we assume that any two transmitters are at at least $\delta(n) > 0$ distance apart. As we will see, under mild assumptions on $\delta(n)$, one can accurately estimate the number of transmitters and their locations with high probability, as $n \to \infty$. Moreover, even if the transmitter are located less than $\delta(n)$ apart, since the whitespace is identified as the area outside the coverage of the the transmitters, the proposed scheme will still identify whitespace available, although the number of transmitters may not be correctly determined.

Let the true location of the transmitters be denoted by $x_j, j = 1, 2, \ldots, M$. Let the estimate of M be denoted by $\hat{M}$, and let the estimate of the location of the i^{th} transmitter using the n sensor readings be denoted by $\hat{x}_i, i = 1, 2, \ldots, \hat{M}$. In the 1-d case, for both the true locations and their estimates, we index the transmitters from left to right on $\mathcal{L}$, such that $x_1 \leq x_2 \leq \cdots \leq x_M$ and $\hat{x}_1 \leq \hat{x}_2 \leq \cdots \leq \hat{x}_{\hat{M}}$. In the 2-d case, we index the true locations and their estimates in increasing order of their distance from the origin. We define the *localization error* metric as

$$\mathsf{err} = \sum_{i=1}^{\max\{M, \hat{M}\}} \|x_i - \hat{x}_i\|_\ell.$$

[2] In particular, if the region is of width greater than $4r_s$ in the 1-d case, or of area greater than $4\pi r_s^2$ in the 2-d case, then it must necessarily contain more than one transmitter.

Here, $\|\cdot\|_\ell$ is the absolute value for the 1-d case, and the squared Euclidean distance in the 2-d case. Now, since M could differ from $\hat{M}$, we follow the convention that for $M < \hat{M}$, x_i is at the origin for $i = M+1, \ldots, \hat{M}$, and for $M > \hat{M}$, $\hat{x}_i$ is at 1 in the 1-d case and at $(1, 1)$ in the 2-d case, for $i = \hat{M}+1, \ldots, M$. This ensures that a mismatch between the actual and estimated number of transmitters is always positively penalized. We note that the above penalty function is not necessarily unique. Other notions of localization error could also be defined, for example, by pairing true transmitter locations with estimated locations based on minimum distance, and computing the corresponding mean squared error in localization. However, for any such notion of error that associates a strictly positive error corresponding the transmitters that are missed or falsely detected, as n gets large, the number of transmitters is estimated correctly, and the minimum error scales as $\log n/n$. Hence, any such notion that leads to a positive penalty when the number of transmitters is incorrectly estimated suffices for our purposes in this paper.

Our goal is to find the minimum error $\epsilon(n)$, transmitter separation $\delta(n)$ and radio range $r_s(n)$, that solve the optimization problem

$$\begin{aligned} &\arg \min_{r_s(n), \delta(n)} \epsilon(n) \\ &\text{such that} \lim_{n\to\infty} P\left(\sum_{i=1}^{\max\{M,\hat{M}\}} \|x_i - \hat{x}_i\|_\ell \leq \epsilon(n) \right) = 1, \end{aligned} \quad (1)$$

where the probability is defined over the uniformly random distribution of binary sensor locations. Note that the localization error $\epsilon(n)$ depends on the chosen radio range $r_s(n)$ and the transmitter separation $\delta(n)$, and (1) is a multi-objective optimization problem. For example, if $r_s(n)$ is too small, then a very small area is sensed, while if $r_s(n)$ is too large, the sensing areas overlap too much, and in both cases the localization accuracy could suffer. We note that it is challenging to exactly solve the above optimization problem for a given number of sensors, i.e., it is difficult to arrive at an explicit sequence of sensing radius, transmitter separation and error that solves (1). However, by letting the number of sensors grow large, we obtain order-optimal results that uncover the fundamental relationships between the different parameters of interest. Thus, the goal here is to jointly find the order-optimal radio range and localization error as n gets large. Similar connections exist for the transmitter separation and localization error as well.

The next two Theorems characterize a lower bound that $\epsilon(n)$, $r_s(n)$ and $\delta(n)$ need to satisfy for accurate estimation of the number and locations of the transmitters with high probability in the 1-d and 2-d cases, respectively.

Theorem 1: For the 1-d case, if $\lim_{n\to\infty} \frac{\epsilon(n)}{\frac{\log n}{n}} = 0$, or $\lim_{n\to\infty} \frac{r_s(n)}{\frac{\log n}{n}} = 0$, or $\lim_{n\to\infty} \frac{\delta(n)}{\frac{\log n}{n}} = 0$ then

$$\lim_{n\to\infty} P\left(\sum_{i=1}^{\max\{M,\hat{M}\}} \|x_i - \hat{x}_i\|_\ell \leq \epsilon(n) \right) < 1.$$

In words, if $\epsilon(n)$, $r_s(n)$, or $\delta(n)$ go to zero faster than $\frac{\log n}{n}$, then the probability that we are able to estimate the number of transmitters and their locations with an error at most $\epsilon(n)$ cannot be made arbitrarily close to 1. Hence, $\frac{\log n}{n}$ is a lower bound on the mean absolute error in location estimation.

Remark 1: Note that in (1), the $\min$ is defined jointly over $r_s(n)$ and $\epsilon(n)$. In Theorem 1, we are essentially showing that if any one of $r_s(n)$ or $\epsilon(n)$ approach zero faster than $\frac{\log n}{n}$, the probability in (1) is strictly smaller than 1. Thus, we have a decoupled lower bound on the optimal $r_s(n)$ and $\epsilon(n)$ for the problem in (1). A similar approach is used to state the all the results in the sequel.

The proof of Theorem 1 requires the following coverage Lemma from [36]:

Lemma *1:* (Theorem 3.11 [36]) Let n sensors be deployed uniformly at random locations on $\mathcal{L} \triangleq [0,1]$, where each sensor has radio range of $r(n)$. A point x on $\mathcal{L}$ is said to be *covered* if there is at least one sensor in the interval $[x-cr(n),\ x+cr(n)]$, where c is a constant. Then, if $\lim_{n\to\infty} \frac{r(n)}{(\log n)/n} = 0$, then $\lim_{n\to\infty} P(\text{all points in } \mathcal{L} \text{ are covered}) < 1$. For a 2-d region $\mathcal{S} \triangleq [0,1]^2$, where again n sensors be deployed uniformly at random locations, and a point is said to be covered if there is a sensor within a circle of radius $cr(n)$ around it, we have that if $\lim_{n\to\infty} \frac{r(n)}{\sqrt{(\log n)/n}} = 0$, then $\lim_{n\to\infty} P(\text{all points in } \mathcal{S} \text{ are covered}) < 1$.

Proof: (Theorem 1) See Appendix A. ■

Next, we present the 2-d counterpart of the lower bound obtained in Theorem 1.

Theorem 2: For a 2-d region $\mathcal{S}$, if $\lim_{n\to\infty} \frac{\epsilon(n)}{\frac{\log n}{n}} = 0$, or $\lim_{n\to\infty} \frac{r_s(n)}{\sqrt{\frac{\log n}{n}}} = 0$, or $\lim_{n\to\infty} \frac{\delta(n)}{\sqrt{\frac{\log n}{n}}} = 0$, then $\lim_{n\to\infty} P\left(\sum_{i=1}^{\max\{M,\hat{M}\}} \|x_i - \hat{x}_i\|_\ell \le \epsilon(n)\right) < 1$.

Proof: See Appendix B. ■

Our next result shows that for the 1-d case, $\epsilon(n) = r_s(n) = \delta(n) = \Theta\left(\frac{\log n}{n}\right)$ is sufficient for estimating the number and location of transmitters with high probability, asymptotically in n.

Theorem 3: For the 1-d case, if $r_s(n) = \delta(n) = \epsilon(n) = \Theta\left(\frac{\log n}{n}\right)$,

$$\lim_{n\to\infty} P\left(\sum_{i=1}^{\max\{M,\hat{M}\}} \|x_i - \hat{x}_i\|_\ell \le \epsilon(n)\right) = 1.$$

In words, if we choose $r_s(n)$ of the order $\frac{\log n}{n}$, then the lower bound on the error $\epsilon(n)$ of order $\frac{\log n}{n}$ is in fact achievable.

Remark 2: Note that the minimum transmitter separation, $\delta(n) = \Theta\left(\frac{\log n}{n}\right)$ is a mild requirement. Since the number of transmitters M is fixed while $\delta(n)$ is monotonically decreasing with n, for n large enough, the minimum separation between the transmitters will exceed $\delta(n)$.

To prove the Theorem, we need the following Chernoff bound.

Lemma *2:* Let $X_1, X_2, \ldots$ be independent and identically distributed Bernoulli random variables, and let $X = \sum_{i=1}^n X_i$, with $\mathbb{E}\{X\} = \mu = n\mathbb{E}\{X_i\}$. Then for $0 < \delta < 1$, we have that $P(X < (1-\delta)\mu) \le \exp\left(-\frac{\delta^2 \mu}{2}\right)$.

Proof: (Theorem 3) See Appendix C. ■

The result for $2-d$ region follows similarly, as follows.

Theorem 4: For a 2-d region, if $r_s(n) = \delta(n) = \Theta\left(\sqrt{\frac{\log n}{n}}\right)$ and $\epsilon(n) = \Theta\left(\frac{\log n}{n}\right)$, then

$$\lim_{n\to\infty} P\left(\sum_{i=1}^{\max\{M,\hat{M}\}} \|x_i - \hat{x}_i\|_\ell \le \epsilon(n)\right) = 1.$$

Proof: See Appendix D ■

In this section, we considered the problem of estimating both the number of transmitters as well as their locations using n sensors making binary measurements. We first showed that if the minimum transmitter separation is less than order $\frac{\log n}{n}$, then the localization error probability cannot go to zero. Conversely, with the minimum transmitter separation of order $\frac{\log n}{n}$, using the Chernoff bound, we showed that if the radio range is of order $\frac{\log n}{n}$, we can partition $\mathcal{L}$ or $\mathcal{S}$ into small enough intervals so that with high probability, no interval contains more than one transmitter, while simultaneously ensuring that there are enough sensors in each interval for detection of the transmitter with high probability. In practice, since transmitters typically have a minimum geographical separation, the separation requirement for our results to hold is easily satisfied, and, hence, the number of transmitters and their locations can be detected efficiently.

Remark 3: Another localization problem of interest is when the number of transmitters scales with the number of sensors n as $M(n)$. Theorem 3 suggests that, in the 1-d case, if $M(n)$ is such that the minimum distance between any two transmitters scales no faster than order $\frac{\log n}{n}$, then a localization error of $M(n)\frac{\log n}{n}$ can be guaranteed with high probability. So, clearly, for $M(n) = \mathcal{O}\left(\sqrt{\frac{n}{\log n}}\right)$, where the minimum distance between any two transmitters scales no faster than $\frac{\log n}{n}$ [29], the localization error scales as $\sqrt{\frac{n}{\log n}}\frac{\log n}{n}$, i.e., as $\sqrt{\frac{\log n}{n}}$. Thus, our results also extend to the case where the number of transmitters scales with n, under certain conditions. In the next section, we consider the problem of finding the whitespace $\mathcal{A}_{\text{void}}$ that contains no transmitters using binary sensors randomly deployed in the area.

III. Whitespace Detection

In cognitive radios, to avoid interference, it is important to find regions that do not contain any primary transmitter; and such regions are called as whitespace. Let $x_1, x_2, \ldots, x_n$, be the sensor locations in $\mathcal{L}$ or $\mathcal{S}$ and $b_1, b_2, \ldots, b_n$, be the corresponding sensor readings, where, as before, $b_i = 1$ if there is at least one transmitter within a distance $r_s(n)$ from it sensor i, and $b_i = 0$ otherwise. The sensor readings are combined at a fusion center to find the region $\mathcal{A}_{\text{void}}$ of $\mathcal{L}$ or $\mathcal{S}$ that is guaranteed not to contain any transmitter. In the 1-d case,

$$\mathcal{A}_{\text{void}} = \bigcup_{i=1}^{n} (1-b_i)[\max(x_i - r_s, 0), \min(x_i + r_s, 1)]. \quad (2)$$

In the 2-d case, letting $\mathcal{S} \triangleq [0,1]^2$, we have

$$\mathcal{A}_{\text{void}} = \bigcup_{i=1}^{n} (1-b_i)\mathbf{B}(x_i, r_s(n)) \cap \mathcal{S}, \quad (3)$$

where $\mathbf{B}(x,r)$ denotes a Euclidean disc of radius r centered at x.

Let $\ell(\mathcal{A}) = \int_{x\in\mathcal{A}} \mathrm{d}x$ denote the Lebesgue measure, i.e., in the 1-d case, the length of the region formed by $\mathcal{A}$. For example, if $\mathcal{A}$ is the union of a finite set of disjoint regions, then $\ell(\mathcal{A})$ is the sum of the lengths of the disjoint regions. Similarly, in the 2-d case, $\ell(\mathcal{A}) = \int_{(x,y)\in\mathcal{A}} \mathrm{d}x\mathrm{d}y$, i.e., it represents the area of the region denoted by $\mathcal{A}$.

We define the recovered whitespace $A_{\text{void}} = \ell(\mathcal{A}_{\text{void}})$ as the Lebesgue measure of the region where no transmitter is located. Note that, since the transmitters are located at distinct points that occupy no area, we would expect that, as $n \to \infty$, the transmitters are perfectly localized, and, $A_{\text{void}} \to 1$. Hence, we want to find the minimum $\epsilon(n)$ and the corresponding optimum radio range $r_s(n)$ that guarantees that $P\left((1 - A_{\text{void}}) \le \epsilon(n)\right) = 1$. Formally, we want to solve

$$\arg\min_{r_s(n)} \epsilon(n) \text{ subject to } \lim_{n\to\infty} P((1 - A_{\text{void}}) \le \epsilon(n)) = 1. \tag{4}$$

The probability in the above equation is evaluated over the distribution of the sensor locations, with the unknown transmitter locations assumed to be fixed but arbitrary. This metric essentially captures the scaling of the relative loss in recovering the whitespace, with increasing n, as a function of $r_s(n)$. So, there are two problems to solve, i) finding the minimum scaling of the error $\epsilon(n)$, and ii) finding the optimal radio range $r_s(n)$, both as a function of n. As before, finding an exact solution to the problem is difficult; hence, we look for order-optimal results that hold as $n \to \infty$.

Clearly, whitespace detection is a special case of localization considered in Section II, since once we know the number of transmitters and their location, we automatically get the whitespace as well. Hence, from Section II, we get the following results.

Theorem 5: For the whitespace recovery problem in a 1-d unit-length region, the optimal radio range and whitespace recovery error scale as $\epsilon(n) = r_s(n) = \Theta\left(\frac{\log n}{n}\right)$. In words, if we choose $r_s(n)$ of the order $\frac{\log n}{n}$, then the whitespace recovery loss also scales as order $\frac{\log n}{n}$, and this is the best scaling that can be achieved.

Proof: Follows from Theorems 1 and 3. We omit the details to avoid repetition. ■

The result for the 2-d region is as follows.

Theorem 6: For the whitespace recovery problem in a 2-d region $\mathcal{S}$, the optimal radio range and whitespace recovery error scale as $\epsilon(n) = \Theta\left(\frac{\log n}{n}\right)$ and $r_s(n) = \Theta\left(\sqrt{\frac{\log n}{n}}\right)$.

Proof: Follows from Theorems 2 and 4. For brevity, we omit the details. ■

In this section, we obtained the optimal scaling laws for the radio range and the whitespace recovery error in the 1-d and 2-d cases. In deriving the results, we assumed that the sensor readings were error-free. Surprisingly, the above results hold even when the sensors are unreliable, as we show in the following section.

IV. Unreliable Sensors

So far in this paper, we have assumed the case of ideal sensors, that make no errors. To study the robustness of this binary sensing model, in this section, we assume that the sensors make an error in their reading with probability $p < \frac{1}{2}$ independently of all other sensors. The errors could be attributed to hardware failures at the node, receiver noise, interference leakage from other frequency bands, etc. Thus, a sensor reading could be 1 even if there is no transmitter within a range of r_s around it (i.e., a false alarm), or a sensor reading could be 0 even if there is a transmitter within a range of r_s around it (i.e., a missed detection), and both events happen with probability at most p. In practice, the error rate could be a function of the distance between the transmitter and the sensor. However, since the transmitter locations are unknown, analyzing that case becomes intractable. To simplify the problem, we consider p to be the largest error probability with which a sensor can make errors, which essentially takes care of the worst case scenario.

We illustrate the results in this section for the slightly simpler problem of whitespace detection, as we have already described in detail how to analyze the localization problem in Section II.

Thus, we are interested in finding the minimum error $\epsilon(n)$ and radio range $r_s(n)$ that solves the optimization problem

$$\arg\min_{r_s(n)} \epsilon(n) \text{ subject to } \lim_{n\to\infty} P((1 - A_{\text{void}}) \le \epsilon(n)) = 1. \tag{5}$$

The following Theorem is the analog of Theorem 5, with unreliable sensors. Its proof follows simply because the lower bound with unreliable sensors cannot be better than the lower bound with reliable sensors derived in Theorem 5.

Theorem 7: When the sensor measurements are in error with probability of error $p < \frac{1}{2}$, and for the whitespace recovery problem in a 1-d unit-length region, if $\lim_{n\to\infty} \frac{\epsilon(n)}{\frac{\log n}{n}} = 0$ or $\lim_{n\to\infty} \frac{r_s(n)}{\frac{\log n}{n}} = 0$, then $\lim_{n\to\infty} P\left((1 - A_{\text{void}}) \le \epsilon(n)\right) < 1$.
The 2-d version follows similarly from Theorem 6.

Next, we show that the lower bound in Theorem 7 is also achievable. The proof is constructive, and follows by proposing a reconstruction strategy and analyzing its error performance.

Theorem 8: For the 1-d case, when the sensor measurements are in error with probability of error $p < \frac{1}{2}$, if $r_s(n) = \Theta\left(\frac{\log n}{n}\right)$, then for $\epsilon(n) = \Theta\left(\frac{\log n}{n}\right)$, $\lim_{n\to\infty} P\left(1 - A_{\text{void}} \le \epsilon(n)\right) = 1$.

Proof: See Appendix E. ■

Theorem 9: For the 2-d case, when the sensor measurements are in error with probability of error $p < \frac{1}{2}$, if $r_s(n) = \Theta\left(\sqrt{\frac{\log n}{n}}\right)$, then for $\epsilon(n) = \Theta\left(\frac{\log n}{n}\right)$, $\lim_{n\to\infty} P\left(1 - A_{\text{void}} \le \epsilon(n)\right) = 1$.

Proof: Let the unit square $\mathcal{S}$ be tiled into smaller squares of side $\sqrt{\frac{c \log n}{n}}$. Using the Chernoff bound, each square contains at least $\frac{c \log n}{2}$ sensors with high probability. Thereafter, the proof follows identically to the proof of Theorem 8. ■

In this section, we considered the case when each sensor makes an error with probability p in the detection of any transmitter within its radio range. The lower bound on the whitespace recovery error is the same as in the case of reliable sensors, since the error with unreliable sensors cannot be better than that with reliable sensors. For finding the matching upper bound on the whitespace recovery error, we let the radio range be of order $\frac{\log n}{n}$, so that each interval of width $\frac{\log n}{n}$ contained more or less $\log n$ sensors with high probability. Then, for each interval of width $\frac{\log n}{n}$, we proposed a majority rule for declaring the presence or absence of transmitter in that interval. Since there are a large number of sensors (roughly $\log n$) in each interval, if $p < 1/2$, it followed from a repetition code argument that the probabilities of false alarm and missed detection go to zero for large n. Thus, we showed that, if the radio range is such that there are enough number of sensors in each small interval, asymptotically, the lack of reliability of the sensors has no effect on the whitespace recovery error.

Remark 4: In our setup, n sensors detect the presence or absence of a primary transmitter in their vicinity, and the binary outputs of the sensors is used to localize the transmitters and determine the available whitespace. The above analysis can handle sensing errors, by setting $p = \max(p_f, p_m)$, where p_f and p_m are the false alarm and missed detection probabilities. Specifically, our approach of using majority decisions across nearby sensors leads to asymptotically optimal localization accuracy of primary transmitters even under sensing errors.

Also note that, once the whitespace has been declared by the n sensors, a location could be within the coverage area of the primary transmitters but declared as being out of the coverage area, or vice versa. This can be quantified in terms of (a) the fraction of the available whitespace that is declared as occupied (which is akin to a false alarm) and (b) the fraction of the area covered by the primary users that is declared as whitespace (which is akin to a missed detection). These quantities represent the probability that, if a cognitive radio deployed uniformly at random in the area of interest follows the whitespace output by the sensors to determine whether or not it can transmit makes an error: a false alarm or missed detection, respectively. From our results, since the optimal localization error scales as $\epsilon(n) = \Theta\left(\frac{\log n}{n}\right)$, the area uncertainty in whitespace identification can be upper bounded by $\frac{2\pi RM\sqrt{\epsilon(n)}}{|\mathcal{A}|}$, where R is the coverage radius of the primary transmitter, M is the number of primary transmitters, and $|\mathcal{A}|$ is the area of the region of interest. Due to this, the average probability of detection error over $\mathcal{A}$ also scales as $\Theta\left(\sqrt{\frac{\log n}{n}}\right)$.

V. Optimum Distribution of the Sensor Locations

Thus far, we assumed that the transmitters are arbitrarily located on $\mathcal{L}$, and obtained bounds on the minimum localization error and the optimal sensing radius in the worst case scenario. In some scenarios, it may be possible to obtain the spatial distribution of the transmitters on $\mathcal{L}$, either as side information from the primary network or from long-term statistics collected by the sensors. In this section, we consider the optimization of the spatial distribution of the sensor locations given the spatial distribution of transmitter. We assume that the transmitters are distributed over $\mathcal{L} = [0, 1]$ with pdf $f_X(x)$, and seek to find the optimum sensor distribution $f_\lambda(x)$ over $\mathcal{L}$ that minimizes the probability of missing a transmitter. Intuitively, minimizing the probability of missing a transmitter can lead to low transmitter localization error, and, in turn, result in good whitespace recovery properties. Mathematically, we wish to solve

$$P_f = \min_{f_\lambda(x):\int_0^1 f_\lambda(x)\mathrm{d}x=1} \int_0^1 \left(1 - \int_{\max\{y-r_s,0\}}^{\{y+r_s,1\}} f_\lambda(z)\mathrm{d}z\right)^n f_X(y)\mathrm{d}y, \quad (6)$$

In (6), given that the location of transmitter is y, $p(y) \triangleq \int_{\max\{y-r_s,0\}}^{\{y+r_s,1\}} f_\lambda(z)\mathrm{d}z$ is the probability that there is a sensor that is able to detect its presence. Hence, $(1-p(y))^n$ represents the probability that none of the sensors lie within the sensing range r_s of the transmitter. By averaging over the distribution of y, P_f captures the probability that all the sensors have their reading equal to 0, and hence completely miss the transmitter at a random location in $\mathcal{L}$. For small r_s, the (6) can be approximated by the more tractable expression

$$\hat{P}_f = \min_{f_\lambda(x):\int_0^1 f_\lambda(x)\mathrm{d}x=1} \int_0^1 (1 - 2r_s f_\lambda(x))^n f_X(x)\mathrm{d}x. \quad (7)$$

The above is obtained by approximating the probability that there is a sensor that that detects the presence of a transmitter located at x by $2r_s f_\lambda(x)$, which is valid for continuous $f_\lambda(x)$, small r_s, and ignoring edge effects.

It might appear at first glance that the optimal i.i.d. sensor location distribution to solve (7) be equal to the transmitter location distribution, however, that is not true, as shown below, since it depends on r_s.

Using elementary results from variational calculus [37] to construct the Lagrangian from (6), differentiating it, setting the derivative equal to zero and solving, the optimum $f_\lambda(x)$ that minimizes (6) must satisfy

$$\int_{\max\{x-r_s,0\}}^{\{x+r_s,1\}} n\left(1 - \int_{\max\{y-r_s,0\}}^{\{y+r_s,1\}} f_\lambda(z)\mathrm{d}z\right)^{n-1} f_X(y)\mathrm{d}y = \mu, \quad (8)$$

$\forall\, x \in [0, 1)$, where the Lagrange multiplier μ has to be chosen to satisfy the constraint $\int_0^1 f_\lambda(x)\mathrm{d}x = 1$. Unfortunately, it is difficult to solve for $f_\lambda(x)$ from the above equation. Hence, we use the more tractable approximation for the miss probability given by (7), which leads to the optimality condition,

$$n\left(1 - 2r_s f_\lambda(x)\right)^{n-1} f_X(x) 2r_s = \mu, \quad (9)$$

where μ is a Lagrange multiplier factor, and is chosen such that $\int_0^1 f_\lambda(x)\mathrm{d}x = 1$. This leads to

$$f_\lambda(x) = \left(1 - \left(\frac{\mu}{2nr_s f_X(x)}\right)^{\frac{1}{n-1}}\right)\frac{1}{2r_s}. \quad (10)$$

In the above, $f_\lambda(x)$ is taken to be 0 when $f_X(x) = 0$ or when the right hand side is negative. In some cases, the above reduces to intuitively satisfying results. For example, when $f_X(x) = 1, 0 \le x < 1$, the above implies that $f_\lambda(x) = 1, 0 \le x < 1$, i.e., the optimum density is also uniform. On the other hand, when $n = 1$, i.e., when only one sensor is deployed, $f_\lambda(x)$ drops out of (9), but since (7) is linear in $f_\lambda(x)$, the optimal solution must occur at an extreme point. This leads to the the optimal distribution $f_\lambda(x) = \delta(x - x_0)$, where $x_0 = \arg\max_x f_X(x)$.

The value of μ that ensures $\int_0^1 f_\lambda(x)\mathrm{d}x = 1$ has to be obtained using numerical techniques. This is not difficult, since the right hand side in (10) is monotonically decreasing in μ, taking the value $1/2r_s > 1$ when $\mu = 0$, and taking the value 0 as μ gets large. Thus, any simple numerical technique such as the bisection method can be used to find the value of μ.

Now, substituting the optimum $f_\lambda(x)$ into (7) and simplifying, we get

$$P_f^{(\mathrm{opt})} = \frac{(1 - 2r_s)^n}{\left[\int_0^1 (f_X(x))^{-\frac{1}{n-1}} \mathrm{d}x\right]^{n-1}}. \quad (11)$$

We recognize the denominator as the ℓ_p quasi-norm of $1/f_X(x)$, with $p = 1/(n-1)$. Note that, substituting the uniform point distribution for $f_\lambda(x) = 1, 0 \le x < 1$ in (7) results in $P_f^{(\mathrm{unif})} = (1 - 2r_s)^n$. Thus, the performance improvement from the optimized point density depends on the magnitude of the denominator in (11). For example, consider the case were X has a triangular distribution: $f_X(x) = 4x$ for $0 \le x < 1/2$, and $= 4(1 - x)$ for $1/2 \le x < 1$. With some algebra, it can be shown that (11) reduces to

$$P_f^{(\mathrm{opt})} = \frac{2(1 - 2r_s)^n}{\left(\frac{n-1}{n-2}\right)^{n-1}} \approx \frac{2(1 - 2r_s)^n}{e\left(\frac{n-1}{n-2}\right)}. \quad (12)$$

Thus, the optimum point density does improve performance over the uniform point density, but both scale as $(1 - 2r_s)^n$ with n. When $r_s = (\log n)/n$, for large n, $(1 - 2r_s)^n \approx 1/n^2$, i.e., the probability of missing a transmitter is inversely proportional to n^2.

VI. Simulation Results

We now present Monte Carlo simulation results to illustrate the analytical results developed in this paper. We consider M transmitters and n sensors deployed uniformly at random locations over $\mathcal{L} = [0, 1]$. Sensors return a 1 if there is a transmitter within the sensing radius r_s around them, and return 0 otherwise. We identify the whitespace as the union of the $2r_s$-width regions around sensors that return 0. To estimate the number of transmitters and their locations, we first identify the occupied space as the union of the $2r_s$ regions around sensors that return 1. Then, for each contiguous occupied region of width smaller than $2r_s$, we identify one transmitter at the center of the region. In contiguous occupied regions of width greater than $2r_s$, we identify $\lfloor \mathrm{width}/2r_s \rfloor$ transmitters, placed uniformly in the region. We compute the probability of the whitespace recovered exceeding $1 - \epsilon(n)$, i.e., the objective function in (4), with $\epsilon(n) = \log(n)/n$, and the probability of

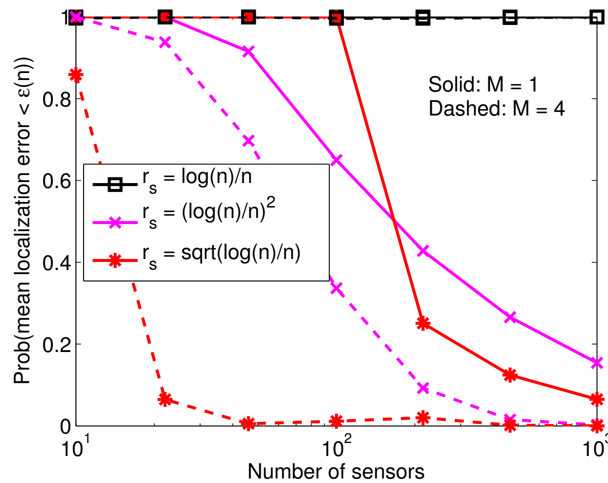

Fig. 1. Probability that the sum absolute error in localizing the transmitters is $< \epsilon(n)$, with $\epsilon(n) = \log(n)/n$.

the transmitter localization error exceeding $\epsilon(n)$ as in (1), by averaging over 10,000 instantiations of transmitter and sensor deployments.

In Fig. 1, we plot the probability that the sum absolute error in localizing the transmitters is $< \epsilon(n)$, given by (1). We set $\epsilon(n) = \log(n)/n$, and compare the performance of three different scalings for r_s: $\log(n)/n$, $(\log(n)/n)^2$, and $\sqrt{\log(n)/n}$, for $M = 1$ and $M = 4$ transmitters. We see that $\log(n)/n$ captures the optimal scaling of the radio range with n, and it significantly outperforms the other scalings considered. Moreover, even at moderate or low values of n, scaling $r_s(n)$ at a rate that is higher or lower than $\log(n)/n$ results in a significant degradation in the performance.

Figure 2 shows the probability of the whitespace recovered exceeding $1 - \epsilon(n)$, i.e., the objective function in (4), versus the number of sensors n, with $M = 1$ and 4 transmitters. To show the behavior over a similar range of values of n, we use $\epsilon(n) = \log(n)/n$ in the $M = 1$ transmitter case, and $\epsilon(n) = 4\log(n)/n$ for the $M = 4$ transmitter case. In Fig. 3, we plot the probability of the whitespace recovered exceeding $1 - \epsilon(n)$ as a function of M, for $n = 250, 500$ and 1000. In both Figs. 2 and 3, we see that radio range scaling of $r_s(n) = \log(n)/n$ outperforms other faster or slower scaling factors, which is in line with the result in Theorem 5. Figure 4 shows an analogous result in the 2-d case, where we compare the scaling factors $r_s(n) = \sqrt{\frac{\log(n)}{n}}, \frac{2}{n^{1/4}}$, and $\left(\frac{\log(n)}{n}\right)^{1/4}$. Again, we see that the optimal scaling factor of $r_s(n) = \sqrt{\frac{\log(n)}{n}}$ outperforms the other scaling factors.

In Fig. 5, we include the effect of lognormal shadowing and Rayleigh-distributed multipath fading into the simulation, as follows. We map the unit length interval to a physical distance of 10 km. We consider the primary transmitters to be transmitting at $P_T = 10$ dB relative to the noise floor at the sensor nodes. We assume a path loss exponent of $\eta = 4$, a reference distance of $d_0 = 10$ m, a lognormal shadowing standard deviation of $\sigma_s = 3.5$ dB, and standard exponentially distributed multipath fading [38]. The received power at the

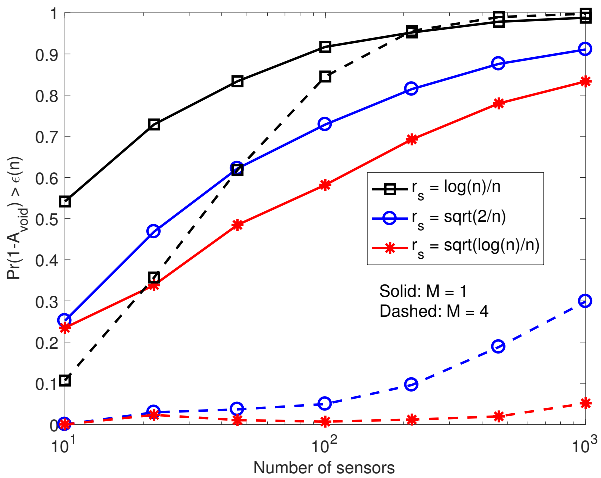

Fig. 2. Probability that the whitespace recovered is $> 1-\epsilon(n)$, with $\epsilon(n) = \log(n)/n$ for $M = 1$ and $\epsilon(n) = 4\log(n)/n$ for $M = 4$.

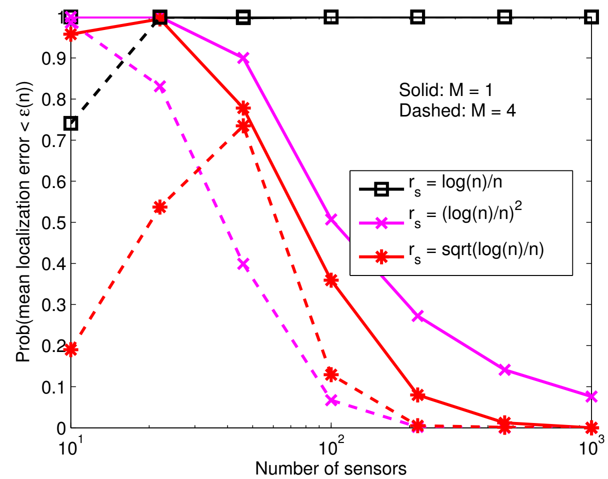

Fig. 5. Probability that the sum absolute error in localizing the transmitters is $< \epsilon(n)$, with $\epsilon(n) = \log(n)/n$. The plot includes the effect of lognormal shadowing with standard deviation 3.5 dB and Rayleigh fading.

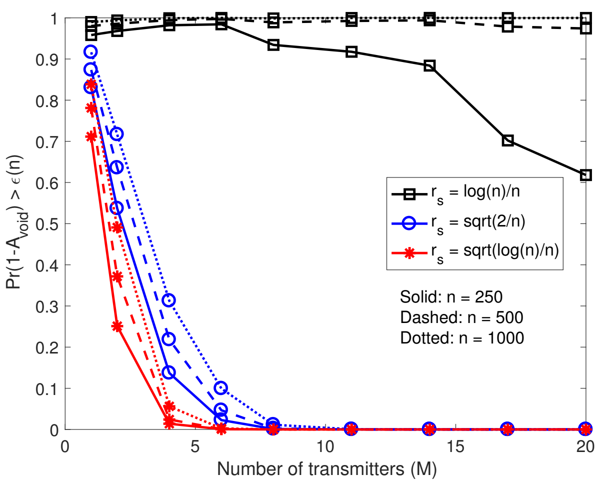

Fig. 3. Probability that the whitespace recovered is $> 1-\epsilon(n)$, as a function of M, the number of primary transmitters, for $n = 250, 500$ and 1000.

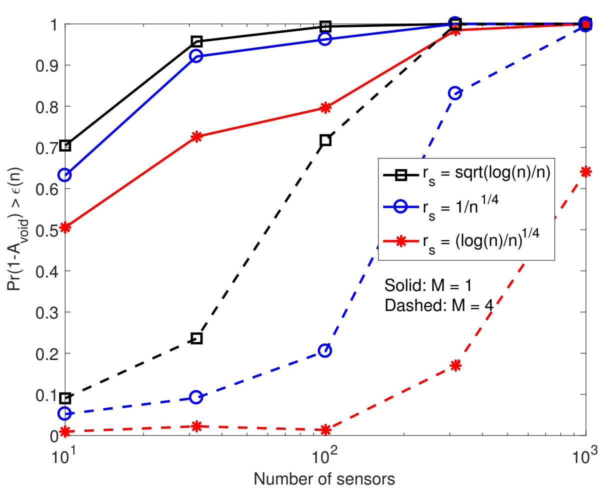

Fig. 4. Probability that the whitespace recovered is $> 1 - \epsilon(n)$ in the 2-d setting, with $\epsilon(n) = \log(n)/n$ for $M = 1$ and $\epsilon(n) = 4\log(n)/n$ for $M = 4$.

sensors is computed as the sum of the powers from the different transmitters. The received power is compared to a threshold based on $r_s(n)$ to determine the sensor readings. The threshold itself is computed as $P_T/(r_s(n)/d_0)^\eta$, i.e., it is the received power at a distance of $r_s(n)$ from a single primary transmitter, in the absence of shadowing and fading. We assume the shadowing and fading are i.i.d. across the sensors, and average the results over 1000 different channel instantiations. Due to the effect of shadowing and fading, sensors within $r_s(n)$ could miss the primary transmitter, and sensors outside $r_s(n)$ could detect the presence of a transmitter. Hence, we divide the interval into small regions of length $\log(n)/n$, and use the majority decision rule proposed in Sec. IV to determine the presence or absence of sensors in each of the small regions. We estimate the number of primary transmitters as the number of contiguous regions containing intervals that detect a primary transmitter. Also, we estimate the transmitter locations as the centroids of each of the contiguous intervals. We plot the probability that the sum absolute error in localizing the transmitters is $< \epsilon(n)$. We see that the scaling of $r_s(n) = \log(n)/n$ outperforms the other scaling rates, even after incorporating the effect of shadowing and multipath fading. Note that, this result also illustrates that the binary sensing model is accurate, even in the presence of multiple transmitters, and after accounting for signal fading and shadowing effects.

Finally, Fig. 6 shows the probability of missing a transmitter uniformly distributed on $[0, 1]$, and n sensors with sensing radius $r_s(n) = \log(n)/n$ are deployed according to the triangular, truncated Gaussian and uniform distributions. For the triangular distribution, we consider $f_\lambda(x) = 4x$ for $0 \leq x < 1/2$, and $= 4(1-x)$ for $1/2 \leq x < 1$. For the truncated Gaussian distribution, we consider the Gaussian distribution with mean 0.5 and standard deviation 0.25, truncated to $[0, 1]$. We see that, as expected, the uniform distribution outperforms the other distributions, and its performance matches with the $P_f \approx 1/n^2$ result derived in Sec. V.

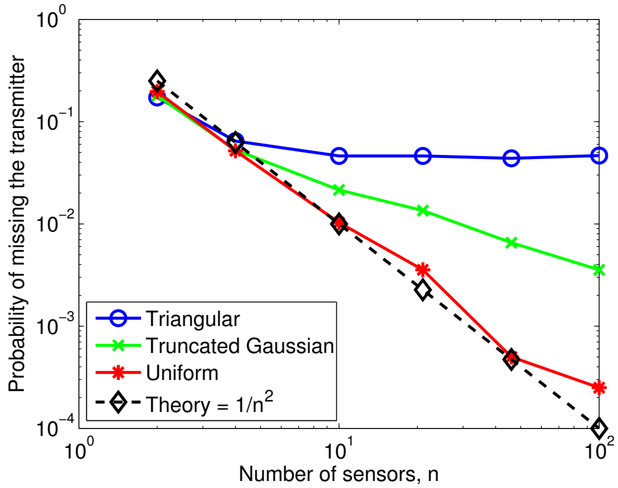

Fig. 6. Probability of missing a transmitter, when n sensors with sensing radius $r_s(n) = \log(n)/n$ are deployed according to the triangular, truncated Gaussian and uniform distributions.

VII. Conclusions

In this paper, we studied the localization of transmitters and recovery of whitespace using n binary sensors that are deployed at random locations within a given geographical area. We derived the limiting behavior of the mean absolute error in localization and the recovered whitespace as a function of n and the sensing radius r_s. Using the sum absolute error in transmitter localization as the metric, we analyzed the optimal scaling of r_s that minimizes the localization error with high probability, as n gets large. We derived the corresponding minimal localization error, and showed that it scales as $\log(n)/n$. We showed that both the whitespace recovery error (loss) and the radio range optimally scale as $\log(n)/n$ as n gets large. We also showed that, surprisingly, the radio range scaling of $\log(n)/n$ is optimal even with unreliable sensors. Finally, we derived the optimal distribution of sensors that minimizes the probability of missing a transmitter, for a given distribution of the transmitters, and analyzed the behavior of the missed detection probability as n is increased. Our results yielded useful insights into the interplay between the number of sensors to be deployed and the corresponding optimal radio range for detecting transmitters, that maximizes the recovered whitespace and accurately localizes the transmitters within the given geographical area.

In our work, we considered that the sensors make one-bit readings. Effectively, this amounts to quantizing the power measured at the sensors to a single bit. In contrast, sensors with multi-bit quantization capability changes the problem considerably. In practice, successively using different values of r_s, one can convey different bits (MSB to LSB) of RSS observed by the sensors. The bit (correspondingly, the value of r_s) that is the most relevant for accurate reconstruction of the whitespace would perhaps correspond to the value of r_s used in our work. In general, with multi-bit quantization, one needs to first come up with an appropriate quantization scheme, and an associated algorithm to identify the whitespace, and then analyze its performance. This is an interesting line for future work. Another possible extension is to consider the problem of tracking temporal variations in the whitespace (see, e.g., [39]).

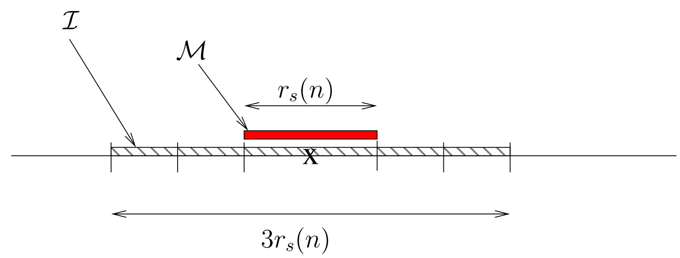

Fig. 7. Illustration of the area that needs to be empty of sensors to get a lower bound of $r_s(n)$ on the localization error. If the region marked $\mathcal{I}$ around any transmitter (whose location is marked by an X) does not contain any sensor, the localization error is at least $r_s(n)$.

Appendix A
Proof of Theorem 1

In order to establish the lower bound, it is sufficient to consider the case of a single transmitter, i.e., $M = 1$, since the error can only be higher for $M > 1$. Consider Fig. 7, where a single transmitter is located at location x, and consider the interval $\mathcal{I}$ of length $3r_s(n)$ centered at x. If $\mathcal{I}$ does not contain any sensors, the middle interval $\mathcal{M}$ of length $r_s(n)$ is beyond the sensing range of any sensor, since the radio range is $r_s(n)$. Therefore, $\mathcal{M}$ lies in the uncertainty region, if $\mathcal{I}$ does not contain any sensors, and hence the localization error is at least $r_s(n)$. Since the location of the transmitter x can be arbitrary, to have localization error $\epsilon(n)$ of at most $r_s(n)$, we need all intervals of width $3r_s(n)$ to contain at least one sensor.

From Lemma 1, if $r_s(n)$ is less than order $\frac{\log n}{n}$, then $\lim_{n\to\infty} P(\text{each point in } \mathcal{L} \text{ is covered}) \leq c_2$, $c_2 < 1$. If any point on $\mathcal{L}$ is not covered, then surely the interval of length $2cr_s(n)$ (for a constant c) around it has no sensor. Hence, if $3r_s(n)$ is less than order $\frac{\log n}{n}$, then there exists an interval of width $3r_s(n)$ that does not have any sensor with probability greater than $1 - c_2$. Hence, if $r_s(n)$ is less than order $\frac{\log n}{n}$, then for any localization error $\epsilon(n)$ less than order $\frac{\log n}{n}$, $\lim_{n\to\infty} P\left(\sum_{i=1}^{\max\{M,\hat{M}\}} \|x_i - \hat{x}_i\|_\ell \leq \epsilon(n)\right) < 1$.

Now we work towards finding the lower bound on $\delta(n)$. Consider $M = 2$ transmitters at locations x_1 and x_2 with distance $\|x_2 - x_1\|_\ell = \delta(n)$ between them. To be able to decide that two transmitters are present, i) at least one sensor has to lie between x_1 and x_2 with a reading of 0, or ii) $r_s(n)$ has to be less than or equal to $\delta(n)$, since otherwise the sensors lying outside the interval (x_1, x_2) cannot discern whether there are one or two transmitters, as both x_1 and x_2 may possibly be in their range r_s.

Since the two transmitters can be arbitrarily located on $\mathcal{L}$, condition i) implies that each interval of length $\delta(n)$ on $\mathcal{L}$ should contain at least one sensor or $r_s(n) \leq \delta(n)$. From Lemma 1, the probability that each interval of length $\delta(n)$ contains at least one sensor is upper bounded by a constant less than 1 if $\lim_{n\to\infty} \frac{\delta(n)}{\frac{\log n}{n}} = 0$. We already know that, for $\lim_{n\to\infty} \frac{r_s(n)}{\frac{\log n}{n}} = 0$, $P\left(\sum_{i=1}^{\max\{M,\hat{M}\}} \|x_i - \hat{x}_i\|_\ell \leq \epsilon(n)\right) <$

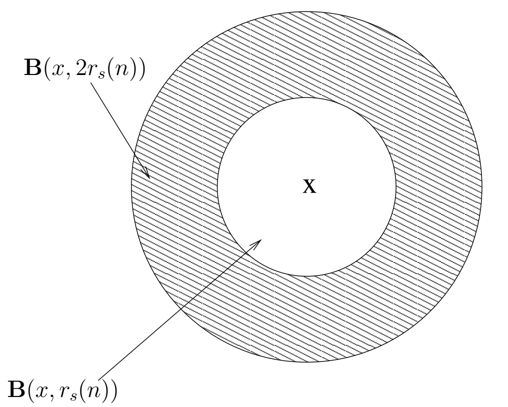

Fig. 8. Empty areas for sensors in 2-dimensions to get a lower bound on localization error. If the ball $\mathbf{B}(x, 2r_s(n))$ does not contain any sensor, the localization error is at least $r_s(n)$.

1. Thus, conditions i) and ii) together imply that for $\lim_{n\to\infty} \frac{\delta(n)}{\frac{\log n}{n}} = 0$, $P\left(\sum_{i=1}^{\max\{M,\hat{M}\}} \|x_i - \hat{x}_i\|_\ell \leq \epsilon(n)\right) < 1$.

APPENDIX B
PROOF OF THEOREM 2

Once again, to find the lower bound, consider the case of a single transmitter, i.e., $M = 1$. Consider Fig. 8, where a single transmitter is located at location x, and consider the disc $\mathbf{B}(x, 2r_s(n))$ of radius $2r_s(n)$ centered at x. If $\mathbf{B}(x, 2r_s(n))$ does not contain any sensors, then clearly, since the radio range is $r_s(n)$, the smaller disc $\mathbf{B}(x, r_s(n))$ of radius $r_s(n)$ is beyond the sensing range of any sensor. Therefore, $\mathbf{B}(x, r_s(n))$ lies in the uncertainty region if $\mathbf{B}(x, 2r_s(n))$ does not contain any sensors, and hence the localization error is at least equal to the area of $\mathbf{B}(x, r_s(n))$. Since the location of the transmitter x can be arbitrary, to have localization error $\epsilon(n)$ of at most $\pi r_s^2(n)$, we need all discs of radius $2r_s(n)$ to contain at least one sensor.

From the 2-d part of Lemma 1, if $r_s(n)$ is less than order $\sqrt{\frac{\log n}{n}}$, for $c = 2$,

$$\lim_{n\to\infty} P(\text{each point in } \mathcal{S} \text{ is covered}) \leq c_2,$$

where $c_2 < 1$. If any point on $\mathcal{S}$ is not covered, then surely the disc of radius $2r_s(n)$ around it has no sensor. Hence, if $2r_s(n)$ is less than order $\sqrt{\frac{\log n}{n}}$, then there exists a disc of radius $2r_s(n)$ that does not have any sensor with probability greater than $1 - c_2$. Hence, if $r_s(n)$ is less than order $\sqrt{\frac{\log n}{n}}$, then for any localization error $\epsilon(n)$ less than order $\frac{\log n}{n}$, $\lim_{n\to\infty} P\left(\sum_{i=1}^{\max\{M,\hat{M}\}} \|x_i - \hat{x}_i\|_\ell \leq \epsilon(n)\right) < 1$.

The lower bound on $\delta(n)$ follows identically along the lines of the proof of Theorem 1.

APPENDIX C
PROOF OF THEOREM 3

Let $r_s(n) = \frac{c \log n}{n}$, $c > 1$, and let the minimum transmitter separation $\delta(n) = \frac{d \log n}{n}$, where $d > 10c$. Divide the region $\mathcal{L}$ into smaller intervals of length $\left(\frac{10c \log n}{n}\right)$, and index these segments as z_1 to $z_{\lceil(\frac{n}{10c \log n})\rceil}$. Hence, each interval z_k contains at most one transmitter.

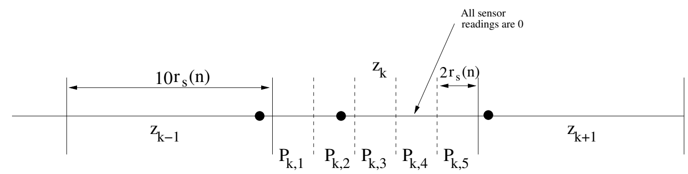

Fig. 9. Worst case positions of transmitters for estimating their locations.

Partition each z_i into five equal parts of width $\left(\frac{2c \log n}{n}\right)$, and index them with $P_{i,1}, \ldots, P_{i,5}$. Let the number of sensors lying in $P_{i,j}$ be $N_{i,j}$. From the Chernoff bound, $P(N_{i,j} < c \log n) \leq n^{-c/4}$, and taking the union bound, $P(N_{i,j} < c \log n$ for any $i, j = 1, \ldots, 5) < c_4 n^{1-c/4}$, where c_4 is a constant. Thus, with high probability, each partition of each interval contains at least $c \log n$ sensors for large enough c.

Consider any interval z_k. If all the sensor readings in z_k are zero, or if only the readings of the sensors lying in left half of $P_{k,1}$ or right half of $P_{k,5}$ are 1, then no transmitter lies in z_k. Otherwise, we know that there is a transmitter lying in z_k, say x_i. Note that, it is hardest to detect the location of transmitter lying in z_k if there are transmitters in both intervals z_{k-1} and z_{k+1}, and they lie closest to the boundary of z_k, as shown in Fig. 9, where black dots represent the transmitters. Let $x_i \in P_{k,j}$. Then, an interval W_i of width $\left(\frac{2c \log n}{n}\right)$ around x_i contains at least $\left(\frac{c \log n}{n}\right)$ sensors with high probability from the Chernoff bound, and all these sensors have reading 1. In addition, irrespective of the index j of partition $P_{k,j}$ to which x_i belongs, there exists l_m, $l_m \in \{1, \ldots, 5\}$ for which all sensors lying in the partition P_{k,l_m} have a reading of 0, since all the sensors lying in P_{k,l_m} are at a distance greater than the radio range $\left(\frac{c \log n}{n}\right)$ from the transmitter x_i in $P_{k,j}$. For example, in Fig. 9, all sensors lying in $P_{k,4}$ have their reading equal to 0. Hence, using the readings from sensors in W_i and P_{k,l_m}, we can localize the transmitter in W_i, and the uncertainty about the i^{th} transmitter location is no more than twice the width of any partition. This equals $\left(\frac{4c \log n}{n}\right)$, and hence, $\|\hat{x}_i - x_i\|_\ell < \left(\frac{4c \log n}{n}\right)$. Since this is true for each transmitter i, $\hat{M} = M$, the total localization error $\sum_{i=1}^{\max\{M,\hat{M}\}} \|x_i - \hat{x}_i\|_\ell \leq \sum_{i=1}^{M} \left(\frac{4c \log n}{n}\right) \leq M\left(\frac{4c \log n}{n}\right)$ with high probability. This concludes the proof.

APPENDIX D
PROOF OF THEOREM 4

The proof follows along the lines of the proof of Theorem 3. We tile the unit square $\mathcal{S}$ into smaller squares $s_{i,j}$ with side $\sqrt{\frac{10c \log n}{n}}$. We further partition each small square into 25 smaller squares, with side $\frac{1}{5}\sqrt{\frac{10c \log n}{n}}$. Using the Chernoff bound (Lemma 2), each smaller square contains at least $c \log n$ sensors for, large enough c. Once again, with minimum separation of $\delta(n) = \Theta\left(\sqrt{\frac{\log n}{n}}\right)$, at most one transmitter is present in any one smaller square. Hereafter, the proof follows identically to the proof of Theorem 3, where the 25 smaller squares within each square, play the role of 5 vertical and 5 horizontal partitions of each interval, defined in the proof of Theorem 3.

Appendix E
Proof of Theorem 8

Let $r_s(n) = \left(\frac{c\log n}{n}\right)$, where $c > 1$ is a constant. Divide the segment $\mathcal{L}$ into smaller intervals of length $\left(\frac{c\log n}{n}\right)$, and index these segments as z_1 to $z_{\lceil(\frac{n}{c\log n})\rceil}$. Let the number of sensors lying in z_k be N_k. From the Chernoff bound, $P(N_k < \frac{c\log n}{2}) \le n^{-c/8}$, and taking the union bound, $P(N_k < \frac{c\log n}{2} \text{ for any } k) < c_4 n^{1-c/8}$, where c_4 is a constant. Thus, with high probability, for large $c > 8$, each smaller interval z_k contains at least $\frac{c\log n}{2}$ sensors.

As before, since there are only M transmitters, at the maximum only M smaller intervals among $z_1 \dots z_{\lceil(\frac{n}{c\log n})\rceil}$ contain any transmitter. Since a transmitter lying in z_k can only influence readings of sensors lying in 3 adjacent intervals z_{k-1}, z_k and z_{k+1}. Therefore, ideally, at least $\left\lceil\left(\frac{n}{c\log n}\right)\right\rceil - 3M$ intervals among $z_1 \dots z_{\lceil(\frac{n}{c\log n})\rceil}$ should have all sensor readings as 0. However, because of errors in sensor readings, some of the sensors in these intervals have readings 1 instead of 0. To resolve this problem, we use the majority rule to decide whether an interval z_k contains a transmitter or not. Thus, a transmitter is declared to be *present* in an interval z_k, if the number of sensors with reading 1 are more than the number of sensors with reading 0, and a transmitter is declared to be *absent* in an interval z_k otherwise.

With this decision rule, $A_{\text{void}} = \cup_{\text{Maj}(z_k)=0} \, z_k$, where the function $\text{Maj}(z_k)$ equals 1 if z_k has a larger number of sensors with a reading of 1 than with a reading of 0, and equals 0 otherwise. We first consider the probability of missed detection, P_{md}, which is the probability that the majority of sensor readings in z_k is 0, given that there is a transmitter in an interval z_k. Recall that each sensor makes an error with probability p independently of all other sensors. From the Chernoff bound, we know that in each interval z_k there are at least $\frac{c\log n}{2}$ sensors with high probability, for large enough c. Let N_k denote the number of sensors in z_k. Without loss of generality, we assume that N_k is odd, as otherwise, we can consider one less sensor for making decisions. Then $P_{md} = \sum_{k=\frac{N_k+1}{2}}^{N_k} \binom{N_k}{k} p^k (1-p)^{N_k-k}$. From an upper bound known in coding theory for repetition codes [40], $\sum_{k=\frac{N_k+1}{2}}^{N_k} \binom{N_k}{k} p^k (1-p)^{N_k-k} \le \left(2\sqrt{p(1-p)}\right)^{N_k}$. Hence, for $p < \frac{1}{2}$, $P_{md} \le a^{N_k}$, where $a < 1$, and N_k is of the order $\log n$ with high probability. Thus, the probability of missed detection P_{md} decreases exponentially with $\frac{c\log n}{2}$. Since there are at the maximum $\left\lceil\left(\frac{n}{c\log n}\right)\right\rceil$ intervals in $\mathcal{L}$, using the union bound, the probability that there is a missed detection in any one of the $\left\lceil\left(\frac{n}{c\log n}\right)\right\rceil$ intervals is at most $\left\lceil\left(\frac{n}{c\log n}\right)\right\rceil a^{\frac{c\log n}{2}}$, where $a < 1$. Thus, for c large enough, $\lim_{n\to\infty} P\left(\left\lceil\left(\frac{n}{c\log n}\right)\right\rceil a^{\frac{c\log n}{2}} = 0\right) = 1$.

Using an identical analysis, we can show that the probability of false alarm in any interval z_k, i.e., the probability that the majority of sensor readings in z_k is 1, given that there is no transmitter in an interval z_k, goes to zero as n goes to infinity.

Thus, with high probability, we have that $\left(\left\lceil\left(\frac{n}{c\log n}\right)\right\rceil - 3M\right)$ intervals not containing any transmitter have their majority of reading equal to 0. Hence, $A_{\text{void}} > \left(\left\lceil\left(\frac{n}{c\log n}\right)\right\rceil - 3M\right)\left(\frac{c\log n}{n}\right) = 1 - 3M\left(\frac{c\log n}{n}\right)$ with high probability, proving the Theorem.

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